A Revival of the Girth Conjecture

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Abstract

We show that for each $\varepsilon > 0$, there exists a number g such that the circular chromatic index of every cubic bridgeless graph with girth at least g is at most $3+\varepsilon$. This contrasts to the fact (which disproved the Girth Conjecture) that there are snarks of arbitrary large girth. In particular, we show that every cubic bridgeless graph with girth at least 14 has the circular chromatic index at most $7/2$.

1 Introduction

We study edge-colorings of cubic graphs. A *proper* edge-coloring of a graph G is a coloring of all edges of G such that every two incident edges receive distinct colors. The smallest number of colors for which there is a proper edge-coloring is called the *chromatic index* and denoted by $\chi'(G)$. The chromatic index of a cubic graph is either 3 or 4 by a theorem of Vizing [20]. In this paper, we study circular edge-colorings of simple cubic graphs.

The circular coloring of graphs was introduced by Vince [19] under the name of "star coloring". A *proper circular p/q-edge-coloring* is a coloring of edges of G by colors $0, \ldots, p-1$ such that the difference modulo p of colors assigned to two incident edges is not among $-(q-1), -(q-2), \ldots, q-1$. A circular p/q-edge-coloring can also be viewed as a coloring by points on a circle of circumference p in such a way that a pair of incident edges receive colors which are at distance at least q on the cycle. The smallest ratio p/q for which there is a proper circular

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p/q-edge-coloring is called the circular chromatic index of G and denoted by $\chi'_c(G)$ (the minimum is always attained [5, 19]). It can be shown that $\chi'(G)-1 < \chi'_c(G) \leq \chi'(G)$. It is also true that for each p and q with $\chi'_c(G) \leq p/q$, there is a proper circular p/q-edge-coloring of G. For further results on the circular coloring, the reader is referred to a recent survey on the subject by Zhu [22].

It can be easily deduced that $3 \leq \chi'_c(G) \leq 4$ for each cubic bridgeless graph G and that $\chi'_c(G) = 3$ iff $\chi'(G) = 3$. Cubic bridgeless graphs whose chromatic index is equal to 4 are known as *snarks* [6, 21] (sometimes an additional connectivity requirement is imposed). Zhu [22, Question 8.4] asked whether there is a cubic bridgeless graph whose circular chromatic index is equal to 4. Afshani, Hatami and Tusserkani [1] have recently proved that the circular chromatic index of each cubic bridgeless graph is at most 11/3. This cannot be further improved, as witnessed by the Petersen graph whose circular chromatic index is 11/3. The condition that the graph is bridgeless cannot be removed because there are cubic graphs whose circular chromatic index is equal to 4. There are also bridgeless graphs of maximum degree 3 whose circular chromatic index is equal to 4.

A conjecture of Jaeger and Swart [12] asserts the existence of a number g such that each snark has girth at most g (actually, they conjectured that $g = 6$ suffices) where the *girth* of a graph is the length of its shortest cycle. This conjecture has become known as the Girth Conjecture and it was disproved by Kochol [13] who constructed cyclically 5-edge-connected snarks with arbitrary large girths. In the present paper, we prove that the Girth Conjecture "holds" for the circular edge-coloring. In particular, our main result is the following: For each $\varepsilon > 0$, there is a number g such that every cubic bridgeless graph with girth at least g has circular chromatic index at most $3 + \varepsilon$ (Corollary 1). This result is best possible in the following sense: For each g, there is a cubic bridgeless graph with girth g with circular chromatic index greater than 3 (just consider a snark with girth g).

We remark that large girth itself does not imply that the circular chromatic number is smaller than the chromatic number. For each integers $k \geq 1$ and $g \geq 3$, there is a graph G with girth at least g with $\chi(G) = \chi_c(G) = k$ as shown by Steffen and Zhu [18]. However, Galluccio, Goddyn and Hell [10] proved the following for graphs avoiding a fixed graph H as a minor: For every graph H and every $\varepsilon > 0$, there is g such that the circular chromatic number of each graph with girth at least g which does not contain H as a minor is at most $2 + \varepsilon$.

Standard graph theory notation, which can be found e.g. in [7], is used throughout this paper. For $X \subseteq V(G)$, $G[X]$ denotes the subgraph of G induced by vertices of the set X. The *order* of a graph is the number of its vertices. All the colorings we construct in this paper are circular $(3p+1)/p$-colorings. If p is an even integer, the colors are denoted by the numbers $-3p/2$, $-3p/2+1$, ..., -1, 0, 1, ..., $3p/2-1$, $3p/2$. The numbers can be viewed as points evenly distributed along the circle whose circumference is $3p+1$. Thus, two incident edges can be assigned numbers x and y only if $p \leq |x-y| \leq 2p+1$.

Proofs of our results are based on a concept of systems of independent segments in cubic graphs which is a modification of the concept of independent systems of representatives. Both these concepts are introduced in Section 2. Our main result is stated and proved in Section 3. A more careful analysis allows us to show that the circular chromatic index of each cubic bridgeless graph with girth at least 14 is at most 7/2 (Theorem 4). We conclude the paper by posing several open problems in Section 5.

2 Systems of Independent Segments

In this section, the concept of systems of independent segments is introduced. This concept is closely related to the concept of independent systems of representatives introduced by Fellows [9]. We recall this concept and state some results related to it. These results are then used to derive several lemmas which are essential for our work.

Fix an integer $k \geq 1$ and consider a graph G whose vertex set is partitioned into sets $V_1, \ldots, V_k$. Vertices $v_1, \ldots, v_k$ form an *independent system of representatives (ISR)* of G with respect to the partition $V_1, \ldots, V_k$ if $v_1, \ldots, v_k$ form an independent set and $v_i \in V_i$ for each $i = 1, \ldots, k$. An argument based on the topological connectivity of certain simplicial complexes was used to derive a sufficient condition on the existence of an ISR in [4]. A more general form of this argument is stated in [14]. Here, we formulate one of its corollaries.

A subset W of the vertex set of a graph G is called *total dominating* if the following holds: For each vertex $v \in V(G)$, there is a vertex $w \in W$ such that vw is an edge of G. Note that we require each vertex v to be adjacent to a vertex of W no matter whether v is in W or not. The smallest size of a total dominating set in G is called the *total domination number* of G and denoted by $\gamma(G)$.

In this paper, we use the following sufficient condition due to Haxell [11] on the existence of an ISR:

Theorem 1 *Let G be a graph whose vertex set is partitioned into sets $V_1, \ldots, V_k$. Then, G has an independent system of representatives with respect to $V_1, \ldots, V_k$*

if the following holds for each non-empty set $I \subseteq \{1, \ldots, k\}$:

$$\gamma\left(G[\bigcup_{i \in I} V_i]\right) \geq 2|I| \ ,$$

where $G[\bigcup_{i \in I} V_i]$ is the graph induced by the vertices of the union $\bigcup_{i \in I} V_i$.

An interested reader can find other conditions on the existence of ISR's and some generalizations of the concept in [2, 3, 15].

Instead of independent systems of representatives, we consider *systems of independent segments* in cubic graphs. Fix a cubic bridgeless graph G with girth g and a 1-factor F of G. We define a graph $G_{F,k}$ for $1 \leq k \leq g$ as follows: Vertices of $G_{F,k}$ are k-tuples $(v_1, \ldots, v_k)$ of vertices of G for which there is a cycle C in $G \setminus F$ such that the vertices $v_1, \ldots, v_k$ are consecutive in the cycle C (they form a path in C). Since $k \leq g$, the graphs G and $G_{F,k}$ have the same order. Vertices of $G_{F,k}$ are called *segments*, or *k-segments* if we want to emphasize the number of vertices in each segment. Two segments $(v_1, \ldots, v_k)$ and $(w_1, \ldots, w_k)$ are joined by an edge in $G_{F,k}$ if there exist v_i and w_j with $v_i w_j \in F$, i.e., a vertex of one of the segments is adjacent to a vertex of the other one. Notice that the maximum degree of $G_{F,k}$ is at most k^2 (but it need not be a k^2-regular graph unless g is sufficiently large).

Let t be the number of the cycles of $G \setminus F$ and let $C_1, \ldots, C_t$ be these cycles. Vertices of $G_{F,k}$ are partitioned into t sets $V_1, \ldots, V_t$, where each set V_i consists precisely of segments of the cycle C_i, $i = 1, \ldots, t$. Hence, the size of V_i is equal to the length of the cycle C_i and the minimum size of a set V_i is at least g. Segments $s_1, \ldots, s_t$ form a *system of r-independent segments* if $s_i \in V_i$ for each $i = 1, \ldots, t$ and the distance between each pair of vertices s_i and s_j with $i \neq j$ in $G_{F,k}$ is at least $r + 1$. Note that a system of 1-independent segments is just a system of independent representatives in the graph $G_{F,k}$ with respect to $V_1, \ldots, V_t$. In general, a system of r-independent segments is a system of independent representatives in $G^r_{F,k}$ where $G^r_{F,k}$ is the r-th power of the graph $G_{F,k}$. Recall that the *r-th power* of a graph is the graph on the same vertex set in which two vertices are joined by an edge if their distance in the original graph is at most r.

We now formulate a sufficient condition on the existence of a system of r-independent segments which is based on Theorem 1:

Lemma 1 *Let r and k be positive integers and let G be a cubic bridgeless graph with girth at least $2k^{2r}$. For each 1-factor F of G, the graph $G_{F,k}$ contains a system of r-independent segments.*

Proof: We need to show that the graph $G^r_{F,k}$ contains an independent system of representatives with respect to the sets $V_1, \ldots, V_s$ corresponding to the factor F in the way described before the statement of this lemma. Note that each V_i has size at least $2k^{2r}$ by the assumption on the girth of G. Since the maximum degree of the graph $G_{F,k}$ is at most k^2, the maximum degree of the graph $G^r_{F,k}$ cannot exceed $k^2(k^2-1)^{r-1} \leq k^{2r}$. Now, let I be a non-empty subset of the set $\{1, \ldots, s\}$. Since the order of the graph $G^r_{F,k}[\bigcup_{i \in I} V_i]$ is at least $2k^{2r} \cdot |I|$ and its maximum degree is at most k^{2r}, its total domination number is at least $\frac{2k^{2r} \cdot |I|}{k^{2r}} = 2|I|$. Then, by Theorem 1, $G^r_{F,k}$ contains an independent system of representatives with respect to the sets $V_1, \ldots, V_s$.

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3 Circular $(3+\varepsilon)$-Edge-Colorings

Theorem 2 *Let G be a bridgeless cubic graph with girth at least $2(2p)^{2p}$ where $p \geq 2$ is an even integer. Then, the circular chromatic index of G is at most $3 + 1/p$.*

Proof: Fix an even integer $p \geq 2$ and a 1-factor F of G. Note that G must have a 1-factor by Petersen's theorem [16] since it is a bridgeless cubic graph. In what follows, we construct a circular $(3p+1)/p$-edge-coloring of G using the colors $-3p/2, -3p/2+1, \ldots, -1, 0, 1, \ldots, 3p/2-1, 3p/2$.

First, certain vertex-disjoint subgraphs called octopuses are found in G. An *octopus* has the following structure: Let C be a cycle of $G \setminus F$. An octopus contains (among others) $2p$ consecutive vertices $v_1, \ldots, v_{2p}$ of C together with all edges incident with them. The cycle C is called the *head cycle* of the octopus and the vertices of the octopus contained in C form the *head* of the octopus. Next, let $v_{i,0}$ be a neighbor of v_i in F for $i = 1, \ldots, 2p$ and let $v_{i,-p+2}, v_{i,-p+3}, \ldots, v_{i,0}, \ldots, v_{i,p-3}, v_{i,p-2}$ be $2p-3$ consecutive vertices of the unique cycle of $G \setminus F$ which contains the vertex $v_{i,0}$. Next, let $v_{i,j,0}$ be a neighbor of $v_{i,j}$ in F for $i = 1, \ldots, 2p$ and $j = -p+2, \ldots, p-2$ and let $v_{i,-p+4}, v_{i,-p+5}, \ldots, v_{i,0}, \ldots, v_{i,p-5}, v_{i,p-4}$ be $2p-7$ consecutive vertices of the cycle of $G \setminus F$ which contains the vertex $v_{i,j,0}$. In this manner, we construct $p/2+1$ levels of vertices (the first level contains vertices of the cycle C, the second level vertices $v_{i,j}$, the third level vertices $v_{i,j,k}$, etc.). Hence, the i-th level of vertices, $2 \leq i \leq p/2+1$, is composed of a number of *blocks* of $2p+5-4i$ vertices consecutive in the cycles of $G \setminus F$ such that the middle vertex of each block is adjacent to a vertex of the preceding level. Note that blocks of the last level are formed by single vertices.

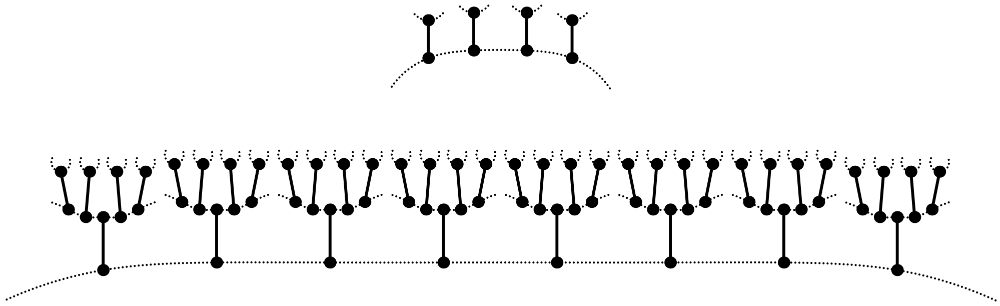

Figure 1: Examples of octopuses for $p = 2$ and $p = 4$.

The octopus itself is formed by the vertices of all levels and all edges incident with at least one such vertex. Examples of octopuses for $p = 2$ and $p = 4$ can be found in Figure 1 (edges of the 1-factor are solid while other edges are dotted; the heads are in the bottom). The edges of cycles of $G \setminus F$ which are incident with two vertices of the octopus are called *inner edges*, while those incident with exactly one vertex of the octopus are called *contact edges*. Both the inner and the contact edges are considered to be parts of the octopus. Since the girth of G is greater than $2p^2 + 2p$, the subgraph induced by an octopus in the graph G is acyclic. Indeed, if the induced subgraph contained a cycle, then its length would be at most $2 \cdot (p/2 + 1) \cdot 2p = 2p^2 + 2p$ (observe that if the subgraph induced by the octopus is not acyclic, then its shortest cycle intersect at most two blocks of each level of the potential octopus).

The crucial property of octopuses is the following:

Claim 1 *Consider a precoloring of all contact edges with the colors $-p/2$ and $p/2$ such that each pair of the contact edges incident with the same block (with the head, respectively) receive opposite colors. Then, this precoloring can be extended to a circular $(3p + 1)/p$-edge-coloring of the whole octopus.*

We postpone the proof of Claim 1 and we begin by showing how it can be used to derive the statement of the theorem.

By Lemma 1, the graph $G_{F,2p}$ contains a system of p-independent segments. Let S be such a system. For each segment s of S, there is an octopus in G whose head is s. Let now O be the set of all octopuses o such that the head of o belongs to S and the head cycle of o has odd length. All the octopuses of O are vertex-disjoint. Indeed, if two of them are not disjoint, then the segments corresponding to their heads have distance in $G_{F,2p}$ at most $2 \cdot p/2 = p$ which is impossible because S is a system of p-independent segments. In particular, two different octopuses of O can only share an edge which is a contact edge of each of them.

We now construct a circular $(3p+1)/p$-edge-coloring of G. Remove from G the edges of the 1-factor F and remove all inner edges of the head of each octopus in O. The remaining graph consists of cycles of even lengths (which are not head cycles) and paths of even length (each odd cycle of $G \setminus F$ is a head cycle of a single octopus of O). Let us color edges of the remaining graph with the colors $-p/2$ and $p/2$ alternately. Observe that the contact edges of each octopus head receive opposite colors. Remove now the colors from all inner edges of octopuses of O. We have just obtained a precoloring of the edges of $G \setminus F$ except for the inner edges of octopuses of O.

Since a block of each octopus consists of an odd number of vertices, it contains an even number of inner edges and the colors of the two contact edges incident with each block are opposite. Hence, the precoloring can be extended to the interiors of all octopuses by Claim 1. The only uncolored edges of G are now the edges of F contained in no octopus of O. Since such edges are incident only with edges of $G \setminus F$ colored with the colors $-p/2$ and $p/2$, they can be arbitrarily colored with the colors $-3p/2$ and $3p/2$. In this way, we obtain a circular $(3p+1)/p$-edge-coloring of the graph G.

Proof of Claim 1: As we said above, we first extend the precoloring to the head, then to the blocks of the second level, then to the blocks of the third level, etc. It will hold that edges of the 1-factor joining the vertices of the i-th and $(i+1)$-th levels of the octopus (recall that the first level is its head) will have one of the following $p+2-2i$ colors:

$$-p-i, \ldots, -3p/2+1, -3p/2 \quad \text{and} \quad p+i, \ldots, 3p/2-1, 3p/2 \; .$$

We first extend the coloring to the head of the octopus. By symmetry, we may assume that the first contact edge is precolored with $-p/2$ and the last one with $p/2$. Then, the coloring of the $2p-1$ inner edges of the head can be as follows:

$$p/2+1, -p/2+1, p/2+2, -p/2+2, \ldots, p, 0, -p, \ldots, p/2-1, -p/2-1 \; .$$

Each of the $2p$ edges of F incident with the head can be colored with one of the colors $p+1, \ldots, 3p/2$ and $-p-1, \ldots, -3p/2$. Extensions of a precoloring to the head for $p=2$, $p=4$ and $p=6$ are depicted in Figure 2. In the figure, the edges of the 1-factor are solid while the other edges are dotted.

Recall that the octopus is formed by several levels of vertices such that the i-th level, $i \geq 2$, of the octopus consists of several blocks of $2p+5-4i$ consecutive vertices. From each block, there is a single edge of F joining its middle vertex

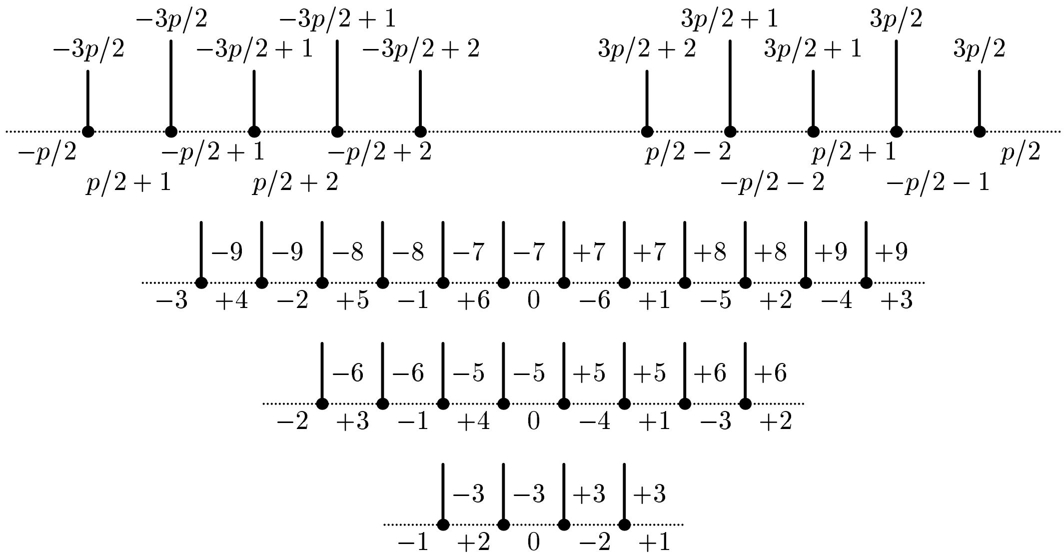

Figure 2: Extending a precoloring to the head of the octopus for $p = 2$, $p = 4$, $p = 6$ and a general p (listed from bottom to top).

to the previous level and all other vertices are joined by edges of F to the next level. We call the edge leading to the previous level an *input edge* of a considered block, while the other edges are *output edges* of it. We establish the following claim:

Claim 2 *Consider a block of the octopus which consists of at least $4k + 1$ vertices, $0 \leq k \leq p/2 - 1$, and whose input edge is colored with the color $-3p/2 + k$ or $3p/2 - k$. Then, any precoloring, which colors one of the two contact edges by the color $-p/2$ and the other by the color $p/2$, can be extended to the whole block in such a way that the output edges avoid the colors $-3p/2 + k, \ldots, 3p/2 - k$.*

If $k = 0$, then color the inner edges by the colors $-p/2$ and $p/2$ alternately and the output edges arbitrarily by the colors $-3p/2$ and $3p/2$. Assume in the rest that $k \geq 1$. We may also assume that the size of the block is exactly $4k + 1$: If the block is larger, simply color the remaining inner edges by the colors $-p/2$ and $p/2$ and the output edges by the colors $-3p/2$ and $3p/2$. By the symmetry, assume that the first contact edge is colored with the color $-p/2$, the last one with $p/2$ and the input edge with the color $-3p/2 + k$. The $4k$ inner edges of the block which are colored with the following colors (each of the following two lines contains $2k$ colors):

$$p/2 + 1, -p/2 + 1, p/2 + 2, -p/2 + 2, \ldots, p/2 + k, -p/2 + k,$$

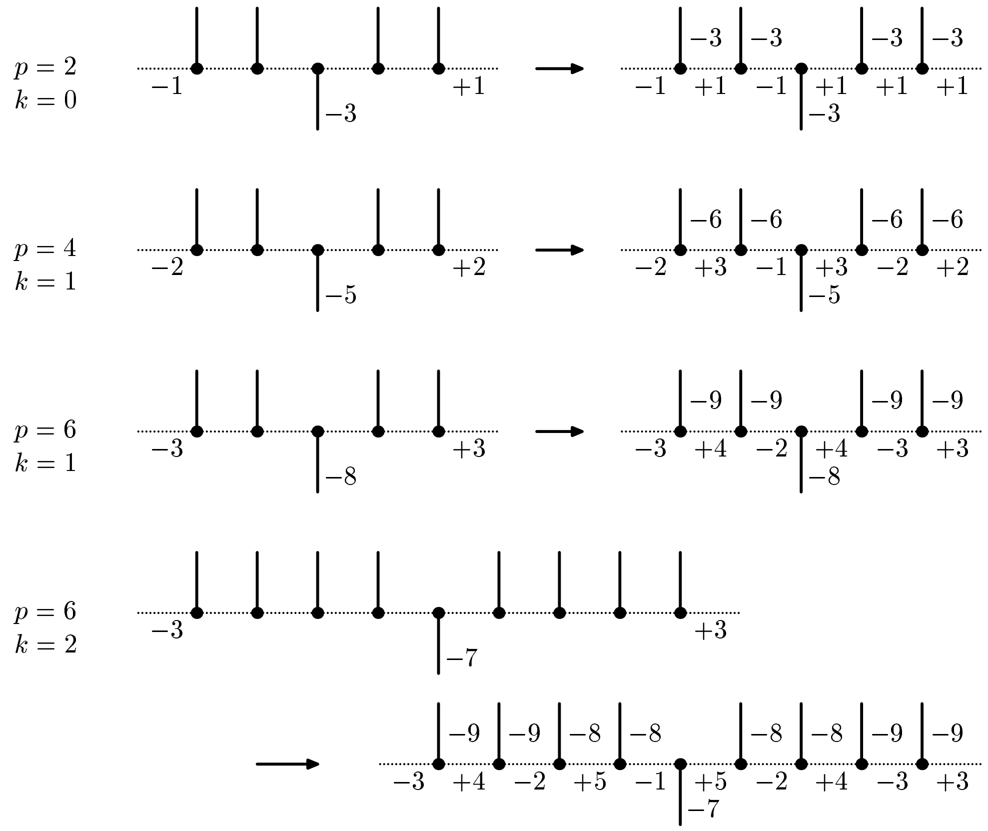

Figure 3: Examples of extensions of a precoloring of contact and output edges to whole blocks for $p = 2$, $p = 4$ and $p = 6$ and for $k = 1$ and $k = 2$.

$$p/2 + k, -p/2 + (k-1), p/2 + (k-1), -p/2 + (k-2), \dots, p/2 + 1, -p/2\ .$$

It is easy to check that the coloring can now be extended using only the colors $-3p/2 + (k-1), \dots, -3p/2$ to the output edges (examples of extending precolorings of contact edges to whole blocks for $p = 2$, $p = 4$ and $p = 6$ can be found in Figure 3). This finishes the proof of Claim 2.

We now use Claim 2 to complete the coloring of the octopus, and hence to complete the proof of Claim 1. The colors of the edges of F leaving the head are among $p+1, \dots, 3p/2$ and $-p-1, \dots, -3p/2$. Since the number of vertices in each block of the second level is $2p - 3 = 4(p/2 - 1) + 1$, Claim 2 implies that the coloring can be extended to each of these blocks in such a way that output edges are colored only with the colors $p+2, \dots, 3p/2$ and $-p-2, \dots, -3p/2$. Now, since the number of vertices in each block of the third level is $2p-7=1+4(p/2-2)$, Claim 2 again implies that the coloring can be extended to these blocks in such a way that output edges are colored only with the colors $p+3,\ldots,3p/3$ and $-p-3,\ldots,-3p/2$. In this way, we extend the coloring to all blocks of the octopus. Note that each block of the last level has a single input edge and this edge gets the color $-3p/2$ or $3p/2$ which is consistent with precoloring the two contact edges by the colors $-p/2$ and $p/2$. Hence, the proof of Claim 1 is completed, and so is the proof of the whole theorem.

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If p is an odd integer, it is possible to use a similar argument, based on octopuses with $\lceil p/2 \rceil + 1$ levels, to prove the following result:

Theorem 3 *Let G be a bridgeless cubic graph with girth at least $2(2p)^{2p+1}$ where $p \geq 3$ is an odd integer. Then, the circular chromatic index of G is at most $3+1/p$.*

The main result of our paper is the following corollary of Theorem 2:

Corollary 1 *For each $\varepsilon > 0$, there exists an integer g such that each cubic bridgeless graph with girth at least g has circular chromatic index at most $3+\varepsilon$.*

4 Circular 7/2-Edge-Colorings

In this section, we refine our arguments from Section 3 to show that the circular chromatic index of each cubic bridgeless graph with girth 14 or more is at most 7/2. Note that Theorem 2 implies this statement for graphs with girth at least 32. We first prove an auxiliary lemma:

Lemma 2 *Let G be a cubic graph with girth at least 14 and let F be a 1-factor of G. Then, it is possible to assign each edge of F the sign $+$ or $-$ in such a way that the following holds: Each odd cycle of $G \setminus F$ contains four consecutive vertices v_1, v_2, v_3 and v_4 such that v_1 and v_2 are incident with edges assigned the sign $+$, while v_3 and v_4 are incident with edges assigned the sign $-$.*

Proof: First, an auxiliary graph $G_{F,\pm}$ is constructed. For each 4-segment $(\alpha,\beta,\gamma,\delta)$ of a cycle C of $G \setminus F$, the graph $G_{F,\pm}$ contains two vertices $v^{\pm}_{\alpha,\beta,\gamma,\delta}$ and $v^{\mp}_{\alpha,\beta,\gamma,\delta}$. The first one, $v^{\pm}_{\alpha,\beta,\gamma,\delta}$, represents the requirement that the edges of F incident with α and β are assigned the sign $+$ and the edges incident with γ and δ are assigned the sign $-$. Similarly, the other vertex $v^{\mp}_{\alpha,\beta,\gamma,\delta}$, represents the requirement that the edges of F incident with α and β are assigned $-$

and the edges incident with γ and δ are assigned $+$. Two vertices of $G_{F,\pm}$ are joined by an edge if the corresponding requirements contradict each other. For example, if α' is the neighbor of α in F and β', γ' and δ' are three consecutive vertices following α' in a cycle of $G \setminus F$, then the vertices $v^{\pm}_{\alpha,\beta,\gamma,\delta}$ and $v^{\mp}_{\alpha',\beta',\gamma',\delta'}$ are joined by an edge, but the vertices $v^{\pm}_{\alpha,\beta,\gamma,\delta}$ and $v^{\pm}_{\alpha',\beta',\gamma',\delta'}$ are not. Formally, $G_{F,\pm}$ contains an edge between the vertices $v^{\pm}_{\alpha,\beta,\gamma,\delta}$ and $v^{\pm}_{\alpha',\beta',\gamma',\delta'}$ if F contains one of the following edges:

$$\alpha\gamma', \alpha\delta', \beta\gamma', \beta\delta', \gamma\alpha', \gamma\beta', \delta\alpha' \text{ or } \delta\beta' \ .$$

The same condition holds for the presence of an edge between the vertices $v^{\mp}_{\alpha,\beta,\gamma,\delta}$ and $v^{\mp}_{\alpha',\beta',\gamma',\delta'}$ in $G_{F,\pm}$. Finally, there is an edge between the vertices $v^{\pm}_{\alpha,\beta,\gamma,\delta}$ and $v^{\mp}_{\alpha',\beta',\gamma',\delta'}$ in $G_{F,\pm}$ if F contains at least one of the following edges:

$$\alpha\alpha', \alpha\beta', \beta\alpha', \beta\beta', \gamma\gamma', \gamma\delta', \delta\gamma' \text{ or } \delta\delta' \ .$$

Note that the order of $G_{F,\pm}$ is twice the order of G and the graph $G_{F,\pm}$ is 16-regular.

Let r be the number of all odd cycles $C_1, \ldots, C_r$ of $G \setminus F$ and let V_i be the vertices of $G_{F,\pm}$ corresponding to segments of the cycle C_i, $1 \leq i \leq r$. Thus, the length of each C_i is at least 15. The size of each V_i is at least $2 \cdot 15 = 30$ by the assumption on the girth of G. We want to find a system of independent representatives for $G[\bigcup_{i=1,\ldots,r} V_i]$ with respect to $V_1, \ldots, V_r$. Indeed, such a system of independent representatives yields an assignment of signs to some edges of F (there is no conflict when assigning signs to edges of F since the vertices of the system represent non-contradictory requirements). This assignment already has the property from the statement of the lemma, and thus an arbitrary extension of it to all edges of F has the claimed property.

We now show that $G[\bigcup_{i=1,\ldots,r} V_i]$ has a system of independent representatives with respect to $V_1, \ldots, V_r$. By Theorem 1, it is enough to verify that for each non-empty set $I \subseteq \{1, \ldots, r\}$, the total domination number of the graph $G[\bigcup_{i \in I} V_i]$ is at least $2|I|$. For the sake of contradiction, assume the opposite and let $I_0 \subseteq \{1, \ldots, r\}$ be a non-empty set such that the total domination number of $G[\bigcup_{i \in I_0} V_i]$ is at most $2|I_0| - 1$. Let W be a total dominating set of size $2|I_0| - 1$. We show that $G[\bigcup_{i \in I_0} V_i]$ contains a vertex which is not total dominated by W.

The dominance of vertices contained in W can be represented in G as follows: Consider a vertex $v^{\pm}_{\alpha,\beta,\gamma,\delta}$ contained in W. Let α', β', γ' and δ' be the neighbors of α, β, γ and δ, respectively, in the 1-factor F. Assign now the sign $+$ to the vertices α' and β' and the sign $-$ to the vertices γ' and δ'. In case that the vertex $v^{\mp}_{\alpha,\beta,\gamma,\delta}$ is contained in W, the signs assigned to the vertices α', β', γ'

and δ' are opposite compared to the previous case. Perform this for each vertex contained in W. Note that some vertices of G may be assigned both signs $+$ and $-$. Now, a vertex $v^{\pm}_{a,b,c,d}$ is dominated by W if one of a and b is assigned $-$ or one of c and d is assigned $+$. A similar statement is true for the vertex $v^{\mp}_{\alpha,\beta,\gamma,\delta}$. By our assumption, both vertices corresponding to each 4-segment of a cycle C_i for $i \in I_0$ are dominated.

Since each vertex of W causes the assignment of signs to four vertices of G, it follows that at most $4|W| \leq 8|I_0| - 4$ signs are assigned altogether. Thus, there is a cycle C_k, $k \in I_0$, whose vertices have been assigned a total of at most 7 signs (if a vertex is assigned both signs $+$ and $-$, then these two signs are counted as two). Let s be the number of the signs assigned to the vertices of C_k and let t be the number of vertices which have been assigned both signs. The length of C_k is at least 15 because G has girth at least 14 and C_k is an odd cycle. Hence, C_k contains at least $15 - (7 - t) = 8 + t$ vertices which have not been assigned any sign. Let us call such vertices *unsigned*. The remaining vertices of C_k are called *signed*.

In what follows, it will be shown by a simple discharging argument that the set V_k in $G_{F,\pm}$ contains a vertex which is not dominated by W. Only the signed vertices will have non-zero initial charge. Each vertex which has been assigned both signs $+$ and $-$ has 3 units of initial charge. Each of the other signed vertices has 1 unit of initial charge. Hence, the total initial charge distributed to all signed vertices together is equal to $s + t$. Note that this is at most $7 + t$ by the choice of C_k. Unsigned vertices induce in C_k a subgraph consisting of several paths. Let U be the set of these paths. Observe that each path in U consists of at most three vertices because the vertex of V_k corresponding to four consecutive unsigned vertices would not be dominated by W.

The initial charge is distributed from each signed vertex to the paths of unsigned vertices by the following simple rule: *Each signed vertex splits its charge to two halves. One half of its charge is sent to the clockwise nearest path of U in C_k and the other half is sent to the anti-clockwise nearest path of U.*

We now establish the following claim:

Claim 3 *The charge received by each path of unsigned vertices receives is at least the number of vertices it contains.*

We distinguish three cases according to the number of unsigned vertices comprising the path. Recall that this number is either 1, 2 or 3.

- If the path consists of a single vertex, then it receives at least a half unit of charge from each of the two signed vertices adjacent to it in C_k. Hence, it receives at least one unit of charge in total.

- Consider now a path comprised of two unsigned vertices. Let u and v be the two signed vertices adjacent to this path in C_k. If u is not assigned both signs, then the neighbor u' of u in C_k not contained in the path must be signed (otherwise, the vertices u', u and the two vertices of the path correspond to a non-dominated vertex of W). Hence, either the group receives $3/2$ units from the vertex u alone (this is the case that u has been assigned both signs) or 1 unit from the vertices u and u' together. Similarly, the path receives at least 1 unit from the signed vertices adjacent to it on its other side. Hence, it receives a total of at least 2 units of charge.

- If the path consists of three vertices, then each of the two vertices adjacent in C_k to this path must have been assigned both signs $+$ and $-$. Otherwise, this vertex and the three unsigned vertices of the path correspond to a non-dominated vertex of V_k. Hence, such a path of vertices receives $3/2$ units from each signed vertex adjacent to it. Hence, 3 units of charge are sent to it in total.

This finishes the proof of Claim 3. Since the total initial charge is at most $7+t$, there can be at most $7+t$ unsigned vertices but there are at least $8+t$ such vertices — a contradiction. We conclude that the total domination number of $G[\bigcup_{i\in I_0} V_i]$ is at least $2|I_0|$ as desired. Hence, $G[\bigcup_{i=1,\dots,r} V_i]$ contains a system of independent representatives with respect to $V_1,\dots,V_r$. This completes the proof of the lemma.

■

Theorem 4 *Every cubic bridgeless graph G with girth at least 14 has circular chromatic index at most $7/2$.*

Proof: Let F be a 1-factor of G. It exists because G is cubic and bridgeless [16]. We construct a circular $7/2$-edge-coloring of G which uses the colors -3, -2, -1, 0, 1, 2 and 3. Consider an assignment $\sigma : F \to \{+,-\}$ of the signs $+$ and $-$ to edges of F with the property described in the statement of Lemma 2.

An edge e of F is colored with the color $+3$ if $\sigma(e) = +$ and it is colored with the color -3, otherwise. Edges of each even cycle of $G \setminus F$ are colored with the colors -1 and $+1$ alternately. Consider now an odd cycle C of $G \setminus F$ and let v_1, v_2, v_3 and v_4 be consecutive vertices of C such that the edges of F incident with v_1 and v_2 have been assigned $+$ and the edges incident with v_3 and v_4 have been assigned $-$. Let us color the edge v_1v_2 with the color -2, the edge v_2v_3 with the color 0 and the edge v_3v_4 with the color $+2$. Finally, color the remaining edges of C with -1 and $+1$ alternately such that the edge incident with v_1 is colored with $+1$. It is easy to verify that the circular 7/2-edge-coloring obtained in this way is proper.

■

5 Open Problems

Several new questions related to our results arise. Theorem 4 shows that every cubic bridgeless graph G with girth at least 14 has circular chromatic index at most 7/2. However, it is not clear that the assumption on the girth cannot be further improved. So far we know only that the largest girth of a cubic bridgeless graph with the circular chromatic index strictly larger than 7/2 is between 5 and 13 (recall that the circular chromatic index of the Petersen graph is 11/3).

It is known [8, 17] that the so-called fractional chromatic index of every k-edge-connected k-regular graph of even order is equal to k. It is also well-known that the circular chromatic index is sandwiched between the fractional chromatic index and the (usual) chromatic index [22]. Every bridgeless 2-regular graph with at least four vertices has circular chromatic index at most 5/2. And, every bridgeless 3-regular graph has circular chromatic index at most 11/3. We pose the following problem:

Problem 1 *Is it true that the circular chromatic index of every k-edge-connected k-regular graph G of even order is smaller than $k+1$? Is it always at most $k+1-1/k$?*

Possibly, the assumption that G is k-edge-connected can be replaced by the weaker assumption that G is bridgeless. It also seems that the assumption that the order of G is even can be omitted for some values of k.

Finally, it seems natural to ask whether Theorems 2 and 3 with Corollary 1 can be generalized to the realm of all regular graphs:

Problem 2 *Is it true that for each k and each $\varepsilon > 0$, there exists an integer g such that every bridgeless k-regular graph with girth at least g has circular chromatic index at most $k + \varepsilon$?*

It is easy to show that the answer is positive if $k = 2$. Theorem 2 implies a positive answer for the case $k = 3$, too. Hence, the first open case is $k = 4$.

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