A note on the pancyclicity of lexicographic products

Tomáš Kaiser* and Matthias Kriesell†

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Abstract

We prove that if G and H are graphs containing at least one edge each, then their lexicographic product $G[H]$ is *weakly pancyclic*, i. e., it contains a cycle of every length between the length of a shortest cycle and that of a longest one. This supports some conjectures on locally connected graphs and on product graphs. We obtain an analogous result on even cycles in products $G[H]$ that are bipartite. We also investigate toughness conditions on G implying that $G[H]$ is hamiltonian (and hence pancyclic).

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*Department of Mathematics, University of West Bohemia, Univerzitní 8, 306 14 Plzeň, Czech Republic, and the Institute for Theoretical Computer Science (ITI), Charles University, Praha, Czech Republic. E-mail: kaisert@kma.zcu.cz. Supported by the project LN00A056 of the Czech Ministry of Education.

†Inst. f. Mathematik (A), Univ. Hannover, Welfengarten 1, D–30167 Hannover, Germany. E-mail: kriesell@math.uni-hannover.de.

1 Introduction

The *lexicographic product* $G[H]$ of two graphs G and H is obtained from G by replacing each vertex with a copy of H and adding all edges between any pair of copies corresponding to adjacent vertices of G. In this paper, we shall be concerned with cycles in $G[H]$. In particular, we shall show that if H contains at least one edge, then $G[H]$ is weakly pancyclic. (Recall that a graph is *weakly pancyclic* if it either is a forest, or it contains cycles of all lengths between the length of the shortest cycle and the length of the longest one.)

If G has no isolated vertices, then $G[H]$ is *locally connected* (every vertex has connected neighborhood). Thus, the above result supports a conjecture of RYJÁČEK [6] stating that every locally connected graph is weakly pancyclic.

Another related conjecture has been made by the second author [5]: the lexicographic product of a nontrivial 1-tough graph with the graph K_2 is hamiltonian. By our result, this would imply that such a product is, in fact, pancyclic.

We also consider products $G[H]$ for general graphs H. If G or H have no edges, we cannot expect $G[H]$ to be pancyclic. However, we show (Theorem 11) that if G is bipartite and contains at least one edge, then $G[H]$ always contains cycles of all even lengths between the length of the shortest cycle and that of the longest one.

In case H does have edges, one may ask whether $G[H]$ is pancyclic given G is 1-tough. We show that this is true under the stronger assumption that G is 4-tough.

We recall the relevant definitions as needed. For terms not defined here, see [1] or [2].

2 Strolls

All graphs considered here are finite, undirected, and contain no multiple edges although they may contain loops. The vertex and edge sets of a graph G are denoted by $V(G)$ and $E(G)$, respectively. Further, we let G^+ denote

G with a loop at every vertex. A *stroll* S in G is a walk in G^+; the stroll is *closed* if it is a closed walk. If each vertex is visited at most k times by S, then S is a *k-stroll*. If, moreover, no loop of $E(G^+) - E(G)$ appears in S more than t times, then S is a *(k,t)-stroll*. The length of a stroll is the total number of vertices in S, counting multiplicities. The stroll S is *spanning* if each vertex of G is visited by S. A closed spanning $(k,0)$-stroll is a *k-walk* in G. Closed and open 1-strolls are referred to as *cycles* and *paths*, respectively. More generally, if a stroll S contains no loops of G^+, then it is said to be *loopless*.

We use a sequence such as $x_0x_1\dots x_n$ to describe an open walk (or stroll) of *length* n through the vertices $x_0,\dots,x_n$. Similarly, $x_0x_1\dots x_nx_0$ denotes a closed walk of length $n+1$. Note also that the leading and the trailing x_0 in this closed walk count as one occurence only. Thus, for instance, the closed walk *abbaca* is a 2-stroll, but not a 1-stroll, as b (or a) is visited twice.

We shall need the following technical device. Assume that $\ell \geq 2$ and $S = x_0\dots x_n$ is a stroll of length at least $(\ell+1)/2$. We set

$$S^{(\ell)} = \begin{cases} x_0 \dots x_{\frac{\ell}{2}} x_{\frac{\ell}{2}-1} \dots x_0 & \text{if } \ell \text{ is even,} \\ x_0 \dots x_{\frac{\ell+1}{2}} x_{\frac{\ell+1}{2}} \dots x_1 & \text{if } \ell \text{ is odd.} \end{cases}$$

Observe that $S^{(\ell)}$ is a closed stroll of length ℓ. Moreover, if S is a $(k,0)$-stroll, then $S^{(\ell)}$ is a $(k,0)$-stroll if ℓ is even, and a $(k+1,1)$-stroll if ℓ is odd.

For brevity, we shall henceforth take every stroll to be closed, unless referred to as an *open* stroll. (There are just one or two occasions, in the proof of Lemma 1, where we need open strolls.)

Lemma 1 *Let a graph G have a (k,t)-stroll of length ℓ, where $\ell, k \geq 2$ and $t \geq 0$.*

- *(i) If $t \geq 1$, then G has a (k,t)-stroll of length $\ell - 1$.*
- *(ii) If $t = 0$ and G is bipartite, then G has a (k,t)-stroll of length $\ell - 2$.*

Proof. Let $S = x_0x_1\dots x_\ell$ be a (closed) (k,t)-stroll of length $\ell \geq 2$ in G. We prove part (i) first. We may assume S to be loopless, for removing a loop, we reduce the length to $\ell - 1$ as desired.

Suppose now that S is a 1-stroll. We have observed that $S^{(\ell-1)}$ is then a $(2,1)$-stroll of length $\ell-1$, and by our assumptions on k and t, these are the required properties.

Thus, some vertex is visited twice by S. We may assume that it is x_0 and choose $p < \ell$ such that $x_p = x_0$ and p is as large as possible. Then $C = x_p \dots x_\ell$ is clearly a (closed) 1-stroll of length at least 2. The k-stroll $S' := x_0 \dots x_p$ visits each vertex in C at most $k-1$ times. The length of S' is at most $\ell - 2$. We shall extend it to the desired stroll of length $\ell - 1$.

Consider the set Y of vertices visited by both C and S'. These vertices break C into a number of open 1-strolls as follows. For a vertex $y \in Y$, let $i \in \{p, \dots, \ell-1\}$ be the unique index such that $y = x_i$. Further, take the smallest j with $i < j \leq \ell$ and such that $x_j \in Y$. Let P_y denote the path $x_i \dots x_{j-1}$.

Similarly, the stroll S' is divided into pieces by Y. Enumerate the vertices in Y as $y_0, \dots, y_q$ (where $q = |Y| - 1$) in the order of their first appearances along S' as traversed from x_0 via x_1 on. (Note that this ordering of the y_j's need not match the ordering in which the vertices appear on C.) Clearly, $y_0 = x_0$. For each $i = 0, \dots, q$, let the open stroll S'_i be the portion of S' between the first appearance of y_i and the first appearance of y_{i+1} inclusive. We set $y_{q+1} = y_0$.

We now extend S' to the desired stroll S''. First note that if the length of the open 1-stroll P_{y_i} is denoted by L_i (where $0 \leq i \leq q$), then all the L_i sum up to $\ell - p$. Choose, for each $i = 0, \dots, q$, a non-negative integer $\ell_i \leq L_i$, in such a way that

$$\sum_{i=0}^{q} \ell_i = \ell - p - 1.$$

Such a choice can certainly be made.

Modify S' to S'' by replacing the first occurence of each $y_i \in Y$ by the closed $(2,1)$-stroll $P_{y_i}^{(\ell_i)}$ of length ℓ_i. Each such replacement increases the total length by ℓ_i, since a vertex is being replaced by $\ell_i + 1$ vertices (the identical starting and ending vertex of the inserted stroll count as two). Thus, the length of S'' is $\ell - 1$ as required. Finally, from our observation that no vertex of C is visited more than $k-1$ times by S', it follows that S'' is a k-stroll. Since S is loopless, and each $P_{y_i}^{(\ell_i)}$ is a $(2,1)$-stroll, S'' is a $(k,1)$-stroll. This concludes the proof of part (i).

Part (ii) is proved in a similar way. The stroll S is loopless now, and we can assume that it is not a 1-stroll, since otherwise $S^{(\ell-2)}$ is a 1-stroll (as ℓ is even) of length $\ell - 2$.

Assume that x_0 is visited twice by S. Define p, the 1-stroll C, the k-stroll S', and the set Y as in the proof of part (i). The difference is that we have to avoid using loops to extend S' to S''. Color the vertices from Y in two colors — black and white — in such a way that the colors alternate as we traverse C. Define the paths P_y as above, and let q_0 (resp., q_1) be the total number of vertices in the paths P_y, where $y \in Y$ ranges over black (resp., white) vertices. Since these paths cover all the $\ell - p$ vertices of C, we may assume that $q_0 \geq (\ell - p)/2$.

The stroll S'' is now defined as follows. For each black vertex $y_i \in Y$, choose an integer $\ell_i \leq L_i$ (where L_i is the length of P_{y_i}) in such a way that these numbers sum up to $(\ell - p)/2 - 1$. Such a choice is made possible by the fact that since G is bipartite and $t = 0$, both ℓ and p are even, so the fraction is integral. Define Q_{y_i} to be the path obtained by extending P_{y_i} one vertex further, to y_{i+1} (recall $y_{q+1} = y_0$). We need this extension in order to be able to obtain a closed stroll of length $2\ell_i$ from this path (see the assumption in the definition of $S^{(\ell)}$).

Replace the first occurence of each black y_i in S' by the $(2, 0)$-stroll $Q_{y_i}^{(2\ell_i)}$. As in part (i), each such replacement increases the total length by $2\ell_i$, and thus the length of S'' will be $p + (\ell - p) - 2 = \ell - 2$. Moreover, S'' is clearly a $(k, 0)$-stroll. The proof is finished. $\square$

The *lexicographic product* of graphs G and H is the graph $G[H]$ whose vertex set is $V(G[H]) = V(G) \times V(H)$ and whose edge set is defined by

$$(x, x')(y, y') \in E(G[H]) \quad \text{if} \quad xy \in E(G) \text{ and } (x = y \text{ or } x'y' \in E(H)).$$

There is the following interplay between strolls in G and cycles in $G[H]$.

Lemma 2 *Let G and H be simple graphs, and let $\ell \geq 3$. Assume that H has n vertices and m edges, and that a longest path in H has p vertices.*

If $G[H]$ has a cycle of length ℓ, then G has an (n, m)-stroll of length ℓ.

Conversely, if G has an (n,p)-stroll of length ℓ, then $G[H]$ has a cycle of length ℓ.

Proof. Let $(x_0, x'_0) \dots (x_{\ell-1}, x'_{\ell-1})(x_0, x'_0)$ be a cycle in $G[H]$. Clearly, every $x \in V(G)$ is the first component of at most n of the (x_i, x'_i), so $W := x_0 \cdots x_{\ell-1} x_0$ is an n-stroll in G'. Furthermore, $x_i = x_{i+1}$ only if $x'_i x'_{i+1} \in E(H)$. Therefore, W is an (n,m)-stroll.

Conversely, choose a path P of length p in H. If $W = x_0 \cdots x_{\ell-1}$ is an (n,p)-stroll in G, then one can find a function $f : \{0, \dots, \ell-1\} \longrightarrow V(P)$ such that $f(i) \neq f(j)$ whenever $x_i = x_j$ and $i \neq j$, and such that $f(i)$ and $f(i+1)$ are neighbors on P if $x_i = x_{i+1}$ (again, the addition is modulo ℓ). It follows that $(x_0, f(0)) \dots (x_{\ell-1}, f(\ell-1))(x_0, f(0))$ is a cycle in $G[H]$. $\square$

Note the following simple but important consequence of Lemma 2, observed already in [4]:

Observation 3 *The product $G[K_2]$ is hamiltonian if and only if G has a 2-walk.*

3 Pancyclicity

A combination of the above two lemmas leads us to a result on the cycle length spectrum of certain product graphs. Let G be a graph containing at least one cycle. Recall that the length of a shortest cycle in G is the *girth* $g(G)$ of G, and the length of the longest cycle is the *circumference* $c(G)$ of G.

The graph G is *weakly pancyclic* if it either has no cycles, or it contains cycles of each length ℓ such that $g(G) \leq \ell \leq c(G)$.

Theorem 4 *If G, H are graphs with at least one edge each, then $G[H]$ is weakly pancyclic.*

Proof. Let H have n vertices. As $G[H]$ may be assumed to contain a cycle, it contains one of length $c(G[H])$. Cycles of smaller lengths are constructed inductively as follows.

Suppose there exists a cycle $C = (x_0, x'_0)\dots(x_{\ell-1}, x'_{\ell-1})(x_0, x'_0)$ of length $\ell > 3$ in $G[H]$. If $x_0 = x_1 = \dots = x_{\ell-1}$ then we may assume that x_0 is not an isolated vertex in G. Taking a neighbor y of x_0 and an arbitrary vertex $y' \in V(H)$, we obtain a cycle $(x_0, x'_0)\dots(x_{\ell-3}x'_{\ell-3})(y, y')(x_0, x'_0)$ of length $\ell - 1$.

If there exists an $i \in \{0, \dots, \ell-1\}$ such that $x_i = x_{i+1}$ (with addition modulo ℓ), then we may choose it in such a way that $x_{i+2} \neq x_i$. After a cyclic relabelling of the vertices of C we may assume $i = \ell - 1$, and $(x_0, x'_0)\dots(x_{\ell-2}, x'_{\ell-2})(x_0, x'_0)$ is a cycle of length $\ell - 1$ in G.

We may thus assume that the x_i are pairwise distinct, so C is a cycle in $G[\bar{K}_n]$, too. By Lemma 2, there exists an $(n, 0)$-stroll of length ℓ in G. By Lemma 1, G has an $(n, 1)$-stroll of length $\ell - 1$, and by Lemma 2, $G[H]$ admits a cycle of length $\ell - 1$. $\square$

This supports a conjecture of Ryjáček on locally connected graphs. Recall that a simple graph G is *locally connected* if the set of neighbors of any vertex $x \in V(G)$ induces a connected subgraph of G.

Observing that $G[H]$ is locally connected for any graph G and any graph H without isolated vertices, we see that Theorem 4 supports the following conjecture.

Conjecture 5 ([6]) *Every locally connected graph is weakly pancyclic.*

Combining Lemma 2 and Theorem 4 with a result on the existence of 2-walks in sufficiently tough graphs, we can provide support for another conjecture, due to one of the authors.

Let us denote by $q(G)$ the number of components of G. Recall that a non-complete graph G is *t-tough* (where t is a real number), if for every $S \subseteq V(G)$, either $q(G - S) \leq 1$ or $t \cdot q(G - S) \leq |S|$. The following theorem has been proved in [3].

Theorem 6 ([3]) *Every 4-tough graph admits a 2-walk.*

Together with Observation 3, Theorem 6 implies that if G is 4-tough, then $G[K_2]$ is hamiltonian. Theorem 4 then shows that $G[K_2]$ contains cycles of each length from the girth $g(G[K_2])$ up to $2n$, the number of vertices of $G[K_2]$. Since, trivially, $g(G[K_2]) = 3$ unless $n = 1$, we infer that $G[K_2]$ is pancyclic. (Recall that a graph is *pancyclic* if it either is a forest or it contains cycles of all lengths from 3 up to the number of vertices in the graph.) In fact, a similar conclusion can be made for all products $G[H]$ provided that H contains edges.

Theorem 7 *If G is 4-tough and H contains at least one edge, then $G[H]$ is pancyclic.*

Proof. It suffices to prove that $G[H]$ is hamiltonian. By [3], the fact that G is 4-tough implies that G contains a spanning subgraph G' of maximum degree at most 3, such that G' contains a 2-factor F of G, and all the cycles of G' are cycles of F. (In the terminology of [3], this is an example of a *1-quasitree* in G.) We show that $G'[H]$ is hamiltonian; since G' is spanning in G, the statement of the theorem follows.

Enumerate all vertices of H as $y_1, \ldots, y_m$ in such a way that y_1y_2 is an edge of H. We proceed by induction on the number of cycles in G', and prove the following stronger statement:

> (*) The product $G'[H]$ has a Hamilton cycle containing the edge $(x, y_1)(x, y_2)$ for any x of degree 2 in G'.

If G' contains just one cycle (i.e., G' is a Hamilton cycle of G), say, $x_1 \ldots x_n x_1$, then the cycle

$$\begin{aligned} C = & (x_1, y_1)(x_1, y_2)(x_2, y_1)(x_2, y_2) \ldots (x_n, y_1)(x_n, y_2) \\ & (x_1, y_3)(x_2, y_3) \ldots (x_n, y_3)(x_1, y_4) \ldots (x_n, y_m)(x_1, y_1) \end{aligned}$$

satisfies the claim.

If G' contains more than one cycle, then it also contains a cut-edge $e = z_1z_2$. Let G'_1 and G'_2 be the components of $G' - e$. We may assume that each G'_i contains z_i. By the induction hypothesis, each $G'_i[H]$ has a Hamilton cycle C_i satisfying the condition of (*). In particular, since the degree of z_i in $G' - e$ is 2, it follows that $(z_i, y_1)(z_i, y_2)$ is an edge of C_i for $i = 1, 2$. We can now easily link C_1 and C_2 into a Hamilton cycle C of $G'[H]$ satisfying (*) by replacing the edges $(z_1, y_1)(z_1, y_2)$ and $(z_2, y_1)(z_2, y_2)$ with the edges $(z_1, y_1)(z_2, y_1)$ and $(z_1, y_2)(z_2, y_2)$. $\square$

One may wonder if the assumption that G is 4-tough in Theorem 7 can be replaced by the assumption that G has a 2-walk. After all, the latter suffices for $H = K_2$. In general, however, $G[H]$ need not be hamiltonian even if G admits a 2-walk and H contains edges. To see this, let G be the path on 3 vertices x_1, x_2, x_3 (in this order), and let H be the complete bipartite graph $K_{1,4}$. We show that $G[H]$ is not hamiltonian.

Assume, for contradiction, that C is a Hamilton cycle in $G[H]$. We write H_i ($i = 1, 2, 3$) for the copy of H corresponding to x_i in $G[H]$. Since H_1 cannot be covered by less than three vertex-disjoint paths, there are at least 6 edges of C with one endvertex in H_1 and the other endvertex in H_2. Similarly, C has at least 6 edges with one endvertex in H_3 and one in H_2. Thus, C has at least 12 edges with precisely one endvertex in H_2, a contradiction with the fact that H_2 has just 5 vertices, each of degree 2 in C.

It could be possible to improve the toughness bound in Theorem 7. In particular, the second author has made the following conjecture, related to the case $H = K_2$.

Conjecture 8 ([5]) *If G is 1-tough then $G[K_2]$ is hamiltonian.*

By Observation 3, Conjecture 8 is equivalent to the case $k = 2$ of the following long-standing conjecture on k-walks:

Conjecture 9 ([4]) *Every $\frac{1}{k-1}$-tough graph admits a k-walk.*

4 Products with $\bar{K}_p$

Theorem 4 asserts that $G[H]$ is weakly pancyclic if H contains some edges. This assumption is clearly necessary, for if H is the graph $\bar{K}_p$ consisting of p isolated vertices and G happens to be bipartite, then $G[H]$ is bipartite as well (and thus it is not weakly pancyclic in general). Moreover, $G[\bar{K}_p]$ may fail to be weakly pancyclic even if G is not bipartite. One example is the product $C_7[\bar{K}_2]$ (where C_7 is the 7-cycle): it contains cycles of length 4 and 6, but no cycle of length 5.

In this section, however, we show that something can still be said about cycles in $G[\bar{K}_p]$.

As observed in [4], the product $G[\bar{K}_p]$ is hamiltonian if and only if the graph G has an *exact* p*-walk*, that is, a p-walk visiting each vertex exactly p times. This is generalized in the following easy observation. (Note also that the observation is a special case of Lemma 2.)

Observation 10 *Let $\ell \geq 3$. A graph G has a (closed) $(p, 0)$-stroll of length ℓ if and only if $G[\bar{K}_p]$ has a cycle of length ℓ.* □

Lemma 1 now implies the following theorem.

Theorem 11 *For any bipartite graph G with at least one edge, the product $G[\bar{K}_p]$ contains cycles of all* even *lengths from 4 up to the circumference of $G[\bar{K}_p]$.*

Proof. Let C be a cycle in $G[\bar{K}_p]$ of even length $\ell > 4$. We show that there is a cycle of length $\ell - 2$. By Observation 10, G has a $(p, 0)$-stroll of length ℓ. Since G is bipartite, Lemma 1 implies that it has a $(p, 0)$-stroll of length $\ell - 2$. Hence, $G[\bar{K}_p]$ has a cycle of length $\ell - 2$ as desired. Since there clearly is a cycle of length 4 in $G[\bar{K}_p]$, the theorem follows. □

We remark that we have no example of a graph showing that Theorem 11 fails to hold for non-bipartite graphs.

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