Extending partial 5-colorings and 6-colorings in planar graphs

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Abstract

Let D be a disc and let X be a finite set of points on the boundary of D. A set C of k-colorings of the points X is called k-feasible if there exists a plane graph G drawn in the disc D with $X \subseteq V(G)$ such that precisely colorings contained in the set C can be extended to proper colorings of G. We show that for each k-feasible set C, there exists such a plane graph G of order at most $|X| + 5^{|X|}$ if $k = 5$ and $17|X| - 48$ if $k = 6$.

1 Introduction

Let X be a finite set of points on the boundary of a disc D. A *k-coloring* of X is a coloring of the points of X by colors $1, \ldots, k$. We do not require that all the colors $1, \ldots, k$ are actually used by the coloring. A set C of k-colorings of X is *k-feasible* if there exists a plane graph G drawn in the disc D with $X \subseteq V(G)$ such that proper colorings of G restricted to the set X are precisely the colorings contained in the set C. In other words, C is the set of all colorings of X which can be extended to the whole graph G. Notion of k-feasible sets was introduced by DeVos and Seymour [5] because of its close relation to the proof of the four color theorem. In particular, they raised the following two questions: "What sets of colorings are k-feasible?" and "Given

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a set of k-colorings of X which is k-feasible, how large is the smallest graph G which admits precisely this set of k-colorings?".

In order to answer the first question, they showed that each set of 3-colorings is 3-feasible which contrasts the fact that only certain structured sets of k-colorings are k-feasible for $k \geq 4$. As an answer to the second question, they showed an upper bound of $O(9^{|X|})$ on the order of the graph G if $k = 3$, a bound of $O(|X|^2)$ if $k = 6$ and a linear bound if $k \geq 7$. They wrote that they were not aware of any upper bound on the order of G for $k = 4, 5$. In this paper, we provide an upper bound of $|X| + 5^{|X|}$ on the order of G if $k = 5$ and we show using Euler's formula by a simple list coloring argument that the order of G does not exceed $17|X| - 48$ if $k = 6$. At this point, we have to say that we did not tune constants involved in either of the presented bounds in order to keep arguments short. We remark that the reader might be also interested to see papers [1, 2, 3, 4] which contain results of a different kind on extensions of colorings in (planar) graphs.

2 Notation and Preliminary Results

We follow the standard graph theory notation and the notation introduced in [5]. All graphs which we consider in this paper are finite, simple and planar. Fix a disc D, a finite set X of points on its boundary and a plane graph G with $X \subseteq V(G)$ which is drawn in the disc D. We identify points contained in X with the corresponding vertices of the graph G and we call these vertices *boundary vertices* of G. The other vertices of G are called *internal vertices* of G. By the *interior* of G, we mean the subgraph of G induced by its internal vertices. Note that some internal vertices may be incident with the outer face of G. We also define the graph $\overline{G}$ to be the (simple) graph obtained from the graph G by adding a cycle formed by the vertices of X (this cycle corresponds to the boundary of the disc D). Finally, two vertices of X adjacent on the boundary of the disc D are said to be *consecutive*.

A *k-coloring* of a graph G is a coloring of vertices of G using integers $1, \ldots, k$ as colors. For a subset W of $V(G)$, the set $\Phi_G(W)$ is the set of all colorings of the vertices of W which can be extended to the whole graph G. Here is the only place where we differ from the notation from [5] where colorings were understood to be vertex partitions and the set $\Phi_G(W)$ to be a set of vertex partitions. Under this notation, the result of DeVos and Seymour on

3-feasible sets which was mentioned in Introduction is the following: Any set of 3-colorings closed under permutations of colors is 3-feasible. The definition of a k-feasible set can now be rephrased to a somewhat more formal form as follows: A set C of k-colorings of X is *k-feasible* if there exists a plane graph G drawn in the disc D with boundary vertices X and with $\Phi_G(X) = C$. Such a graph G is said to be *C-minimal* if its order is the smallest possible. Note that the order of G is equal to the sum of the cardinality of the set X and the number of its internal vertices.

We now state two lemmas on structure of C-minimal graphs. The first one was used in [5] to show the existence of upper bounds on the order of C-minimal graphs for k-feasible sets C with $k \geq 6$:

Lemma 1 *Let D be a disc, X a finite set of points on the boundary of D and C a k-feasible set of colorings of X where $k \geq 2$. If G is a C-minimal graph, then each internal vertex of G has degree at least k.*

Before stating the next lemma, we introduce some additional notation. Consider a cycle γ in the plane. The cycle γ splits the plane into two parts. The infinite part is called the *exterior* of the cycle γ and it is denoted by $\mathrm{Ext}(\gamma)$. The other part of the plane is called the *interior* of the cycle and it is denoted by $\mathrm{Int}(\gamma)$. A *seperating triangle* of a plane graph G is a 3-cycle uvw contained in G such that both $\mathrm{Int}(uvw)$ and $\mathrm{Ext}(uvw)$ contain a vertex of G. The second lemma states that a C-minimal graph cannot contain a separating triangle:

Lemma 2 *Let D be a disc, X a finite set of points on the boundary of D and C a k-feasible set of colorings of X where $k \geq 4$. If G is a C-minimal graph, then the graph $\overline{G}$ does not contain a separating triangle.*

Proof: Assume that $\overline{G}$ contains a separating triangle uvw. Let G' be the subgraph of G consisting of the vertices and edges drawn in $\mathrm{Ext}(uvw)$ and G_{uvw} the subgraph consisting of the vertices and edges drawn in $\mathrm{Int}(uvw)$. The vertices u, v and w together with all edges between them are included to both graphs G' and G_{uvw}. In addition, add to G_{uvw} also the remaining edges of the triangle uvw.

Fix now a coloring c of the graph G'. Consider first the case that the coloring c colors the vertices u, v and w by three mutually distinct colors. Then, the coloring c can be extended to a proper k-coloring of the whole graph G because the graph G_{uvw} is 4-colorable (and each its 4-coloring colors the vertices u, v and w by mutually distinct colors). If the coloring c colors some of the vertices u, v and w by the same color, we first contract in G_{uvw} the vertices colored with the same color to a single vertex (removing loops and multiple edges which arise) and then we apply the same argument as in the previous case. Since the choice of the coloring c was arbitrary, each coloring c of G' can be extended to G and thus we showed $\Phi_{G'}(X) \subseteq \Phi_G(X)$. On the other hand, it clearly holds $\Phi_G(X) \subseteq \Phi_{G'}(X)$ and hence in fact $\Phi_{G'}(X) = \Phi_G(X) = C$. This contradicts the fact that G is C-minimal.

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3 The List Coloring

In this paper, we use several arguments based on the list coloring. Given a graph G and a set $L(v)$ of colors for each $v \in V(G)$, an *L-coloring* of G is a proper coloring c of G with the additional constraint that $c(v) \in L(v)$. A mapping L is called a *list assignment* of the graph G. If a coloring c with the additional constraint exists, we also say that G is *L-colorable*.

In general, if a graph is k-colorable, there could exist a list assignment L with $|L(v)| \geq k$ for each $v \in V(G)$ such that G is not L-colorable. Even more, for each $k \geq 1$, there is a (non-planar) bipartite graph for which lists of sizes k do not suffice [6].

We now remind two fundamental theorems on the list coloring. The first one was proved by Erdős, Rubin and Taylor [6] and it is an analogue of Brooks' theorem in the realm of list colorings. At this point, let us remind that a *Gallai tree* is a connected graph whose each block (maximal 2-connected subgraph) is a complete graph or an odd cycle. The second theorem which is due to Thomassen [7] forms the core of the proof that each planar graph can be colored from any lists with sizes equal to five.

Theorem 1 *Let G be a connected graph which is not a Gallai tree and L a list assignment such that $|L(v)| \geq \deg_G(v)$ for each $v \in V(G)$. Then, G is L-colorable.*

Theorem 2 *Let G be a plane graph and L a list assignment. If L assigns a list consisting of at least three colors to each vertex incident with the outer face and a list consisting of at least five colors to each of the remaining vertices, then G is L-colorable.*

4 Extending 5-colorings

First, we realize that if G is a C-minimal graph which contains internal vertices, then G must contain an internal vertex adjacent to at least three boundary vertices:

Lemma 3 *Let D be a disc, X a finite non-empty set of points on the boundary of D and C a 5-feasible set colorings of X. Suppose that G is a C-minimal graph. If G contains at least one internal vertex, then G contains an internal vertex adjacent to at least three boundary vertices.*

Proof: Let us assume the opposite, i.e., G is a C-minimal graph such that each internal vertex of G is adjacent to at most two boundary vertices of G. Let G' be the subgraph of G induced by the boundary vertices and G_0 the subgraph of G induced by the internal vertices (note that both G' and G_0 could be disconnected graphs).

We claim that each proper coloring of G' can be extended to the whole graph G. Let c be a proper coloring of G'. Consider now the following list assignment L of G_0: If a vertex $v \in V(G_0)$ is adjacent to some boundary vertices in G, then the list $L(v)$ consists of the colors which c does not use for any boundary vertex adjacent to v in G. If a vertex v is not adjacent to a boundary vertex, then its list consists of all five available colors $1, \ldots, 5$. Since each vertex of G_0 is adjacent to at most two vertices of G', the size of each list $L(v)$, $v \in V(G)$, is at least three. In addition, if v is not incident with the outer face of G_0, then v is adjacent to no boundary vertices in G and $|L(v)| = 5$. By Theorem 2, the graph G_0 is L-colorable. Such a coloring of G_0 completes the coloring c to a proper coloring of the whole graph G. Hence, we have $\Phi_{G'}(X) = \Phi_G(X) = C$. This contradicts the fact that G is C-minimal since G' has clearly a smaller order than the graph G.

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We now would like to apply Lemma 3 to get a proof of an upper bound on the order of C-minimal graphs G for 5-feasible sets C. However, the following obstacle arises in a straightforward proof proceeding by induction on the number of boundary vertices: If w_0 is an internal vertex of G which is adjacent to three consecutive boundary vertices w_1, w_2 and w_3, then w_0 with edges w_0w_1, w_0w_2 and w_0w_3 need not to split the graph G into three graphs each with less boundary vertices. Indeed, if w_1, w_2 and w_3 are consecutive vertices, then the subgraph of G drawn in the interior of the cycle formed by the path from w_3 to w_1 and the path $w_1 w_0 w_3$ has the same number of boundary vertices as G. This leads us to the following definition of a claw: If the interiors of triangles $w_1 w_2 w_0$ and $w_2 w_3 w_0$ are empty, then the subgraph of G induced by the vertices w_0, w_1, w_2 and w_3 is called a *claw*. The vertex w_0 is said to be the *center* of the claw and the vertex w_2 is its *middle nail*.

Consider now a graph G drawn in a disc D with boundary vertices X. Assume that G contains a claw with vertices w_0, w_1, w_2 and w_3. Observe that $G \setminus w_2$ can be understood as a graph with the boundary vertices $(X \setminus \{w_2\}) \cup \{w_0\}$ (the vertex w_2 is replaced by the vertex w_0). Using this, we introduce the following definition: A sequence of vertices $v_1, \ldots, v_k$ is called a *claw-elimination sequence* of a graph G if the following holds:

1. The graph $G \setminus \{v_1, \ldots, v_{i-1}\}$, $i = 1, \ldots, k$, does not contain an edge joining two non-consecutive boundary vertices, and
2. the vertex v_i, $i = 1, \ldots, k$, is the middle nail of a claw in the graph $G \setminus \{v_1, \ldots, v_{i-1}\}$, and
3. the final graph $G \setminus \{v_1, \ldots, v_k\}$ does not contain a claw or it contains an edge joining two non-consecutive boundary vertices.

We slightly abuse notation here in the following sense: If $i = 1$, then the graph $G \setminus \{v_1, \ldots, v_{i-1}\}$ in the first and second condition is G and similarly if $k = 0$, the graph $G \setminus \{v_1, \ldots, v_k\}$ in the third condition is G. Note that the main consequence of the first condition is that no two boundary vertices of the graph $G \setminus \{v_1, \ldots, v_{i-1}\}$ collapse to a single boundary vertex when removing the vertex v_i.

We now prove that the length of each claw-elimination sequence of a C-minimal graph G for a 5-feasible set C is bounded by a function of $|X|$:

Lemma 4 *Let D be a disc, X a finite non-empty set of points on the boundary of D and C a 5-feasible set of colorings of X. Let $v_1, \ldots, v_k$ be a claw-elimination sequence of a C-minimal graph G. Then, $k \leq 5^{|X|-1} - 1$.*

Proof: Let G_i, $1 \leq i \leq k$, be the graph obtained from G by removing the vertices $v_1, \ldots, v_i$ and let $G_0 = G$. Let s be the size of the set $|X|$. We now recursively define vertices $w_{i,j}$, $i = 0, \ldots, k$ and $j = 1, \ldots, s$ as follows: The vertices $w_{0,j}$, $j = 1, \ldots, s$, are vertices of the set X in the clockwise order on the boundary of D. Assume now that for some $i_0 \geq 1$, we have already

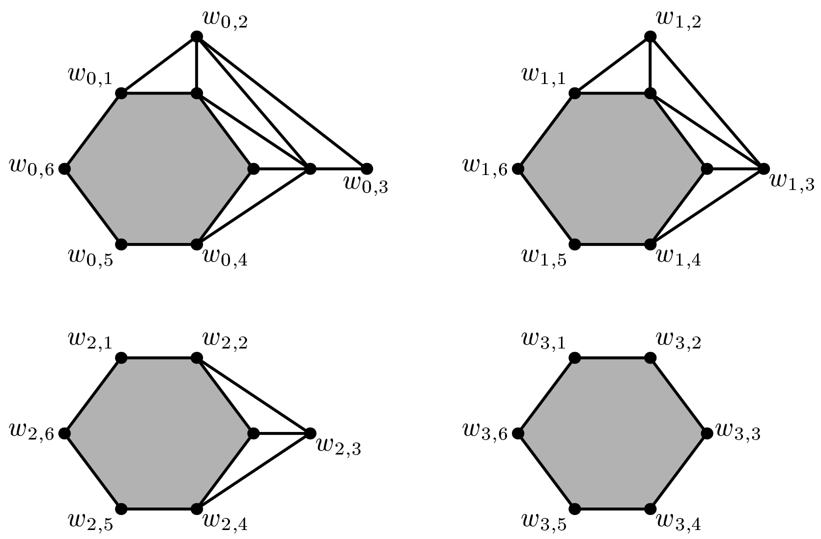

Figure 1: Notation used in Lemma 4 in case of a claw-elimination sequence of length three. The gray area is the graph G_3 with its interior.

defined vertices $w_{i,j}$ for all $i = 0, \ldots, i_0 - 1$ and $j = 1, \ldots, s$. Let j_0 be the index such that $v_{i_0} = w_{i_0-1,j_0}$, i.e., w_{i_0-1,j_0} is the middle nail of the i_0-th eliminated claw. Then, the vertex $w_{i_0,j}$ is the vertex $w_{i_0-1,j}$ if $j \neq j_0$ and the vertex w_{i_0,j_0} is the center of the eliminated claw, i.e., the claw with the middle nail $w_{i_0-1,j_0} = v_{i_0}$ (consult Figure 1). Set $W_i = \{w_{i,1}, \ldots, w_{i,k}\}$ for $i = 0, \ldots, k$. In particular, $W_0 = X$. And finally, let C_i, $0 \leq i \leq k$, be the set $\Phi_{G_i}(W_i)$. Note that $C_0 = C$.

Observe that each G_i is C_i-minimal (otherwise, G_i could be replaced by a smaller graph which would contradict that G is C-minimal). Fix now an integer $i \in \{1, \ldots, k\}$ and let j be such an index that $v_i = w_{i-1,j}$, i.e., $G_i = G_{i-1} \setminus w_{i-1,j}$. Fix a coloring c_0 of the vertices of $W_i \setminus \{w_{i,j}\}$. Let α be the number of extensions of c_0 to a coloring contained in Φ_{i-1} and β to a coloring contained in Φ_i. We show that $\alpha \geq \beta$. Moreover, if $\alpha = \beta$, then the extensions of c_0 contained in Φ_{i-1} and in Φ_i are the same (identifying the vertices $w_{i-1,j}$ and $w_{i,j}$ in the sets W_{i-1} and W_i, respectively).

Let e_{j-1} be an edge joining the vertices $w_{i-1,j-1}$ and $w_{i-1,j}$ and e_{j+1} be an edge joining the vertices $w_{i-1,j+1}$ and $w_{i-1,j}$ (these edges need not to be contained in the graph G_{i-1}). If $\beta = 0$, then $\alpha = 0$ and the claim is true. If $\beta = 1$, then the coloring c_0 can be extended to a proper coloring of G_i with the color of $w_{i,j}$ uniquely determined. Such a coloring can be extended in at least two different ways to the whole graph G_{i-1} and hence $\alpha \geq 2$ (and $\alpha > \beta$ as desired). If $\beta \geq 2$, then c_0 can be extended to at least two (and at most four because the vertex $w_{i,j}$ must have a color different from the color of the vertex $w_{i,j-1}$) proper colorings of G_i, each of them assigning a distinct color to the vertex $w_{i,j}$. Then, the set Φ_{i-1} contains all extensions of c_0 which do not conflict with the assignment of colors to the vertices $w_{i-1,j-1}$ and $w_{i-1,j+1}$ in G_{i-1}. We now distinguish several cases:

- **Neither the edge e_{i-1} nor e_{i+1} is contained in G_{i-1}.**
  We have $\alpha = 5$ since all five extensions of c_0 are in Φ_{i-1}. Hence, $\alpha > \beta$.

- **Precisely one of the edges e_{j-1} and e_{j+1} is contained in G_{i-1}.**
  Assume that G_{i-1} contains the edge e_{j-1} (the other case is symmetric). We have $\alpha = 4$ in this case. Note that β does not exceed four as argued above. Hence if the inequality $\alpha > \beta$ does not hold, then $\alpha = \beta = 4$. In such case, the extensions of c_0 contained in Φ_{i-1} and in Φ_i are the same since they must be simply all extensions of c_0 assigning to the vertex $w_{i-1,j}$, $w_{i,j}$, respectively, a color different from the color of the vertex $w_{i,j-1}$.

- **Both edges e_{i-1} and e_{i+1} are contained in G_{i-1} and $c_0(w_{i-1,j-1}) = c_0(w_{i-1,j+1})$.**
  We have again $\alpha = 4$ in this case. Again, if the inequality $\alpha > \beta$ does not hold, then we have $\alpha = \beta = 4$ and the extensions of c_0 contained in Φ_{i-1} and in Φ_i are the same. In this case, they are all extensions of c_0 assigning to the vertex $w_{i-1,j}$, $w_{i,j}$, respectively, a color different from the color $c_0(w_{i-1,j-1}) = c_0(w_{i-1,j+1})$.

- **Both edges e_{i-1} and e_{i+1} are contained in G_{i-1} and $c_0(w_{i-1,j-1}) \neq c_0(w_{i-1,j+1})$.**
  We have $\alpha = 3$. Note that β cannot exceed three because the vertex $w_{i,j}$ must have a color different from the colors $c_0(w_{i-1,j-1})$ and $c_0(w_{i-1,j+1})$. Hence, if $\alpha = \beta$, we have then $\alpha = \beta = 3$. In such case, the extensions of c_0 contained in Φ_{i-1} and in Φ_i are the same: They are all extensions of c_0 which assign to the vertex $w_{i-1,j}$, $w_{i,j}$, respectively, a color different from the colors $c_0(w_{i-1,j-1})$ and $c_0(w_{i-1,j+1})$.

Since the choice of a coloring c_0 was arbitrary, we have $|C_{i-1}| \geq |C_i|$ and if $|C_{i-1}| = |C_i|$, then $C_{i-1} = C_i$ (identifying the vertices $w_{i-1,j}$ and $w_{i,j}$ in the sets W_{i-1} and W_i). Actually, the latter, i.e., $C_{i-1} = C_i$, is impossible because G_{i-1} is C_{i-1}-minimal. Hence, $|C_0| > |C_1| > \cdots > |C_k|$. Since $|C_0| \le 5^{|X|}$ and the size of each C_i is divisible by 5 (consider colorings obtained by permutations of colors), we have $k \le 5^{|X|-1} - 1$ as desired.

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Now, we prove the main theorem of this section:

Theorem 3 *Let D be a disc, X a finite non-empty set of points on the boundary of D and C a 5-feasible set of colorings of X. If G is a C-minimal graph, then G contains at most $5^{|X|}$ internal vertices.*

Proof: The proof proceeds by induction on the size of the set $|X|$. It is easy to show that if $|X| \le 3$, then a C-minimal graph contains no internal vertices. Assume now that $|X| \ge 4$. Let $v_1, \ldots, v_k$ be a claw-elimination sequence of G. By Lemma 4, the length k of this sequence is at most $5^{|X|-1} - 1$. Let G' be the graph obtained from G by removing the vertices $v_1, \ldots, v_k$ and let X' be the set of boundary vertices of G'. If G' is not $\Phi_{G'}(X')$-minimal, we can replace G' in G by a smaller graph preserving $\Phi_{G'}(X')$ and hence also $\Phi_G(X) = C$. This clearly contradicts the fact that G is C-minimal. Thus, G' is $\Phi_{G'}(X')$-minimal. By the definition of the claw-elimination sequence, either two non-consecutive boundary vertices are adjacent in G' or the graph G' does not contain a claw.

We first consider the case that two non-consecutive boundary vertices x_1 and x_2 are joined by an edge. The edge x_1x_2 splits the graph G' into two parts G_1 and G_2. Say for the sake of accuracy, that both G_1 and G_2 contain the edge x_1x_2. Let further X_i, $i = 1, 2$, consist of the vertices of X contained in G_i. Note that each $|X_1|$ and $|X_2|$ is smaller than $|X|$ because the vertices x_1 and x_2 are not consecutive on the boundary of G'. And finally, let C_i, $i = 1, 2$, be the set $\Phi_{G_i}(X_i)$. Since G' is $\Phi_{G'}(X')$-minimal, it follows that G_i is C_i-minimal for $i = 1, 2$. By the induction hypothesis, each G_i contains at most $5^{|X_i|} \le 5^{|X|-1}$ internal vertices. Since internal vertices of G are precisely internal vertices of G_1 and G_2 plus the centers of the eliminated claws, the graph G can contain at most the following number of internal vertices:

$$5^{|X|-1} + 5^{|X|-1} + k \le 2 \cdot 5^{|X|-1} + 5^{|X|-1} - 1 \le 3 \cdot 5^{|X|-1} - 1 \le 5^{|X|} .$$

We now consider the case that G' does not contain a claw. If G' contains no internal vertices, then the number of internal vertices of G is at most

$k \leq 5^{|X|-1} - 1 \leq 5^{|X|}$. So, we may assume that G' contains some internal vertices. Then by Lemma 3, G' contains an internal vertex y which is adjacent to at least three vertices of X'.

Assume first that y is adjacent only to three boundary vertices and let x_1, x_2 and x_3 be these vertices in the clockwise order on the boundary. Let now G_{ij}, $ij \in \{12, 23, 31\}$, be the subgraph of G' contained in the interior of the triangle $x_i x_j y$ (note that the edge $x_i x_j$ might not exist in G). The graph G_{ij} contains also the vertices x_i, x_j and y together with edges between them which are present in G'. Let X_{ij} be the set consisting of the vertices of X contained in G_{ij}. By Lemma 2, if $x_1 x_2 y$ is a triangle in $\overline{G'}$, then its interior is empty. The same holds for $x_1 x_3 y$ and $x_2 x_3 y$. Since G' does not contain a claw, at least two of the sets X_{12}, X_{23} and X_{31} contain three or more elements. Thus, the size of each of X_{12}, X_{23} and X_{31} is at most $|X|-2$. Since G' is $\Phi_{G'}(X')$-minimal, each G_{ij} is $\Phi_{G_{ij}}(X_{ij} \cup \{y\})$-minimal. By the induction hypothesis, each G_{ij} contains at most $5^{|X_{ij} \cup \{y\}|} \leq 5^{|X|-1}$ internal vertices. Finally, the internal vertices of G are precisely the internal vertices of G_{12}, G_{23} and G_{31}, the vertex y and the centers of the eliminated claws. Thus, the number of internal vertices of G is at most the following:

$$3 \cdot 5^{|X|-1} + 1 + k \leq 3 \cdot 5^{|X|-1} + 1 + 5^{|X|-1} - 1 \leq 4 \cdot 5^{|X|-1} \leq 5^{|X|} .$$

The final case is that y is adjacent to at least four boundary vertices of G'. Let x_1, x_2, x_3 and x_4 be such four boundary vertices in the clockwise order and let G_{ij}, $ij \in \{12, 23, 34, 41\}$, be the subgraph of G' contained in the interior of the triangle $x_i x_j y$. As in the previous case, the subgraph G_{ij} contains the vertices x_i, x_j and y together with edges between them which are present in G'. Let further X_{ij} be the set consisting of the vertices of X contained in G_{ij}. Clearly, the size of each X_{ij} is at most $|X|-2$. Since G' is $\Phi_{G'}(X')$-minimal, it follows that each G_{ij} is $\Phi_{G_{ij}}(X_{ij} \cup \{y\})$-minimal. Thus, $G_i j$ contains at most $5^{|X_{ij}|+1} \leq 5^{|X|-1}$ internal vertices by the induction hypothesis. Now, we may conclude that the number of internal vertices of G is at most the following:

$$4 \cdot 5^{|X|-1} + 1 + k \leq 4 \cdot 5^{|X|-1} + 1 + 5^{|X|-1} - 1 \leq 5 \cdot 5^{|X|-1} \leq 5^{|X|} .$$

This establishes the theorem.

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A simple corollary of Theorem 3 is the following:

Corollary 1 *Let D be a disc, X a finite non-empty set of points on the boundary of D and C a 5-feasible set of colorings of X. If G is a C-minimal graph, then its order is at most $|X| + 5^{|X|}$.*

5 Extending 6-colorings

In the case of 6-feasible sets of colorings, we use a simple argument based on Euler's formula. In order to apply the argument, we need the following auxiliary lemma whose proof uses the analogue of Brooks' theorem for list colorings. Let us remind that a *near-triangulation* is a plane graph, whose all faces except possibly for the outer face, are triangles.

Lemma 5 *Let D be a disc, X a finite set of points on the boundary of D and C a 6-feasible set of colorings of X. Let further G be a C-minimal graph and T a near-triangulation obtained from $\overline{G}$ by adding edges to the interior of G. Then, each internal vertex of the triangulation T of degree six is adjacent in T to a boundary vertex or to a vertex of degree at least seven.*

Proof: Assume the opposite and let v be an internal vertex of T which is adjacent in T neither to a boundary vertex nor to a vertex of degree at least seven. Let N be the set of the six neighbors of v in T. By Lemma 1, the set N coincides with the set of neighbors of the vertex v in the graph G. Again by Lemma 1, the degree of each neighbor of v must be at least six in G. By the choice of v, the degree of each neighbor of v in T is at most six and hence each neighbor of v has degree six both in G and T. Let W be the 6-wheel formed by the vertex v (as the central vertex) and its six neighbors. Consider now the graph G' obtained from G by removing the vertex v.

We claim that each coloring of G' can be modified and extended to a coloring of G in such a way that the colors of the vertices of X are preserved. Fix a coloring c of G' and uncolor the vertices of N (recall that the sets N and X are disjoint). Let c' be the obtained partial coloring of G' (and hence also of G). The coloring c' can be extended to the whole graph G. Indeed, consider the following list assignment L: The set $L(v)$ consists of all six available colors and $L(w)$ for $w \in N$ is the set containing all colors which are not used by c for coloring any neighbor of w in T. Note that $|L(w)| \geq 3$ because the degree of each vertex $w \in N(v)$ is six in T. The graph induced by $N(v) \cup \{v\}$ in G is a subgraph of the 6-wheel W. Otherwise, the graph G contains an edge joining two vertices v' and v'' which are not consecutive on the 6-cycle of W and then the triangle $vv'v''$ of G is a separating triangle in the graph $\overline{G}$. But this is impossible by Lemma 2.

Observe now that the 6-wheel W is L-colorable by Theorem 1. Such an L-coloring is also a proper coloring of the subgraph of G induced by $N \cup \{v\}$ and thus it extends the coloring c' to a proper coloring of the whole graph G. Since the choice of a coloring c was arbitrary, we can conclude $\Phi_{G'}(X) = \Phi_G(X) = C$. This contradicts the fact that the graph G is C-minimal.

■

Now, it is easy to show the desired linear bound on the order of C-minimal graphs for 6-feasible sets C of colorings:

Theorem 4 *Let D be a disc, X a finite set of at least three points on the boundary of D and C a 6-feasible set of colorings of X. Then, the number of internal vertices of a C-minimal graph is at most $16|X| - 48$.*

Proof: Fix a C-minimal graph G and consider a near-triangulation T of $\overline{G}$ obtained from $\overline{G}$ by adding edges to its interior. Let n_i, $i \geq 6$, be the number of internal vertices of T of degree i and let s be the size of the set X. By Lemma 1, the minimum degree among internal vertices of G (and hence of T) is at least six. Hence, $n = \sum_{i \geq 6} n_i$ is the number of internal vertices of G. Let further s_i, $i \geq 0$, be the number of vertices of X adjacent to i internal vertices in T. Note that $s = \sum_{i \geq 0} s_i$.

Consider now the triangulation T_0 obtained from the near-triangulation T by inserting a new vertex w inside the outer face of T and joining w with all vertices of the set X. The sum of degrees of vertices of T_0 is at most the following (note that two boundary vertices can be joined by an edge):

$$\sum_{i=6}^{\infty} i\, n_i + \sum_{i=0}^{\infty} (i+3)\, s_i + s = 6n + 4s + \sum_{i=7}^{\infty} (i-6)\, n_i + \sum_{i=0}^{\infty} i\, s_i$$

On the other hand, by Euler's formula, the sum of degrees of all vertices of the triangulation T_0 is $6\,(n+s+1) - 12$ and thus, we have:

$$\begin{aligned} 6n + 4s + \sum_{i=7}^{\infty} (i-6)\, n_i + \sum_{i=0}^{\infty} i\, s_i &\leq 6\,(n+s+1) - 12 \\ \sum_{i=7}^{\infty} (i-6)\, n_i + \sum_{i=0}^{\infty} i\, s_i &\leq 2s - 6 \qquad (1) \end{aligned}$$

By Lemma 5, each internal vertex of T has degree at least seven or it is adjacent to an internal vertex of degree at least seven or to a boundary vertex. Hence, each internal vertex of T is contained in T in a neighborhood of an internal vertex of degree at least seven or in a neighborhood of a boundary vertex. This yields the following upper bound on the number of i nternal vertices of T (and hence of G):

$$\sum_{i=7}^{\infty}(i+1)\cdot n_i+\sum_{i=0}^{\infty} i\cdot s_i \tag{2}$$

Finally, by combining (1) and (2), we get the desired upper bound on the number of internal vertices of G:

$$\sum_{i=7}^{\infty}(i+1)\,n_i+\sum_{i=0}^{\infty} i\,s_i \le 8\left(\sum_{i=7}^{\infty}(i-6)\,n_i\right)+\sum_{i=0}^{\infty} i\,s_i \le 16s-48$$

■

And again, a simple corollary with an upper bound on the order of C-minimal graphs for 6-feasible sets C follows:

Corollary 2 *Let D be a disc, X a finite set of at least three points on the boundary of D and C a 6-feasible set of colorings of X. Then, the order of G is at most* $17|X|-48$.

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