On the least exponential growth admitting uncountably many closed permutation classes

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Abstract

We show that the least exponential growth of counting functions which admits uncountably many closed permutation classes lies between 2^n and $(2.33529\ldots)^n$.

1 Introduction

Let S_n be the set of $n!$ permutations of $[n] = \{1, 2, \ldots, n\}$, $S = \bigcup_{n=0}^{\infty} S_n$ be the set of all finite permutations, and $\prec$ be the usual containment of permutations (defined below). It is well-known that the partial ordering $(S, \prec)$ has infinite antichains, see [11], [13], [16], and [18]. Equivalently, $(S, \prec)$ has uncountably many lower order ideals $X \subset S$; these are called *closed permutation classes* or, for short, CPC's. In this article we want to localize the least exponential growth of the counting function $n \mapsto |X \cap S_n|$ which admits uncountably many CPC's X.

More precisely, if

$$K_\alpha = \{X :\ X \text{ is a CPC such that } |X \cap S_n| < \alpha^n \text{ for all } n > n_0\},$$

what can be said about the number

$$\kappa = \inf\{\alpha > 1 :\ \text{the set } K_\alpha \text{ is uncountable}\}.$$

We prove the following bounds.

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Theorem 1.1 *Let κ determine the least exponential growth of uncountably many CPC's, as defined above. Then*

$$2 \le \kappa \le 2.33529\ldots$$

where the upper bound is the only real root of $x^5 - x^4 - 2x^3 - 2x^2 - x - 1$.

When the base α in α^n is increased, the "phase transition" from countably to uncountably many CPC's with growth $< \alpha^n$, $n > n_0$, occurs somewhere in the interval $[2, 2.33529\ldots]$. It would be interesting to narrow it or to determine κ exactly.

In the proof of Theorem 1.1 we build on previously obtained results. In Kaiser and Klazar [10, Theorem 3.8] we have proved that the exponential growths of CPC's X such that $|X \cap S_n| < 2^{n-1}$ for at least one n form a discrete hierarchy α_i^n, $i = 2, 3, 4, \ldots$, where $\alpha_2 = 1.61803\ldots < \alpha_3 < \alpha_4 < \ldots < 2$, $\alpha_i \uparrow 2$, and α_i is the largest positive real root of $x^i - x^{i-1} - \cdots - 1$. It follows from the proof, with some additional arguments from the wqo theory, that the structure of the corresponding CPC's is so restricted that each set $K_{2-\varepsilon}$ must be countable. In Section 2 we give a proof of this fact. On the other hand, Spielman and Bóna [16] constructed an infinite antichain $(R, \prec)$ such that $123 \not\prec \pi$ for every $\pi \in R$. Thus, denoting $S(123)$ the set of 123-avoiding permutations, there are uncountably many CPC's X with $X \subset S(123)$. Since $|S(123) \cap S_n| = \frac{1}{n+1}\binom{2n}{n}$ (Rogers [14], Simion and Schmidt [15], ...), we obtain the bound $\kappa \le 4$. The enumeration of $S(123, 3214)$, due to West [19], and the infinite antichain U due to Atkinson, Murphy and Ruškuc [5] give the improvement $\kappa \le 2.61803\ldots$. In Section 3 we lower this further to the upper bound in Theorem 1.1.

Closed permutation classes and permutation avoidance (containment) are related to computer science mainly via sorting problems. The set of permutation π which, when inputed to some sorting device, can be sorted to the identical permutation, is often a CPC. Indeed, this was the very first motivation to introduce $\prec$ in the works of Pratt [13] and Tarjan [18]. Recent works on closed permutation classes and permutation containment with motivation in computer science (sorting, complexity of recognizing $\prec$) are, for example, Ahal and Rabinovich [1], Albert et al. [2], Atkinson [4], Atkinson, Murphy and Ruškuc [6, 7], and Bose, Buss and Lubiw [8].

Now we review the definition of $\prec$ and basic facts on CPC's. Further definitions will be given throughout next two sections.

For $\pi \in S_n$, n is the *length* of π and we define $|\pi| = n$. For $A, B \subset \mathbf{N} = \{1, 2, \ldots\}$ the notation $A < B$ means that $a < b$ for every $a \in A$ and $b \in B$. Interval $\{a, a+1, a+2, \ldots, b\}$, where $a, b \in \mathbf{N}$, is denoted $[a, b]$. Instead of $[1, n]$ we write $[n]$. Two m-term sequences $a_1 a_2 \ldots a_m$ and $b_1 b_2 \ldots b_m$ over $\mathbf{N}$ are *order-isomorphic* if $b_k < b_l \Leftrightarrow a_k < a_l$ for all $k, l \in [m]$. A permutation π is *contained* in another permutation ρ, written $\pi \prec \rho$, if ρ (as a sequence) has a subsequence that is order-isomorphic to π; in the opposite case ρ is π-*avoiding*. Visually, the graph of π (as a discrete function) can be obtained from that of ρ by omitting points. If $\pi \in S_n$ and $A \subset [n]$, the *restriction* $\pi|A$ is the permutation order-isomorphic to the corresponding subsequence of π. For $X \subset S$, $M(X)$ is the set of all $\prec$-minimal permutations not in X, and $S(X)$ is the set of all permutations not containing any member of X. We define $S_n(X) = S(X) \cap S_n$. For finite $X = \{\pi_1, \ldots, \pi_r\}$ we write $S(\pi_1, \ldots, \pi_r)$ and $S_n(\pi_1, \ldots, \pi_r)$ instead of $S(\{\pi_1, \ldots, \pi_r\})$ and $S_n(\{\pi_1, \ldots, \pi_r\})$. Clearly, each proper restriction of each $\pi \in M(X)$ lies in X. A set $X \subset S$ is a CPC (closed permutation class) if $\pi \prec \sigma \in X$ implies $\pi \in X$. Each $S(X)$ is a CPC and for each CPC X we have $X = S(M(X))$. Each $M(X)$ is an antichain (its elements are mutually incomparable by $\prec$) and for each antichain $X \subset S$ we have $X = M(S(X))$. Thus the mapping $X \mapsto M(X)$, with the inverse $X \mapsto S(X)$, is a bijection between the set of all CPC's and the set of all antichains of permutations.

2 The lower bound of Theorem 1.1

In this section we mostly follow the notation of [10]. A permutation σ is *alternating* if $\sigma(\{1, 3, 5, \ldots\}) > \sigma(\{2, 4, 6, \ldots\})$. For $\pi \in S$ we let $\mathrm{al}(\pi)$ be the maximum length of an alternating permutation σ such that $\sigma \prec \pi$ or $\sigma \prec \pi^{-1}$. For a set of permutations X we denote $\mathrm{al}(X) = \max\{\mathrm{al}(\pi) :\ \pi \in X\}$.

Lemma 2.1 *If X is a CPC with $\mathrm{al}(X) = \infty$, then $|X \cap S_n| \geq 2^{n-1}$ for every $n \in \mathbf{N}$.*

Proof. We suppose that X contains arbitrarily long alternating permutations; the other case with inverses is treated similarly. Using the closeness of X and the pigeonhole principle, we deduce that either for every $n \in \mathbf{N}$ there is an alternating $\pi \in X \cap S_n$ such that $\pi(1) < \pi(i)$ for every odd $i \in [2, n]$ or for every odd $n \in \mathbf{N}$ there is an alternating $\pi \in X \cap S_n$ such that $\pi(n) < \pi(i)$ for every odd $i \in [n-1]$. We assume that the former case occurs, the latter one is similar. It follows that for every $n \in \mathbf{N}$ and every subset $A \subset [2, n]$ there is a permutation $\pi_A \in X \cap S_n$ such that $\pi_A(i) < \pi_A(1) \Leftrightarrow i \in A$. For distinct subsets A we get distinct permutations π_A and $|X \cap S_n| \geq 2^{n-1}$. $\Box$

If $\sigma \in S_n$ and $\tau \in S_m$, then $\pi = \sigma \oplus \tau \in S_{n+m}$ is the permutation defined by $\pi(i) = \sigma(i)$ for $i \in [n]$ and $\pi(i) = n + \tau(i-n)$ for $i \in [n+1, n+m]$. Similarly, $\pi = \sigma \ominus \tau$ is defined by $\pi(i) = m + \sigma(i)$ for $i \in [n]$ and $\pi(i) = \tau(i-n)$ for $i \in [n+1, n+m]$. Note that if $\sigma' \prec \sigma$ and $\tau' \prec \tau$, then $\sigma' \oplus \tau' \prec \sigma \oplus \tau$; similarly for $\ominus$. If $\pi \in S$ has no decomposition $\pi = \sigma \oplus \tau$ for any nonempty σ and τ, we say that π is *up-indecomposable.* The subset of up-indecomposable permutations in S_k is denoted Ind_k^+. Each $\pi \in S$ has a unique *up-decomposition* $\pi = \sigma_1 \oplus \sigma_2 \oplus \ldots \oplus \sigma_k$ where each σ_i is up-indecomposable; σ_i's are called *up-blocks.* The maximum size of an up-block in the up-decomposition of π is denoted $h^+(\pi)$. For the operation $\ominus$, the down-(in)decomposability, sets Ind_k^-, down-decompositions, down-blocks, and function $h^-(\cdot)$ are defined in an analogous way.

The proof of the next lemma is left to the reader as an exercise (or see [10, Lemma 3.7]).

Lemma 2.2 *For every $\pi \in \mathrm{Ind}_n^+$, $n > 1$, there is a $\sigma \in \mathrm{Ind}_{n-1}^+$ such that $\sigma \prec \pi$. The same holds for down-indecomposable permutations.*

Lemma 2.3 *If X is a CPC with the property that for every $k \in \mathbf{N}$ there is a permutation $\sigma \in \mathrm{Ind}_k^+$ such that $\sigma \oplus \sigma \oplus \ldots \oplus \sigma \in X$ (k summands), then $|X \cap S_n| \geq 2^{n-1}$ for every $n \in \mathbf{N}$. An analogous result holds for down-decompositions.*

Proof. Using the assumption and Lemma 2.2, we obtain that for every $n \in \mathbf{N}$ there is a set $\Sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ such that $\sigma_i \in \mathrm{Ind}_i^+$ and every permutation of the form $\pi = \rho_1 \oplus \rho_2 \oplus \ldots \oplus \rho_r$, where $\rho_i \in \Sigma$ and $r \leq n$, is in X. Since the up-decomposition uniquely determines π, there are exactly 2^{n-1} such permutations π in $X \cap S_n$ (as compositions of n) and $|X \cap S_n| \geq 2^{n-1}$. $\Box$

Let $H_k^+ = \{\pi \in S : h^+(\pi) < k\}$ and similarly for H_k^-. For $k \in \mathbf{N}$ and $\pi \in S_n$, we let $s_k(\pi)$ be the number r of intervals $I_1 < I_2 < \ldots < I_r$ in this unique decomposition of $[n]$: I_1 is the longest initial interval in $[n]$ such that $\pi|I_1 \in H_k^+ \cup H_k^-$, I_2 is the longest following interval such that $\pi|I_2 \in H_k^+ \cup H_k^-$ and so on. We call $I_1 < I_2 < \ldots < I_r$ the *k-decomposition* of π. Note that each restriction $\pi|I_i$ has up-decomposition or down-decomposition composed of blocks of lengths at most $k-1$ and that each restriction $\pi|I_i \cup I_{i+1}$ contains both an element from Ind_k^+ and an element from Ind_k^-. For $k \in \mathbf{N}$ and X a set of permutations we define $s_k(X) = \max\{s_k(\pi) : \pi \in X\}$. We let $s_1(\pi) = s_1(X) = \infty$ for every permutation π and set X.

Proposition 2.4 *If X is a CPC such that $|X \cap S_n| < 2^{n-1}$ for some $n \in \mathbf{N}$, then $\mathrm{al}(X) < \infty$ and, for some $k \in \mathbf{N}$, $s_k(X) < \infty$.*

Proof. If $\mathrm{al}(X) = \infty$, we have $|X \cap S_n| \geq 2^{n-1}$ for all $n \in \mathbf{N}$ by Lemma 2.1, which is a contradiction. Suppose that $s_k(X) = \infty$ for every $k \in \mathbf{N}$. By the remark after the definition of $s_k(\cdot)$, the pigeonhole principle and the closeness of X, for every $k \geq 2$ there are permutations $\sigma_k \in \mathrm{Ind}_k^+$, $\tau_k \in \mathrm{Ind}_k^-$ and $\pi_k \in X \cap S_r$, $k^2 \leq r \leq 2k^2$, with the property that $[r]$ can be decomposed into k intervals $I_{k,1} < I_{k,2} < \ldots < I_{k,k}$, $k \leq |I_{k,i}| \leq 2k$, so that each of the k restrictions $\pi_k|I_{k,i}$ contains both σ_k and τ_k. For $k \in \mathbf{N}$ and $1 \leq i \leq k$, we consider the interval

$$J_{k,i} = [\min \pi_k(I_{k,i}), \max \pi_k(I_{k,i})].$$

Using the Ramsey theorem and Lemma 2.2, we may assume that either for every $k \in \mathbf{N}$ the k intervals $J_{k,1}, \ldots, J_{k,k}$ intersect each other or for every $k \in \mathbf{N}$ these k intervals are mutually disjoint. In the former case, they must always have one point in common, and it follows that $\mathrm{al}(X) = \infty$. We have again the contradiction by Lemma 2.1. In the latter case, using again Ramsey theorem (or Erdős–Szekeres theorem) and Lemma 2.2, we may assume that either for every $k \in \mathbf{N}$ we have $J_{k,1} < J_{k,2} < \ldots < J_{k,k}$ or for every $k \in \mathbf{N}$ we have $J_{k,1} > J_{k,2} > \ldots > J_{k,k}$. Then for every $k \in \mathbf{N}$ we have $\sigma_k \oplus \sigma_k \oplus \ldots \oplus \sigma_k \in X$ (k summands) or for every $k \in \mathbf{N}$ we have $\tau_k \ominus \tau_k \ominus \ldots \ominus \tau_k \in X$ (k summands). By Lemma 2.3, we get the contradiction that $|X \cap S_n| \geq 2^{n-1}$ for all $n \in \mathbf{N}$. $\square$

Every bijection $f : X \to Y$, where $X = \{x_1 < x_2 < \ldots < x_n\}$ and $Y = \{y_1 < y_2 < \ldots < y_n\}$ are subsets of $\mathbf{N}$, defines a unique $\pi \in S_n$ *order-isomorphic* to f: $\pi(i) = j \Leftrightarrow f(x_i) = y_j$. An interval in X is a subset of the form $\{x_i, x_{i+1}, \ldots, x_j\}$, $1 \leq i \leq j \leq n$.

Lemma 2.5 *Let $X, Y \subset \mathbf{N}$ be two n-element subsets, $f : X \to Y$ be a bijection, and $\pi \in S_n$ be order-isomorphic to f. Suppose $\pi \in H_k^+ \cup H_k^-$.*

Then every interval partition $J_1 < J_2 < \ldots < J_r$ of X can be refined by an interval partition $I_1 < I_2 < \ldots < I_s$ such that $s \le r + (k-1)(r-1)$ and each image $f(I_i)$ is an interval in Y. Similarly, every partition of Y in r intervals can be refined by a partition in at most $r + (k-1)(r-1)$ intervals which under f^{-1} map to intervals in X.

Proof. It suffices to prove only the first part because $\pi \in H_k^+ \cup H_k^-$ implies that $\pi^{-1} \in H_k^+ \cup H_k^-$. Without loss of generality we can assume that $X = Y = [n]$ and $f = \pi$. Let $\pi \in S_n \cap H_k^+$ (the case with H_k^- is similar) and $J_1 < J_2 < \ldots < J_r$ be an interval partition of $[n]$. We call an up-block in the up-decomposition $\pi = \sigma_1 \oplus \sigma_2 \oplus \ldots \oplus \sigma_t$ intact if its domain lies completely in some J_i and we call it split otherwise. Clearly, there are at most r maximal runs of intact up-blocks and at most $r-1$ split up-blocks. We partition $[n]$ in the intervals $I_1 < I_2 < \ldots < I_s$ so that each I_i is either the domain of a maximal run or a singleton in the domain of a split up-block. Since $|\sigma_i| < k$ for each i, we have $s \le r + (k-1)(r-1)$. This is a refinement of the original interval partition and $\pi(I_i)$ is an interval for every i. $\square$

We will need a continuity property of the functions $\mathrm{al}(\cdot)$ and $s_k(\cdot)$.

Lemma 2.6 *Let $\sigma \in S_n$, $\tau \in S_{n+1}$, and $\sigma \prec \tau$. Then $\mathrm{al}(\tau) \le \mathrm{al}(\sigma) + 2$ and, for every $k \in \mathbf{N}$, $s_k(\tau) \le s_k(\sigma) + 2$.*

Proof. Let $\rho \in S$ be alternating, $|\rho| = \mathrm{al}(\tau)$, and $\rho \prec \tau$ (the case $\rho \prec \tau^{-1}$ is similar). The permutation σ arises by deleting one point from the graph of τ. If this point does not lie in the embedding of ρ in τ, we have $\rho \prec \sigma$ and $\mathrm{al}(\sigma) \ge \mathrm{al}(\tau)$. If it does, we can delete one more point from the graph of ρ so that the resulting ρ' is alternating. But $\rho' \prec \sigma$ and $|\rho'| = |\rho| - 2$, so $\mathrm{al}(\sigma) \ge \mathrm{al}(\tau) - 2$.

Let $k \ge 2$ be given, $\pi \in S_n$ be arbitrary, and $I_1 < I_2 < \ldots < I_s$ be any decomposition of $[n]$ into s intervals satisfying, for every $i = 1, \ldots, s$, $\pi|I_i \in H_k^+ \cup H_k^-$; this can be called a weak k-decomposition of π. We claim that $s_k(\pi) \le s$. This follows from the observation that each interval of the k-decomposition of π must contain the last element of some I_i. Now τ arises by inserting a new point p in the graph of σ. The domain $\{p_0\}$ of p is inserted in an interval J_j of the k-decomposition $J_1 < J_2 < \ldots < J_r$ of σ and splits it into three intervals J_j', $\{p_0\}$, and J_j'' (J_j' or J_j'' may be empty). Replacing J_j by J_j', $\{p_0\}$, and J_j'', we get a weak k-decomposition of τ with at most $r+2$ intervals. Thus $s_k(\tau) \le r + 2 = s_k(\sigma) + 2$. $\square$

Recall that a partial ordering $(Q, \leq_Q)$ is a *well partial ordering*, briefly wpo, if it has no infinite strictly descending chains and no infinite antichains. The first condition is in $(S, \prec)$ satisfied but the second one is not and therefore $(S, \prec)$ is not a wpo. Let $(Q, \leq_Q)$ be a partial ordering. The set $\mathrm{Seq}(Q)$ of all finite tuples $(q_1, q_2, \ldots, q_m)$ of elements from Q is partially ordered by the derived Higman ordering $\leq_H$: $(q_1, q_2, \ldots, q_m) \leq_H (r_1, r_2, \ldots, r_n) \Leftrightarrow$ there is an increasing mapping $f : [m] \to [n]$ such that $q_i \leq_Q r_{f(i)}$ for every $i \in [m]$. For the proof of the following theorem see Higman [9] or Nash-Williams [12].

Theorem 2.7 (Higman, 1952) *If $(Q, \leq_Q)$ is a wpo then $(\mathrm{Seq}(Q), \leq_H)$ is a wpo as well.*

If $\sigma \in S_m$ and $\tau_i \in S_{n_i}$, $i = 1, \ldots, m$, the permutation $\pi = \sigma[\tau_1, \ldots, \tau_m] \in S_{n_1+\cdots+n_m}$ is defined, for $i \in [n_1 + \cdots + n_m]$ and setting $k = \max(\{j : n_1 + \cdots + n_j < i\} \cup \{0\})$ and $n_0 = 0$, by

$$\pi(i) = n_0 + n_1 + \cdots + n_k + \tau_{k+1}(i - n_0 - n_1 - \cdots - n_k).$$

Visually, for $i = 1, \ldots, m$ the i-th point (counted from the left) in the graph of σ is replaced by a downsized copy of the graph of τ_i; the copies are small enough not to interfere horizontally and vertically each with the other. This operation generalizes $\oplus$ and $\ominus$: $\sigma \oplus \tau = 12[\sigma, \tau]$ and $\sigma \ominus \tau = 21[\sigma, \tau]$. If $\tau_i' \prec \tau_i$, $i = 1, \ldots, m$, then $\sigma[\tau_1', \ldots, \tau_m'] \prec \sigma[\tau_1, \ldots, \tau_m]$. If P and Q are sets of permutations, we define

$$P[Q] = \{\pi[\sigma_1, \ldots, \sigma_m] : \ m \in \mathbf{N}, \pi \in P \cap S_m, \sigma_i \in Q\}.$$

The next lemma is an immediate consequence of Higman's theorem or of the easier result that the Cartesian product of two wpo's also is a wpo.

Lemma 2.8 *Let P and Q be sets of permutations such that P is finite and $(Q, \prec)$ is a wpo. Then $(P[Q], \prec)$ is a wpo.*

Let $\pi \in S_n$ and $J_1 < J_2 < \ldots < J_r$ be an interval partition of $[n]$. Observe that if each image $\pi(J_i)$ is also an interval, then there is a permutations $\sigma \in S_r$ such that $\pi = \sigma[\pi|J_1, \ldots, \pi|J_r]$.

Lemma 2.9 *For every fixed $k, K \in \mathbf{N}$ there is a finite set of permutations P such that*

$$\{\pi \in S : \ \mathrm{al}(\pi) < K \ \& \ s_k(\pi) < K\} \subset P[H_k^+ \cup H_k^-].$$

Proof. We show that

$$P = S_1 \cup S_2 \cup \ldots \cup S_{kK^*}$$

works where $K^* = (K-1)\binom{K}{2} + 1$. Let $\pi \in S_n$ satisfy $\mathrm{al}(\pi) < K$ and $s_k(\pi) < K$. Since $s_k(\pi) < K$, $[n]$ can be partitioned in r intervals $J_1 < J_2 < \ldots < J_r$, $r < K$, so that always $\pi|J_i \in H_k^+ \cup H_k^-$ (we will not need the other property of k-decomposition of π). We show that $[n]$ can be partitioned in at most kK^* intervals so that their images under π^{-1} are intervals refining $J_1 < J_2 < \ldots < J_r$. Then we are done because $\pi|I \in H_k^+ \cup H_k^-$ for every interval (in fact, every subset) $I \subset J_i$.

We consider two words u and u' over $[K]$. The word $u = a_1 a_2 \ldots a_n$ is defined by $a_i = j \Leftrightarrow \pi^{-1}(i) \in J_j$ and u' arises from u by contracting each maximal run of one letter in one element. For example, if $u = 2221331111$ then $u' = 2131$. Let l be the length of u' which is also the number of maximal runs in u. Clearly, u' has no two consecutive identical letters. Since $\mathrm{al}(\pi) < K$, u and u' have no alternating subsequence $\ldots a \ldots b \ldots a \ldots b \ldots$, $a \neq b$, of length $K+1$. A pigeonhole argument implies that $l \le K^* = (K-1)\binom{K}{2} + 1$.

We partition $[n]$ in l intervals $L_1 < L_2 < \ldots < L_l$ according to the maximal runs in u. Each $\pi^{-1}(L_i)$ is a subset of some J_j but in general is not an interval. Let $j \in [r]$ and $M_j \subset [n]$ be the union of i_j intervals L_i corresponding to all i_j maximal runs of j in u; $\pi^{-1}(M_j) = J_j$. Applying Lemma 2.5 to the restricted mapping $\pi : J_j \to M_j$ and to the partition of M_j into i_j intervals L_i, we can refine the partition by at most $i_j + (k-1)(i_j - 1)$ intervals in M_j (but they are also intervals in $[n]$) whose images by π^{-1} are intervals in J_j (and so in $[n]$). Taking all these refinements for $j = 1, 2, \ldots, r$, we get a partition of $[n]$ in at most $\sum_{j=1}^{r}(i_j + (k-1)(i_j - 1)) < \sum_{j=1}^{r} k i_j = kl \le kK^*$ intervals whose images by π^{-1} are intervals in $[n]$ refining the partition $J_1 < J_2 < \ldots < J_r$. $\square$

Proposition 2.10 *For every fixed $k, K \in \mathbf{N}$, the set*

$$\{\pi \in S : \ \mathrm{al}(\pi) < K \ \& \ s_k(\pi) < K\}$$

is a wpo with respect to $\prec$.

Proof. In view of Lemmas 2.8 and 2.9, it suffices to show that $(H_k^+ \cup H_k^-, \prec)$ is a wpo. It is enough to show that $(H_k^+, \prec)$ is a wpo. Using k-decompositions, we represent each $\pi \in H_k^+$ by a word over $\Sigma = \mathrm{Ind}_1^+ \cup$

$\ldots \cup \operatorname{Ind}^+_{k-1}$. Now, denoting $\leq_s$ the ordering by subsequence, it follows from Theorem 2.7 that $(\Sigma^*, \leq_s)$ is a wpo and this implies that $(H^+_k, \prec)$ is a wpo. $\square$

Proposition 2.11 *For every $0 < \varepsilon \leq 1$, the set $K_{2-\varepsilon}$ is countable.*

Proof. Let an ε, $0 < \varepsilon \leq 1$, and a CPC $X \in K_{2-\varepsilon}$ be given. It suffices to show that the antichain of permutations $M(X)$ is finite. We have $|X \cap S_n| < 2^{n-1}$ for some $n > 1$ and, by Proposition 2.4, $\mathrm{al}(X) < K$ and $s_k(X) < K$ for some constants $k, K \in \mathbf{N}$. By Lemma 2.6, $\mathrm{al}(M(X)) < K + 2$ and $s_k(M(X)) < K + 2$. By Proposition 2.10, $M(X)$ is finite. $\square$

This finishes the proof of the inequality $\kappa \geq 2$. In fact, we have proved that the set

$$\{X : X \text{ is a CPC such that } |X \cap S_n| < 2^{n-1} \text{ for some } n \in \mathbf{N}\}$$

is countable. It is likely that K_2 is countable.

3 The upper bound of Theorem 1.1

Atkinson, Murphy and Ruškuc [5] introduced an infinite antichain of permutations

$$U = \{\mu_7, \mu_9, \mu_{11}, \ldots\}$$

where

$$\begin{aligned}
\mu_7 &= 4,7,6|1,5,3,2 \\
\mu_9 &= 6,9,8|4,7|1,5,3,2 \\
\mu_{11} &= 8,11,10|6,9,4,7|1,5,3,2 \\
&\vdots \\
\mu_{2k+5} &= 2k+2, 2k+5, 2k+4|2k, 2k+3, 2k-2, 2k+ \\
&\quad +1, \ldots, 6, 9, 4, 7|1,5,3,2 \\
&\vdots
\end{aligned}$$

The initial segment in μ_{2k+5} is $2k+2, 2k+5, 2k+4$, the final segment is $1, 5, 3, 2$, and in the middle segment the sequences $2k, 2k-2, \ldots, 4$ and $2k+$

$3, 2k+1, \ldots, 7$ are interleaved. (In fact, we have reversed the permutations of [5]). We reprove, using a different argument than in [5], that μ_i form an antichain. We associate with $\pi \in S_n$ a graph $G(\pi)$ on the vertex set $\{(i, \pi(i)) : i \in [n]\}$, in which $(i, \pi(i))$ and $(j, \pi(j))$ are adjacent if and only if $i < j$ and $\pi(i) < \pi(j)$. It is clear that $\pi \prec \sigma$ implies $G(\pi) \leq_g G(\sigma)$ where $\leq_g$ is the subgraph relation (this holds even with the induced subgraph relation). A *double fork* F_i is the tree on i vertices, $i \geq 6$, that is obtained by appending pendant vertex both to the second and to the penultimate vertex of a path with $i-2$ vertices. It is easy to see that $(\{F_i : i \geq 6\}, \leq_g)$ is an antichain.

Lemma 3.1 *$(U, \prec)$ is an antichain. Moreover,*

$$(\{123, 3214, 2143, 15432\} \cup U, \prec)$$

is an antichain.

Proof. For every $i = 7, 9, 11, \ldots$, $G(\mu_i) = F_i$. Since double forks form an antichain to $\leq_g$, so do the permutations μ_i to $\prec$. It is clear that the four new short permutations form an antichain and none contains any μ_i. $G(123)$ is a triangle, $G(2143)$ is a quadrangle and $G(15432)$ has a vertex of degree 4, and therefore none of the three permutations is contained in any μ_i. That $3214 \not\prec \mu_i$ for every i is easily checked directly. $\square$

Proposition 3.2 *Let $s_n = |S_n(123, 3214, 2143, 15432)|$. Then*

$$\sum_{n \geq 1} s_n x^n = \frac{x^5 + x^4 + x^3 + x^2 + x}{1 - x - 2x^2 - 2x^3 - x^4 - x^5}.$$

As $n \to \infty$, $s_n \sim c(2.33529\ldots)^n$ where $c > 0$ is a constant and $2.33529\ldots$ is the only real root of $x^5 - x^4 - 2x^3 - 2x^2 - x - 1$.

Proof. We denote $S_n^* = S_n(123, 3214, 2143, 15432)$ and partition S_n^* in five sets $A_n, \ldots, E_n$ as follows. For $n \geq 2$ and $\pi \in S_n^*$, we let $\pi \in A_n \Leftrightarrow \pi(1) = n-1$, $\pi \in B_n \Leftrightarrow \pi(1) = n-2$, $\pi \in C_n \Leftrightarrow \pi(1) \leq n-3$, $\pi \in D_n \Leftrightarrow \pi(1) = n$ & $\pi(2) \geq n-3$, and $\pi \in E_n \Leftrightarrow \pi(1) = n$ & $\pi(2) \leq n-4$. We denote $|A_n| = a_n, \ldots, |E_n| = e_n$. Notice that for every $n \in \mathbf{N}$ and $\pi \in S_n^*$, $\pi^{-1}(n) \leq 3$. For if $\pi^{-1}(n) \geq 4$, the first three values of π have an ascend or all are descending, and $123 \prec \pi$ or $3214 \prec \pi$. Thus every $\sigma \in S_{n+1}^*$ arises from some $\pi \in S_n^*$ by inserting the value $n+1$ on one of the three sites: in front of the whole π (site 1), between the first two values of π (site 2) or between the second and the third value of π (site 3). We discuss the cases depending on in which set π lies.

In all five cases we can insert $n+1$ on site 1. With the exception of the case $\pi \in D_n$, we cannot insert $n+1$ on site 3 because this would give $123 \prec \sigma$ or $2143 \prec \sigma$ or $15432 \prec \sigma$. If $\pi \in C_n$, we cannot insert $n+1$ on site 2 because this would give $123 \prec \sigma$ or $15432 \prec \sigma$. One can check that there are no other restrictions on the insertion of $n+1$. Hence $\pi \in A_n$ produces two σ's, one in D_{n+1} and the other in B_{n+1}; $\pi \in B_n$ produces also two σ's, one in D_{n+1} and the other in C_{n+1}; $\pi \in C_n$ produces one σ in E_{n+1}; $\pi \in D_n$ produces three σ's, one in D_{n+1} and two in A_{n+1}; and $\pi \in E_n$ produces two σ's, one in D_{n+1} and the other in A_{n+1}. From this we obtain the recurrences $a_{n+1} = 2d_n + e_n$, $b_{n+1} = a_n$, $c_{n+1} = b_n$, $d_{n+1} = a_n + b_n + d_n + e_n$, and $e_{n+1} = c_n$.

We set $(a_1, b_1, c_1, d_1, e_1) = (0, 0, 0, 0, 1)$, which gives correctly $(a_2, b_2, c_2, d_2, e_2) = (1, 0, 0, 1, 0)$. Let $v = (0, 0, 0, 0, x)$ be the vector of initial conditions for $n = 1$ and M be the 5×5 transfer matrix

$$M = \begin{pmatrix} 0 & 0 & 0 & 2x & x \\ x & 0 & 0 & 0 & 0 \\ 0 & x & 0 & 0 & 0 \\ x & x & 0 & x & x \\ 0 & 0 & x & 0 & 0 \end{pmatrix}.$$

For the generating functions $A = \sum_{n\geq 1} a_n x^n, \ldots, E = \sum_{n\geq 1} e_n x^n$, the recurrences give relation

$$(A, B, C, D, E)^T = (I + M + M^2 + \cdots)v^T = (I - M)^{-1}v^T.$$

From this, since $s_n = a_n + b_n + c_n + d_n + e_n$,

$$\begin{aligned} \sum_{n\geq 1} s_n x^n &= A + B + C + D + E = (1, 1, 1, 1, 1)(I - M)^{-1}v^T \\ &= \frac{x(x^4 + x^3 + x^2 + x + 1)}{1 - x - 2x^2 - 2x^3 - x^4 - x^5}. \end{aligned}$$

One can check that $2.33529\ldots$ is the dominant root of the reciprocal polynomial $x^5 - x^4 - 2x^3 - 2x^2 - x - 1$ of the denominator. The asymptotics of s_n follows from the standard facts on asymptotics of coefficients of rational functions. $\square$

We obtain the recurrence $s_1 = 1$, $s_2 = 2$, $s_3 = 5$, $s_4 = 12$, $s_5 = 28$, and $s_n = s_{n-1} + 2s_{n-2} + 2s_{n-3} + s_{n-4} + s_{n-5}$ for $n \geq 6$. The first values of s_n are:

$$(s_n)_{n \geq 1} = (1, 2, 5, 12, 28, 65, 152, 355, 829, 1936, 4521, 10558, \ldots).$$

Proposition 3.3 *For every $\varepsilon > 0$, the set $K_{2.33529\ldots+\varepsilon}$ is uncountable.*

Proof. The set of CPC's

$$\{S(\{123, 3214, 2143, 15432\} \cup V) : V \subset U\}$$

is uncountable, due to Lemma 3.1 and the 1-1 correspondence between CPC's and antichains of permutations, and

$$|S_n(\{123, 3214, 2143, 15432\} \cup V)| \leq |S_n(123, 3214, 2143, 15432)| = s_n.$$

By Proposition 3.2 we know that for any $\varepsilon > 0$, $s_n < (2.33529\ldots + \varepsilon)^n$ for every $n > n_0$. $\square$

Thus $\kappa \leq 2.33529\ldots$ and the proof of Theorem 1.1 is complete. More restrictions can be added to the $\{123, 3214, 2143, 15432\}$-avoidance and the bound $\kappa \leq 2.33529\ldots$ can be almost surely improved but the question is by how much. It seems not very likely that one could prove this way that $\kappa \leq 2$.

We conclude with some comments on our choice of the four permutations 123, 3214, 2143, and 15432. By the results in [5], if $(S(\pi, \rho), \prec)$ is not a wpo, where $\pi \in S_3$, $\rho \in S_4$ and $\pi \not\prec \rho$, then (π, ρ) equals, up to obvious symmetries, to $(123, 3214)$ or $(123, 2143)$. In [5] it is also observed that $S(123, 3214, 2143) \supset U$ and so $(S(123, 3214, 2143), \prec)$ is not a wpo. We have employed one more restriction: From the 28 permutations in $S_5(123, 3214, 2143)$, only 15432 is not contained in infinitely many μ_i. The enumeration $|S_n(123)| = C_n$, where C_n is the n-th Catalan number, is a classic result (see Stanley [17]); C_n have exponential growth 4^n. West [19] proved that $|S_n(123, 3214)| = |S_n(123, 2143)| = F_{2n}$ where $(F_n)_{n \geq 1} = (0, 1, 1, 2, 3, 5, 8, 13, \ldots)$ are Fibonacci numbers. F_{2n} grow as $((3+\sqrt{5})/2)^n = (2.61803\ldots)^n$. Using simpler arguments than those in the proof of Proposition 3.2, we can prove that the numbers $t_n = |S_n(123, 3214, 2143)|$ follow the recurrence $t_1 = 1$, $t_2 = 2$ and $t_n = 2t_{n-1} + t_{n-2}$ for $n \geq 3$. Thus t_n grow as $(1 + \sqrt{2})^n = (2.41421\ldots)^n$.

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