The Erdős-Turán Property for a Class of Bases

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Abstract

Given a set A of positive integers, $r_A(n)$ denotes the number of representations of n as a sum of two elements in A. It was conjectured by Erdős and Turán in 1942 that, if A is an asymptotic basis of order 2, then $\limsup_{n\to\infty} r_A(n) = \infty$. A set $A \subset \mathbb{N}$ is 2-bounded if there is a function f such that, for each $n \in A + A$ there is a pair $x, y \in A$ such that $n = x + y$ and $|S(x) \cup S(y)| \leq f(|S(n)|)$ where $S(n)$ denotes the binary support of n, that is, $n = \sum_{i\in S(n)} 2^i$. We prove the Erdős-Turán conjecture for any 2-bounded asymptotic basis of order $h \geq 2$. We list several other variations and strengthenings of this conjecture and we highlight the differences between versions for integers and positive integers, and additive and multiplicative versions.

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1 Introduction

Given a set $A \subset \mathbb{N}$ let $r_A(n)$ denote the number of ordered pairs $(a, a') \in A \times A$ such that $a + a' = n$. Then A is an *asymptotic additive basis of order* 2 (a basis for short in what follows) if there is $n_0 = n_0(A)$ such that $r_A(n) \geq 1$ for each positive integer $n \geq n_0$. It was conjectured by Erdős and Turán in 1942 [5] that, if A is a basis, then

$$\limsup_{n\to\infty} r_A(n) = \infty.$$

An excellent account of this and related problems is given by Sárközy and Sós [11].

Not much is known about this famous conjecture. It can be easily shown that if a set of positive integers A satisfies $\limsup_{n\to\infty} |A(n)|n^{-1/2} = \infty$, where $A(n) = A \cap [1, n]$, then $r_A(n)$ can not be bounded. Indeed, if $r_A(n) \leq g$ for all $n \in \mathbb{N}$ and $A_n(x) = \sum_{a\in A(n)} x^a$, then $A_n^2(x) = \sum_{i\geq 0} r_{A(n)}(i)x^i$ and

$$|A(n)|^2 = A_n^2(1) = \sum_{i\geq 1} r_{A(n)}(i) \leq ng.$$

On the other hand, the counting function of a basis must satisfy $|A(n)| = \Omega(n^{1/2})$, since $|A(n)|^2 > |A(n) + A(n)| \geq n - n_0(A)$ for some $n_0(A)$. More explicit quantitative expressions of both facts can be found for instance in [7]. Therefore, the Erdős-Turán conjecture is open for bases whose counting function satisfies $0 < \limsup |A(n)|/n^{1/2} < \infty$, the so-called *thin* bases. The first examples of thin bases were given by Cassels, Stöhr and Raikov, see e.g. [6]. More recent constructions were provided by Hofmeister [7]. In all these examples the constructed basis either contain arbitrarily large arithmetic progressions or contain elements of the form $\sum_{i\in E} x_i d^i$, $d \geq 2$, for arbitrarily large subsets $E \subset \mathbb{N}$ and all possible choices of $x_i \in \{0, r\}$ for some $r \neq 0$. In both cases we trivially obtain that $r_A(n)$ is not bounded.

Erdős [2] showed that the function $r_A(n)$ can grow slowly. More precisely, he shows that there exist bases A for which

$$c_1 \log n \leq r_A(n) \leq c_2 \log n,\ n \geq n_0(A)$$

for some constants $c_1, c_2 > 0$. The Erdős-Fuchs Theorem This result may explain the difficulties involved in the Erdős-Turán conjecture.

In this note we prove the validity of Erdős -Turán conjecture for a class of "bounded" bases. The binary support of a positive integer n is the subset $S(n) \subset \mathbb{N} \cup \{0\}$ of its binary expansion

$$n = \sum_{i\in S(n)} 2^i.$$

We say that $A \subset \mathbb{N}$ is *bounded* if there is a function f such that, for each $n \in A+A$ there is a pair $x, y \in A$ with

$$n = x + y \text{ and } |S(x) \cup S(y)| \leq f(|S(n)|).$$

For instance, if the binary expansion of each element in A has no two consecutive 1's, then A is a bounded set with $f(n) = n$.

We prove the following result.

Theorem 1 *Let A be a bounded asymptotic basis of order 2. Then*

$$\limsup_{n\to\infty} r_A(n) = \infty.$$

In fact we prove a more general form of Theorem 1. Let us fix a positive integer $h \geq 2$ and let $r_A^{(h)}(n)$ denote the number of solutions of $n = a_1 + \cdots + a_h$ with $a_1, \ldots, a_h \in A$. We say that A has the *h-Erdős-Turán property* (h-ET for short) if $r_A^{(h)}(n)$ is unbounded. Set A is an asymptotic additive basis of order h if every sufficiently large integer can be expressed as the sum of h elements in $A \cup \{0\}$ and h is the minimum positive integer with this property. We shall omit the reference to h in the above definitions when $h = 2$. We prove the next result, for which the precise (and more technical) notion of (d, h)-bounded basis is explained in Section 2.

Theorem 2 *Let A be an asymptotic additive basis for $\mathbb{N}$ of order $h \geq 2$ and $d \geq 2$. If A is (d, h)-bounded then it has the h-Erdős-Turán property.*

This generalizes Theorem 1 as we shall see later that every $(2, 2)$-basis is a bounded basis.

We tested bounded bases in the context of other bases-related results, both additive and multiplicative. For example we have the following:

The Erdős-Newman problem [3] asked for the existence of a set A with $\limsup_{n\to\infty} r_A(n) = k$ such that, for every finite partition $A = A_1 \cup \cdots \cup A_r$ we have $\limsup r_{A_i}(n) = k$ for some i. The problem has a positive answer as proved by Nešetřil and Rödl [10]. A similar question can be asked when $\limsup r_A(n) = \infty$. The method we use in the proof of Theorem 2 gives the following result.

Theorem 3 *Let $A = A_1 \cup \ldots \cup A_r$ be a partition of a (d, h)-bounded additive basis of order $h \geq 2$ into a finite number of parts. Then one of the parts has the h-ET property.*

Erdős also proposed the following strenghtening of Erdős-Turán conjecture: if A is a set of positive integers satisfying $|A(n)| = \Omega(n^{1/2})$ then A has the ET property, regardless of the fact that it is a basis or not. If true then Theorem 3 follows for basis of order 2.

The method we use to prove Theorems 2 and 3 is inspired by the proof that Nešetřil and Rödl [10] gave for the multiplicative analog of the Erdős-Turán conjecture. This was first proved by Erdős [1] in 1964:

Theorem 4 *Suppose that every positive integer can be written as a product of two elements in a set $A \subset N$. Then, for every $m \in \mathbb{N}$, there exists a positive integer n which can be written as a product of two elements in A in at least m different ways.*

It is worth noting that the proof of Theorem 4 given in [10] can be extended to multiplicative basis of $\mathbb{Z}$. However the situation is strikingly different for additive basis of $\mathbb{Z}$ or, more generally, in any linearly ordered Abelian group. Nathanson [9] proves that, for any arbitrary function $r : \mathbb{Z} \cup \{\infty\} \to \mathbb{N}_0$ with $r^{-1}(0)$ finite, there is an additive basis A of order 2 such that $r_A(x) = r(x)$ for all but a finite number of $x \in \mathbb{Z}$. As shown by Nathanson, a greedy algorithm provides basis for $\mathbb{Z}$ with $r_A(n) = 1$ for all $n \in \mathbb{Z}$. A similar result holds when restricted to bounded basis.

Theorem 5 *There is a bounded basis of $\mathbb{Z}$ satisfying $r_A(n) = 1$ for each $n \in \mathbb{Z}$.*

Thus while the definition of bounded basis and the proofs of Theorems 2 and 3 were inspired by the multiplicative versions of the Erdős-Turán conjecture which is valid both in $\mathbb{N}$ and $\mathbb{Z}$, this notion is sensitive enough to separate $\mathbb{N}$ and $\mathbb{Z}$ in the additive case.

In the same vein, Theorem 3 is no longer true in $\mathbb{Z}$. We can construct an additive basis A for $\mathbb{Z}$ with a prescribed function $r_A(n)$, even possesing the ET-property, such that it can be partitioned into two B_2^1 sequences.

Theorem 6 *Let $r : \mathbb{Z} \to \mathbb{N} \cup \{\infty\}$ be a given function. There is a set A of integers such that $r_A(n) = r(n)$ for each integer n and a partition $A = A_1 \cup A_2$ of A such that $r_{A_i}(n) = 1$, for each n, $i = 1, 2$.*

2 Proofs of Theorems 2 and 3.

Let $d \geq 2$ be an integer. We consider the d-adic expansion of each positive integer $n = \sum_{i \geq 0} n_i d^i,\ \ 0 \leq n_i < d$. The d-support of n is the subset $S_d(n) \subset \mathbb{N}_0$ such that

$$n = \sum_{i \in S_d(n)} n_i d^i, \quad 0 < n_i < d.$$

Let $0 \in A$ be an additive basis of order $h \geq 2$, that is, every sufficiently large integer can be expressed as a sum of h elements from A. We say that A is a (d,h)*- bounded set on* $X \subset \mathbb{N}$ if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that, for each $n \in X$, there is an h-tuple $\tau(n) = (a_1, \dots a_h) \in A^h$ satisfying

$$|S_d(a_1) \cup \cdots \cup S_d(a_h)| \leq f(|S_d(n)|). \tag{1}$$

We call a set X of positive integers (N,d)*-good* if there is an infinite set $Y \subset \mathbb{N}_0$ all of whose N-subsets are supports of some element in X, that is, for every N-subset $K \subset Y$ there is $n \in X$ such that $S_d(n) = K$. Set X is d*-good* if it is (N,d)-good for infinitely many values of N. A basis A is (d,h)*- bounded* if it is (d,h)- bounded on some d-good set X. In particular, a 2-bounded basis A is $(2,2)$-bounded on $X = [n_0(A), \infty)$. This explains the notions involved in Theorem 2 and it also shows that Theorem 2 implies Theorem 1.

Proof of Theorem 2. Let $0 \in A$ be an additive basis of order $h \geq 2$ which is (d,h)-bounded on a good set X. Given N for which X is (N,d)-good, let $Y' \subset \mathbb{N}_0$ be an infinite set each of whose N-subsets is the d-support of some element in X. By the definition, there is a function f such that for each N-subset $K \subset Y'$ there is $n \in X$ and a h-tuple $\tau(n) = (a_1, \dots, a_h)$ satisfying inequality (1).

Denote by $M = \lceil \log_d h \rceil$ and consider an infinite subset $Y = \{y_1 < y_2 < \cdots\} \subset Y'$ such that

$$|y_{r+1} - y_r| > \max\{f(N), M\}, \quad r \geq 1.$$

We color the N-subsets of Y as follows. For each N-subset $K = \{k_1 < \cdots < k_N\} \subset Y$, choose $n = n(K)$ with $S_d(n) = K$ and let $\tau(n) = (a_1, \dots, a_h)$. Note that,

$$\max S_d(n) \leq \max\{S_d(a_1), \dots, S_d(a_h)\} + M.$$

Since $k_{r+1} - k_r > M$, we have

$$(S_d(a_1) \cup \cdots \cup S_d(a_h)) \cap [k_r - M, k_r] \neq \emptyset, \quad r = 1, \dots, N.$$

We may therefore assume that $S_d(a_1)$ intersects $s \geq N/h$ of the intervals $[k_r - M, k_r]$. On the other hand, since $k_{r+1} - k_r > f(N)$ we have $S_d(a_1) \subset \cup_{r=1}^{N} [k_r - M, k_r]$. Let $a_1 = \sum_{i \geq 0} a_{1i} d^i$ be the d-adic expansion of a_1. Then we define the coloring $\mathbf{c}$ on the N-subsets of Y as follows

$$c_r(K) = (a_{1i}, i \in [k_r - M, k_r]) \text{ and } \mathbf{c}(K) = (c_1(K), \dots, c_N(K)). \tag{2}$$

By the definition of $\mathbf{c}$, if $c_r(K) = (x_{r0}, \dots, x_{rM})$, $r = 1, \dots, N$, then

$$x(K, \mathbf{c}) = \sum_{i=k_1-M}^{k_1} x_1 i d^{k_1-M+i} + \cdots + \sum_{i=k_N-M}^{k_N} x_i d^{k_N-M+i} = a_1 \in A. \tag{3}$$

The coloring $\mathbf{c}$ uses less than Nd^{M+1} colors. By Ramsey theorem, there is an infinite subset $Y_0 \subset Y$ all of whose N-subsets receive the same color $\beta = (\beta_1, \dots, \beta_N)$. Let $J \subset \{1, \dots, N\}$ be the set of subscripts for which $\beta_i \neq (0, \dots, 0)$. By the choice of a_1 in the definition of $\mathbf{c}$ we have $|J| \geq s \geq N/h$. Choose a subset $S = \{z_1 < z_2 < \ldots < z_{2s}\} \subset Y_0$ of cardinality $2s$ such that between any two consecutive elements of S there are N elements of Y_0. Set $V_j = \{z_{2j-1}, z_{2j}\}$, $j = 1, \dots, s$.

Let $U' \subset S$ be a subset of cardinality s obtained by picking out one element in each V_j. Since between any two elements in S there are N elements of Y_0, U' can be completed to a set $U = \{u_1 < \cdots < u_N\}$ such that $u_i \in U'$ whenever $i \in J$ and $u_i \in Y_0 \setminus S$ otherwise. By the choice of Y_0, set U has color β. Therefore, the element $x(U, \mathbf{c})$ as defined in (3) belongs to the base A. By the same token, the complement $W' = S \setminus U'$ of U' in S can be completed to an N-subset $W = \{w_1 < \ldots < w_N\}$ of Y_0 by adding $N - s$ elements in $Y_0 \setminus S$ which fill positions with value $(0, \dots, 0)$ in β. Therefore, the element $x(W, \mathbf{c})$ belongs to the base A as well.

Note that, by the choice of Y, the supports of $x(U, \mathbf{c})$ and $x(W, \mathbf{c})$ are disjoint. Moreover, for any other choice of an s-subset L of S obtained by picking one element in each V_j, we have

$$x(U) + x(S \setminus U) = x(L) + x(S \setminus L).$$

Therefore, we get $2^{s-1} \geq 2^{(N/h)-1}$ different expressions of $m = x(U) + x(S \setminus U)$ as sums of two elements in the base. Since there are infinitely many choices for N, the base A has the ET-property. $\square$

Proof of Theorem 3. The proof is similar to the above proof of Theorem 2. Given n and $\tau(n) = (a_1, \dots, a_h)$ consider the following coloring of $S(n)$,

$$\mathbf{c}'(S(n)) = (\mathbf{c}(S(n)); \alpha(n)),$$

where $\mathbf{c}(S(n))$ is the coloring given in (2) and $\alpha(n) = i$ if $a_1 \in A_i$. By the same argument as in the proof of Theorem 2, for each $N > 0$ there is an infinite set $Y_0 \subset \mathbb{N}$ such that all its N-subsets have the same color $\beta' = (\beta_1, \cdots, \beta_N; \alpha)$, where $\alpha = \alpha(N)$ identifies the set of the partition where the corresponding a_1 belongs to. Hence, there is a positive integer $m \in \mathbb{N}$ which admits at least $2^{(N/h)-1}$ different expressions as a sum of two elements in

$A_{\alpha(N)}$. Since A is partitioned into a finite number of parts, there are infinitely many values of N with the same value $\alpha(N)$. □

Let us now consider additive basis in $\mathbb{Z}$. For $n \in \mathbb{Z}$ let $S(n)$ denote the 2-adic support of n in the sense that $|n| = \sum_{i \in S(n)} 2^i$. The notion of 2-bounded sets is extended to subsets of $\mathbb{Z}$ in the obvious way.

3 Additive bases for $\mathbb{Z}$

The notion of (d,h)-bounded basis extends naturally to bases in $\mathbb{Z}$. For simplicity we only consider $(2,2)$-bounded basis. In order to preserve the unicity of binary expansions, we define the binary support of a negative integer as $S(x) = -S(|x|)$, that is, $x = -\sum_{i \in S(x)} 2^{-i}$. A set B of integers is *bounded* if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that, for each $n \in B + B$ there is a pair $x, y \in B$ satisfying

$$n = x + y \text{ and } |S(x) \cup S(y)| \leq f(|S(n)|).$$

We can now proceed to prove Theorem 5.

Proof of Theorem 5. We construct an increasing family $\{A_k,\ k \geq 1\}$ of sets of integers which satisfy the following properties:

1. $r_{A_k}(n) \leq 1$ for all $n \in \mathbb{Z}$.
2. $A_k + A_k \supseteq (-k/2, k/2)$ for all $k \geq 1$.
3. For each pair $x, y \in A_k$ we have $|S(x) \cup S(y)| \leq 4|S(x+y)|$.

¿From (1) and (2) it follows that $A = \cup_{k \geq 1} A_k$ is a unique representation basis of $\mathbb{Z}$ and, by (3), it is 2-bounded.

For each integer $n \in \mathbb{Z}$ define $\sigma(n)$ by $S(\sigma(n)) = \{i \in S(n) : i - 1 \notin S(n)\}$. Then $|S(n + \sigma(n))| = |S(\sigma(n))| \leq |S(n)|$ and $S(n + \sigma(n))$ has no two consecutive integers.

Let $A_1 = \{0\}$, $A_2 = \{-10, 0, 9\}$ and suppose we have constructed A_k satisfying properties (1)-(3) above and such that the support of each element in A_k has no two consecutive integers.

Let n_k be an element not in $A_k + A_k$ with smallest absolute value. Suppose that $n_k > 0$, the other case being similar. Choose a subset U_k of positive integers with cardinality

$|U_k| = |S(n_k)|$ such that $\min U_k > 1 + \max_{x\in A_k+A_k} \max\{i \in S(|x|)\}$ and U_k has no two consecutive integers. Define

$$a_k = n_k + \sigma(n_k) + \sum_{i\in U_k} 2^i, \quad b_k = n_k - a_k \text{ and } A_{k+1} = A_k \cup \{a_k, b_k\}.$$

We have

$$A_{k+1} + A_{k+1} = \{n_k\} \cup (A_k + A_k) \cup (A_k + a_k) \cup (A_k + b_k),$$

where, by the choice of a_k and b_k, the four sets in the righthand side of the equality are pairwise disjoint. Hence A_{k+1} verifies (1). Since $n_k \in A_{k+1} + A_{k+1}$, (2) is also verified. Let us check that (3) also holds. Since $|S(a_k)| \le 2|S(n_k)|$ and $|S(b_k)| \le 2|S(n_k)|$ then $|S(a_k) \cup S(b_k)| \le 4|S(n_k)| = 4|S(a_k + b_k)|$. Let $x \in A_k$. Since $S(x)$ has no two consecutive integers, we have $|S(x) \cup S(a_k)| \le |S(x + a_k)|$ if $x > 0$ and it can be easily checked that $|S(x) \cup S(a_k)| \le 4|S(x + a_k)|$ if $x < 0$. Indeed, let $V = S(x)$ and $V' = S(|x|) \setminus S(a_k)$. Then

$$|S(x) \cup S(a_k)| \le |V| + 2|S(n_k)| \le |V'| + 3|S(n_k)| \le 4(|V'| + |U_k| - 1) \le 4|S(x + a_k)|,$$

where in the last inequality we used the facts that V' has no two consecutive integers, that $\max S(|x|) < \min U_k$ and that, for each positive integers r and s, $2^{r+s} - 2^r = 2^{r+s-1} + \cdots + 2^r$. Similar arguments show that $|S(x) \cup S(b_k)| \le 4|S(x + b_k)|$. $\square$

Proof of Theorem 6. Denote by $Z = r^{-1}(\infty)$ and $Z' = \mathbb{Z} \setminus Z$. Consider the set $F \subset \mathbb{Z} \times \mathbb{N}$ of ordered pairs

$$F = \{(i, j) : i \in Z,\ j = 1, 2, \ldots\} \cup \{(i, j) : i \in Z',\ j = 1, \ldots, r(i)\},$$

with the ordering defined as

$$(i,j) \preceq (i',j') \Leftrightarrow \begin{cases} |i| + j < |i'| + j', & \text{or} \\ |i| + j = |i'| + j' \text{ and } j < j', & \text{or} \\ |i| + j = |i'| + j' \text{ and } j = j' \text{ and } |i| < |i'|, & \text{or} \\ |i| + j = |i'| + j' \text{ and } j = j' \text{ and } |i| = |i'| \text{ and } i < i'. & \end{cases}$$

so that $F = \{f_1 = (0, 1), f_2 = (-1, 1), f_3 = (1, 1), f_4, \cdots\}$ in this ordering.

We construct an non decreasing family $\{A_k, k \ge 1\}$ of sets of integers verifying

(i) $r_{A_k}(n) = \max\{j : (n, j) \preceq f_k\}$ if $(n, 1) \preceq f_k$ and $r_{A_k}(n) \le 1$ otherwise, and

(ii) $r_{A_k^+}(n) \le 1$ and $r_{A_k^-}(n) \le 1$ for all $n \in \mathbb{Z}$, where $A_k^+ = A_k \cap [0, \infty)$ and $A_k^- = A_k \setminus A_k^+$.

Once such a family of sets has been constructed, then $A = \cup_{k\geq 1} A_k$ verifies

$$r_A(n) = \lim_{k\to\infty} r_{A_k}(n) = r(n) \text{ for all } n \in \mathbb{Z}$$

while both $A^+ = A \cap [0,\infty)$ and $A^- = A \setminus A^+$ are B_2^1 sequences.

Let $A_1 = \{0\}$ and suppose that A_{k-1} satisfies conditions (i) and (ii) above. Let $f_k = (i,j)$. Then we have $r_{A_{k-1}}(i) = j-1$ if $j > 1$ and $r_{A_{k-1}}(i) \leq 1$ if $j = 1$.

If $r_{A_{k-1}}(i) = j = 1$ then we define $A_k = A_{k-1}$.

If $r_{A_{k-1}}(i) = j-1$, let $x_k = \max\{|x| : x \in A_{k-1} + A_{k-1}\} + |A_{k-1}| + |i|$ and define

$$a_k = i + x_k, \; b_k = -x_k \text{ and } A_k = A_{k-1} \cup \{a_k, b_k\}.$$

Then,

$$A_k + A_k = ((A_{k-1} + A_{k-1}) \cup \{i\}) \cup (A_{k-1} + a_k) \cup (A_{k-1} + b_k),$$

where, by the choice of x_k, the three sets on the righthand side of the equality are pairwise disjoint. Therefore $r_{A_k}(n) \leq r_{A_{k-1}}(n) + 1$ for all $n \in \mathbb{Z}$, where the equality holds either if $n = i$, so that $r_{A_k}(i) = j$, or when $n \in (A_{k-1} + a_k) \cup (A_{k-1} + b_k)$, in which case $f_k \prec (n,1)$ and $r_{A_k}(n) = 1$. Thus A_k verifies (i). Moreover, since x_k is large enough, then both $A_k^+ = A_{k-1}^+ \cup \{a_k\}$ and $A_k^- = A_{k-1}^- \cup \{b_k\}$ are still B_2^1 sets. This completes the proof. $\square$

4 Remarks and Open Problems

It is pleasing to note that the multiplicative version of Erdős-Turán conjecture is valid both in $\mathbb{N}$ and in $\mathbb{Z}$. However the additive version for 2-bounded basis is true in $\mathbb{N}$ while it fails to be true in $\mathbb{Z}$ even in the stronger form of Theorem 6 (we note that the basis in this Theorem can be required to be bounded as well). The proof of Theorem 5, though, involves constructing a very sparse basis for the set of all integers. Thicker bases for the integers without the ET property have been constructed by Nathanson [9] and Łuczak and Schoen [8] with counting function of order $n^{1/3}$. It might be true that unboundedness of the representation function appears when the lower density of the set has order $n^{1/2}$. In that respect the following may be an easier problem.

Problem 1 *Let A be an additive basis of $\mathbb{N}$ and assume that it has the ET property. Is it true that in any finite partition of A one of the parts still has the ET property.?*

Perhaps the following perspective may contribute to a better understanding of the Erdős-Turán conjecture. Given a function $f : \mathbb{N} \to \mathbb{N}$, we say that a base $A \subset \mathbb{Z}$ is *f-restricted* if for any $n \in \mathbb{Z}$ there exist $x, y \in X$ such that

$$n = x + y \text{ and } \max\{|x|, |y|\} \leq f(n).$$

A basis for the set of positive integers is f-restricted with $f(n) = n$, so that this case for basis of $\mathbb{Z}$ seems to be closely related to the Erdős-Turán conjecture. One possible formulation of the link between the two problems is the following:

Problem 2 *Is it true that for every thin additive basis A of $\mathbb{N}$ there exists a basis B of $\mathbb{N}$ and a constant c which is bounding the number of solutions $n = a - b, a \in A, b \in B$ for every $n \in \mathbb{N}$?*

A positive answer implies that proving the ET property for f-restricted basis in $\mathbb{Z}$ with $f(n) = n$ gives a proof of the Erdős-Turán conjecture.

On the one side of the spectrum the function $f(n) = n/2 + c$ yields a positive solution of the Erdős-Turán conjecture for f-restricted bases.

Proposition 1 *For any positive integer c a f-restrictive basis in $\mathbb{Z}$ with $f(n) = n/2 + c$ has the Erdős-Turán property.*

Proof. Let A be a f-restricted base of $\mathbb{Z}$. Define sets $A_i, i = 0, 1, \ldots, 2c$ by $n \in A_i$ iff there exists $x \in A$ such that $n/2 - x = i/2$. Using Van der Waerden Theorem, one of the sets A_i contains an arithmetic progression of any length. Since an arithmetic progression of length k yields $k/2$ pairs of elements with the same sum, the basis A has the Erdős-Turán property.

Of course the same proof gives the Erdős-Turán property for f-restricted bases for functions $f(n) = n/2 + \epsilon(n)$ where $\epsilon(n)$ is a very slowly growing function (essentially the inverse function to the Van der Waerden's function $W(k, k)$; by Shelah and Gowers this is not as astronomically slowly growing as it seems on the first glance.) These remarks also apply to f-restricted bases in $\mathbb{N}$.

By using a greedy algorithm like in the above proof of Theorem 5 we easily get examples of f-restricted basis for $\mathbb{Z}$ without the ET-property for any function $f(n) > 2^n$. It would be interesting to find examples for more slowly growing functions as suggested by the following problem.

Problem 3 *Fix a positive $c \in \mathbb{R}$. Let A be a basis for $\mathbb{Z}$ such that, for each sufficiently large $n \in A + A$ there is $x, y \in A$ with $n = x + y$ and $\max\{|x|, |y|\} \leq cn$. Is it true that A has the ET property?*

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