The Subchromatic Index of Graphs

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Abstract

In an edge coloring of a graph, each color class forms a subgraph without path of length two (a matching). An edge subcoloring generalizes this concept: Each color class in an edge subcoloring forms a subgraph without path of length three. While every graph with maximum degree at most two is edge 2-subcolorable, we point out in this paper that recognizing edge 2-subcolorable graphs with maximum degree three is NP-complete, even when restricted to triangle-free graphs. As by-products, we obtain NP-completeness results for the star index and the subchromatic number for several classes of graphs. It is also proved that recognizing edge 3-subcolorable graphs is NP-complete.

Moreover, edge subcolorings and subchromatic index of various basic graph classes are studied. In particular, we show that every unicyclic graph is edge 3-subcolorable and edge 2-subcolorable unicyclic graphs have a simple structure, allowing an easy linear time recognition. We also present an algorithm for testing edge $k$-subcolorability for graphs of bounded treewidth.

1 Introduction

Let $G = (V, E)$ be a graph. An independent set (a clique) is a set of pairwise nonadjacent (adjacent) vertices. For $W \subseteq V$, the subgraph of $G$ induced by

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$W$ is denoted by $G[W]$. For $F \subseteq E$, $V(F)$ denotes the set of endvertices of edges from $F$, and $G(F) = (V(F), F)$ is the subgraph of $G$ induced by the edge set $F$.

A (proper) vertex $r$-coloring of $G$ is a partition $V_1, \ldots, V_r$ into disjoint independent sets, called color classes of the coloring. The chromatic number $\chi(G)$ is the smallest number $r$ for which $G$ admits a vertex $r$-coloring. One of the most interesting generalizations of the classical vertex coloring is the notion of vertex subcoloring; see [2, 9, 12, 15]. A vertex $r$-subcoloring is a partition $V_1, \ldots, V_r$ of $V$ where each color class $V_i$ consists of disjoint cliques (of various sizes). The smallest number $r$ for which $G$ has a vertex $r$-subcoloring is called the subchromatic number $\chi_s(G)$ of $G$.

Note that a partition $V_1, \ldots, V_r$ of $V$ is a vertex $r$-coloring of $G = (V, E)$ if and only if, for each $i$, $G[V_i]$ does not contain a $P_3$ as an (induced) subgraph, and the partition is a vertex $r$-subcoloring if and only if, for each $i$, $G[V_i]$ does not contain a $P_3$ as an induced subgraph ($P_k$ denotes the path on $k$ vertices).

A (proper) edge $r$-coloring is a partition $E_1, \ldots, E_r$ of $E$ into color classes $E_i$ in which every two distinct edges do not have an endvertex in common, i.e., each $E_i$ forms a matching. The chromatic index $\chi'(G)$ is the smallest number $r$ for which $G$ admits an edge $r$-coloring. Clearly, a partition $E_1, \ldots, E_r$ of $E$ is an edge $r$-coloring of $G = (V, E)$ if and only if, for each $i$, $G(E_i)$ does not contain a $P_3$ as a (not necessarily induced) subgraph. This observation leads to the following natural generalization of the classical edge coloring.

**Definition.** An edge $r$-subcoloring of the edges of a graph $G = (V, E)$ is a partition $E_1, \ldots, E_r$ of $E$ into disjoint color classes $E_i$ such that each $G(E_i)$ does not contain a $P_3$ as a (not necessarily induced) subgraph. The subchromatic index $\chi'_s(G)$ is the smallest number $r$ for which $G$ admits an edge $r$-subcoloring.

**Remark.** Obviously, a partition $E_1, \ldots, E_r$ of $E(G)$ is an edge $r$-coloring of $G$ if and only if, for each $i$, the connected components of $G(E_i)$ are stars or triangles, where a star is a complete bipartite graph $K_{1,s}$ for some $s \geq 1$.

A related notion that has been studied in the literature is as follows. A partition $E_1, \ldots, E_r$ of $E(G)$ is a star partition of $G$ if, for each $i$, the connected components of $G(E_i)$ are stars. The star index $\chi^*(G)$ of $G$ is the smallest number $r$ for which $G$ has a star partition into $r$ subsets $E_i$; cf. [1, 3, 4, 5, 11, 16, 19].

Clearly, for all graphs $G$, $\chi'_s(G) \leq \chi^*(G)$, and $\chi'_s(G) = \chi^*(G)$ whenever
$G$ is triangle-free.

Recall that the line graph $L(G)$ of a graph $G$ has the edges of $G$ as vertices and two distinct edges $e, e'$ are adjacent in $L(G)$ whenever they have an endvertex in common. It is well-known that proper edge colorings of $G$ correspond to proper vertex colorings of $L(G)$ and vice versa. In particular, $\chi'(G) = \chi(L(G))$. Likewise, we have the following easy to see fact.

**Fact 1** Edge subcolorings of a graph $G$ correspond to vertex subcolorings of the line graph $L(G)$ of $G$ and vice versa. In particular, $\chi'_s(G) = \chi_s(L(G))$.

Our terminology of edge subcoloring is intended to recall this fact.

## 2 Basic properties and examples

First, since the subchromatic index of a graph is the maximum subchromatic index among those of its connected components, so we assume throughout this paper that all graphs are connected.

The next proposition means that subcolorings provide monotone property.

**Proposition 1** For any graph $G$ and any vertex $v$ of $G$, \[\chi'_s(G) \leq \chi'_s(G \setminus v) + 1.\]

**Proof:** Since any subcoloring of $G \setminus v$ can be extended to a subcoloring of $G$ by using an extra new color on all edges incident with $v$. \qed

General lower and upper bounds for the subchromatic index are given below. Let $\Delta(G)$ be the maximum degree of a vertex in the graph $G$.

**Proposition 2** For any graph $G$ with $m$ edges on $n$ vertices, \[\frac{m}{n} \leq \chi'_s(G) \leq \Delta(G).\]

Moreover, if $G$ is triangle-free, $\chi'_s(G) \geq \frac{m}{n} + 1$.

**Proof:** Since every color class consists of stars and triangles, it may contain at most $n$ edges. In the subcoloring each edge has to be colored and the lower bound follows.
Note that a color class in a graph on \( n \) vertices can have \( n \) edges, iff \( n \) is a multiple of 3 and the class itself is a covering of the vertices by \( \frac{n}{3} \) disjoint triangles. Hence for triangle-free graphs the lower bound can be shifted by at least 1.

To obtain the upper bound, we have from Fact 1: \( \chi'_s(G) = \chi'_s(L(G)) \leq \left\lfloor \frac{\Delta(L(G))}{2} \right\rfloor + 1 = \left\lfloor \frac{2\Delta(G) - 2}{2} \right\rfloor + 1 = \Delta(G) \), where the inequality for the subchromatic number was shown in \([2]\).

To find a valid subcoloring using at most \( \Delta(G) \) edge colors efficiently we may proceed greedily on the vertex set: With each new vertex \( u \) assign colors to its adjacent edges as follows: for an edge \((u,v)\) pick a color that is not used on no already colored edge incident with \( v \). Such a subcoloring is triangle-free and all stars have the property that the center of the star is the latest vertex of the star in the order. \( \square \)

Observe that the upper bound is attained e.g. for the 5-cycle \( \chi'_s(C_5) = 2 = \Delta(C_5) \) or the Petersen graph \( \chi'_s(P) = 3 = \Delta(P) \). The last property follows for any cubic graph which contains \( C_5 \) as an induced subgraph: it is impossible to extend a valid 2-subcoloring to all edges incident to the cycle \( C_5 \). See also Figure 1.

**Corollary 1** For any \( r \)-regular graphs \( G \), \( \frac{r}{2} \leq \chi'_s(G) \leq r \). Moreover, if \( G \) is triangle-free, \( \frac{r}{2} + 1 \leq \chi'_s(G) \leq r \).

### 2.1 Trees, cycles

For trees and cycles, the subchromatic index can be determined explicitly as follows:

**Proposition 3**

(i) For any tree \( T \), \( \chi'_s(T) = \chi^*(G) \leq 2 \); \( \chi'_s(T) = 2 \) if and only if \( T \) is not a star;

(ii) \( \chi'_s(C_3) = 1 \), \( \chi'_s(C_n) = \chi^*(G) = 2 \) for all \( n \geq 4 \).

**Proof:** Color greedily. \( \square \)
2.2 Cacti

A cactus is a connected graph in which every block (maximal 2-connected subgraph) is an edge or a cycle. Equivalently, a graph $G$ is a cactus if and only if every two cycles in $G$ are edge-disjoint.

**Proposition 4** For all cacti $G$, $\chi'_s(G) \leq \chi^*(G) \leq 3$. Moreover, an edge 3-subcoloring can be found in linear time.

**Proof:** Let $T$ be a breadth-first search (bfs) tree of $G$, rooted at vertex $v$. We claim that

all edges of $G$ outside $T$ form a matching. \hfill (1)

During searching the graph, its vertices are be arranged into levels, based on the distance from the initial vertex. Since $G$ is a cactus, the tree $T$ misses from each odd cycle the edge connecting the two vertices at the highest level, while similarly in an even cycle one of the two edges incident with the vertex at the highest level remains outside $T$.

Then, given two edges $e, e'$ of $E(G) \setminus E(T)$, either $e$ is separated from $e'$ by the lowest vertex of the cycle containing $e$, or vice-versa.

Now, color $T$ with two colors and the matching $E(G) \setminus E(T)$ with the third color showing $\chi'_s(G) \leq 3$. Since a bfs-tree can be performed in linear time, Proposition 4 follows. \phantom{a}

We leave as an open problem whether cacti with subchromatic at most 2 by allow a simple structural description. Since all cacti have treewidth bounded by 2, their subchromatic index can be computed in polynomial time as we will show later in section 3.3.

Observe, that in any edge 2-subcolored graph $G$ each vertex of degree at least 3 is either a center of a monochromatic star or it belongs to a monochromatic triangle. Let us further direct the edges of monochromatic stars $K_{1,k}$, $k \geq 2$ towards its center. (The other edges remain undirected.) Clearly, no vertex of degree at least 3 in $G$ is of outdegree 2 or more. Also each directed cycle is of even length, since the colors of stars must alternate. It follows straightforwardly that each component of the subgraph of $G$ induced by vertices of degree at least 3 may only contain at most one cycle. (By the argument with directions, a little more discussion is needed to encompass also triangles.) Figure 1 shows some cacti $G$ that do not fulfill these necessary conditions and hence have $\chi'_s(G) = 3$.\hfill (2)
Moreover, if some vertices of degree at least 3 induce a cycle in a graph of subchromatic index 2, its subcoloring can be almost uniquely determined:

**Corollary 2** Let vertices of degree at least three induce a cycle $C$ in a graph $G$ of $\chi'_s(G) = 2$. Then either $C$ forms a monochromatic triangle in any edge 2-subcoloring of $G$, or $C$ is of even length and colors of its edges alternate.

### 2.3 Unicyclic graphs

A (connected) graph is *unicyclic* if it contains exactly one cycle. As we show now all edge 2-subcolorable unicyclic graphs have a simple structure and hence can be easily recognized in linear time.

**Theorem 1** For any unicyclic graph $G$, $\chi'_s(G) \leq 3$. Moreover, $\chi'_s(G) = 3$ if and only if the only cycle $C$ of $G$ has length $2k + 1$, $k \geq 2$ and all vertices of $C$ are of degree at least 3 in $G$.

**Proof:** Since unicyclic graphs are cacti, the first part follows from Proposition 4. We have shown above that if an odd cycle $C_{2k+1}$, $k \geq 2$ contains no vertex of degree 2, $G$ cannot be edge 2-subcolorable.

It suffices to construct an edge 2-subcoloring in all other cases. First consider the case when $C$ is a triangle. We make it monochromatic (say white) and then distribute all remaining edges of $G$ into two color classes, such that two edges belong to the same class if their distance from $C$ has the same parity modulo 2. (The edge-distance being viewed as the distance between the corresponding vertices in the line graph.)

Now assume that $C_{2k+1}$ contains a vertex of degree 2. Let denote the vertices of $C$ by $v_1,v_2,\ldots,v_{2k+1}$, where $\deg(v_1) = 2$. We use the white color
on the pair of edges incident with $v_1$ and color the other edges of the cycle
alternately by black and white.

The remaining edges form a forest. We color them in a greedy manner,
such that the edges incident with a vertex $v_i$, $i$ odd should be all white,
whereas for even $i$ these should be all black. Similarly as above, inside the
same tree of the forest edge colors alternate in between different levels.

For an even cycle $C$, we first use alternate coloring of $C$. Then we choose
colors of the remaining edges by the same parity principle as in the previous
case.

Observe first, that in both color classes, the edges outside $C$ induce a
disjoint union of stars. Moreover, this edge subcoloring is also valid if we
encounter the edges of $C$. \hfill \Box

Clearly, $C$ can be found in $G$ in linear time, Theorem 1 implies that
unicyclic graphs with subchromatic index at most 2 can be recognized in
linear time.

### 2.4 Complete bipartite graphs

Since complete bipartite graphs contain no triangle, subchromatic index co-
incides with the star chromatic index and we get the following results:

Proposition 5 ([11, 19]) For any $n \geq 1$,

$$
\chi'_s(K_{n,n}) = \chi^*(K_{n,n}) = \begin{cases} 
  n, & n \leq 4 \\
  4, & n = 5 \\
  \left\lceil \frac{n}{2} \right\rceil + 2, & n \geq 6
\end{cases}
$$

### 2.5 Cubes

Recall that the $d$-dimensional Cube $Q_d$ has all 0,1 $d$-tuples as vertices and
two such $d$-tuples are adjacent in $Q_d$ if and only if they differ in exactly one
position. Note that $Q_d$ is $d$-regular, bipartite and has $2^d$ vertices.

Proposition 6 ([19])

(i) For $k \geq 2$, $\chi'_s(Q_{2^k-2}) = \chi^*(Q_{2^k-2}) = 2^{k-1}$;

(ii) For $k \geq 3$, $2^{k-3} + 2 \leq \chi'_s(Q_{2^k-2}) = \chi^*(Q_{2^k-2}) \leq 2^{k-3} + k - 2$;
2.6 Complete graphs

**Proposition 7** For any $n \geq 1$, the subchromatic index of the complete graph $K_n$ is bounded by:

$$\frac{n-1}{2} \leq \chi'_s(K_n) \leq \chi^*(K_n) = \left\lceil \frac{n}{2} \right\rceil + 1.$$  

Moreover the lower bound is attained whenever $n = 3^k$ for some integer $k$.

**Proof:** The upper bound by $\chi^*(K_n) = \left\lceil \frac{n}{2} \right\rceil + 1$ was given in [1]. To prove the lower bound, we first construct optimal subcolorings for the case when $n$ is a $k$-th power of 3. We proceed by induction on $k$. Let the vertices of $K_n$ be denoted by $v_0, \ldots, v_{n-1}$.

For each $i \in \{1, \ldots, \frac{n-1}{2} : 3 \not| i\}$ we define a color class $E_i = \{(v_j, v_{j+i}), (v_j, v_{j-i}), (v_{j+i}, v_{j-i}) : j = 0, 3, 6, \ldots, n - 3\}$ where all indices are counted modulo $n$.

These $\frac{n}{3}$ color classes cover all edges by disjoint sets of packing triangles. The yet uncolored edges connect only vertices that are at distance divisible by 3 and induce three vertex disjoint copies of the graph $K_{n/3}$. By induction hypothesis each of them can be colored independently by $\frac{n/3 - 1}{2} = \frac{n-3}{6}$ colors. In total we get a subcoloring using $\frac{n}{3} + \frac{n-3}{6} = \frac{n-1}{2}$ distinct edge colors.

A valid subcoloring of $K_9$ is depicted in Fig. 2

![Figure 2: Showing $\chi'_s(K_9) = 4$](image-url)

This allows us to classify the subchromatic index for all complete graphs of order at most ten. We first show that $\chi'_s(K_6) = 4$. Assume for a contradiction that a valid 3-edge-subcoloring of $K_6$ exists. Then each color class may have at most 6 edges, hence at least two of these two classes must have at least 5 edges since $|E_{K_6}| = 15$. By a case study it is easy to determine that a color class with at least five edges may only induce in $K_6$ a subgraph of one of the possible three types: $K_3 \cup K_3$ or $K_3 \cup P_3$ or $K_{1,5} \cup K_1$. (Here $\cup$ stands for disjoint union.) But it is impossible to find a pair of not necessarily distinct types that would be edge disjoint, a contradiction.
We summarize the values of chromatic index of small complete graphs
the following table:

<table>
<thead>
<tr>
<th>Graph $G$</th>
<th>$K_1$</th>
<th>$K_2$</th>
<th>$K_3$</th>
<th>$K_4$</th>
<th>$K_5$</th>
<th>$K_6$</th>
<th>$K_7$</th>
<th>$K_8$</th>
<th>$K_9$</th>
<th>$K_{10}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi'_s(G)$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>

The values for $K_7$ and $K_8$ are majorized by $\chi'_s(K_9)$, the remaining values follow from the monotone property of $\chi'_s$ and other bounds which were discussed above.

3 Computational complexity

Formally we define Edge $k$-Subcolorability as a decision problem which for a given graph $G$ (the instance) answers the question: “Is $\chi'_s(G) \leq k$?”

In the following sections we will show, that the problem is NP-complete in general (obviously Edge $k$-Subcolorability $\in$ NP), but for a restricted class of graphs of bounded treewidth a linear time algorithm exists. This also imply that cacti of subchromatic index at most 2 can be recognized in linear time.

3.1 NP-hardness of edge 2-Subcoloring

In this section we prove the following negative result.

**Theorem 2** Edge 2-Subcolorability is NP-complete, even when restricted to triangle-free graphs of maximum degree three.

As all graphs with maximum degree at most 2 are edge 2-subcolorable, Theorem 2 is best possible with respect to degree constraint.

**Proof:** We prove Theorem 2 by showing a reduction from the Not-all-equal 3-Satisfiability (NAE-3SAT) problem, which has been shown to be NP-complete by Schaefer [18] (see also [14, Problem LO3]).

This problem decides whether a Boolean formula $\Phi$ in conjunctive normal form satisfying such that each clause is a disjunction of three not necessarily distinct variables allows a satisfying assignment for $\Phi$ such that each clause in $\Phi$ contains at least one negatively valued literal. We denote the class of all formulas that allow such an assignment by NAE-3SAT.
Let a formula $\Phi$ be an instance for the $\text{NAE-3SAT}$ problem. Assume that $\Phi$ consists of $m$ clauses $C_1, C_2, \ldots, C_m$ over variables $x_1, x_2, \ldots, x_n$ such that every clause $C_j$ contains exactly three variables, $C_j = (x_{j_1} \lor x_{j_2} \lor x_{j_3})$.

We will construct a triangle-free graph $H = H(\Phi)$ of maximum degree three such that $H$ has an edge 2-subcoloring if and only if $\Phi \in \text{NAE-3SAT}$.

**Clause gadget.** Consider the graph $G_C$ depicted in Figure 3 (left) with three labelled vertices $a, b, c$.

![Figure 3: Clause gadget $G_C$ (left) with an edge 2-subcoloring (right)](image)

**Fact 2**

(i) *The graph $G_C$ is edge 2-subcolorable. In any edge 2-subcoloring, the edges $(a, b), (b, c)$ receive different colors.*

(ii) *Every coloring of the two edges $(a, b), (b, c)$ with two distinct colors can be extended into an edge 2-subcoloring of the entire graph $G_C$, such that the two edges incident with the vertex $a$ and the two edges incident with $c$ receive different colors (cf. Figure 3 (right)).*

Fact 2 can be seen quickly by inspection, see also Corollary 2.

**Variable gadget.** Let $k \geq 2$ be an integer. Let $G_V^k$ be the graph depicted in Figure 4 with $k$ labelled vertices $a_1, \ldots, a_k$.

![Figure 4: Variable gadget $G_V^k$](image)
Fact 3 The graph $G_V^k$ is edge 2-subcolorable. In any edge 2-subcoloring, the thick edges receive the same color.

Again, Fact 3 can be seen easily by using Corollary 2.

Now, for the construction of $H$ we take the incidence graph $G_\Phi$. Vertices of $G_\Phi$ represent variables and clauses of $\Phi$, and edges represent the incidence relation. Since the same variable may appear in the same clause, multiple edges may appear in $G_\Phi$.

For the construction of $H$ we replace each clause-representing vertex $v_j$ of degree three with a unique copy of $G_C$, such that the three edges incident with $v$ are one-to-one incident with the vertices $a, b$ and $c$ (i.e. each edge chooses exactly one vertex).

Let further each vertex $u_i$ representing a variable $x_i$ with $k_i$ occurrences be replaced by a unique copy $G_i$ of $G_{V_i}^{k_i}$. Similarly as in the previous case, the $k_i$ edges incident with $u_i$ become one-to-one incident with the vertices $a_1, \ldots, a_{k_i}$.

The construction of $H$ is completely described. Since all gadgets have maximum degree three and no triangles, and since labelled vertices in the gadgets have degree one or two, $H$ has maximum degree three and no triangles, as well.

Suppose now that the edges of $H$ can be subcolored by two colors, say red and blue. For each $i$, define $\phi(x_i) = \text{true}$ if the thick edges of $G_i$ are red; and $\phi(x_i) = \text{false}$ otherwise. By Fact 3, $\phi$ is well defined. For each variable gadget, the edges leaving the gadget $G_i$ from vertices $a_1, \ldots, a_k$ must obtain the same color, complementary to the color of the thick edges of $G_i$. On the other side, the edges pendant from a clause gadget cannot have all the same color, since then a monochromatic $P_4$ would appear by Fact 2. Hence, in each clause at least one variable is positively valued and at least one is valued negatively by $\phi$ and $\Phi \in \text{NAE-3SAT}$.

In the opposite direction assume that $\Phi \in \text{NAE-3SAT}$ for an assignment $\phi$. We derive an edge 2-subcoloring of $H$ as follows. If $\phi(x_i) = \text{true (false)}$, color the thick edges of the variable gadget $G_i$ red (blue, respectively). Then extend this coloring into an edge 2-subcoloring of $G_i$. This is always possible by Fact 3. As was mentioned in the previous paragraph, then the edges stemming from the clause gadgets (i.e. the original edges of $G_\Phi$) allow a unique 2-subcoloring extension. Finally, complete the 2-subcoloring of $H$ on the clause gadgets according to Fact 2 (ii).

This argument completes the proof of Theorem 2. \[\square\]
**Corollaries.** Since for triangle-free graphs $G$, $\chi'_3(G) = \chi^*(G)$, Theorem 2 implies

**Corollary 3** Deciding if the star index of a given graph is two is NP-complete, even for triangle-free graphs with maximum degree three.

We remark that it was first proved in [16] that deciding if the star index of a triangle free graph is two is NP-complete. However, the graph constructed in [16] does not have bounded degree while our NP-complete result for the star index is best possible with respect to degree constraint.

In [12] it was shown that recognizing vertex 2-subcolorable graphs is NP-complete, even for triangle-free planar graphs with maximum degree 4. Theorem 2 and Fact 1 imply

**Corollary 4** Recognizing vertex 2-subcolorable graphs is NP-complete, even for line graphs (of triangle-free graphs) with maximum degree 4.

An $(r_1, \ldots, r_k)$-subcoloring of a graph $G = (V, E)$ is a partition $V_1, \ldots, V_k$ of $V$ such that each $V_i$ consists of disjoint cliques each of which has cardinality at most $r_i$. In [2] it was shown that all cubic graphs are $(2, 2)$-subcolorable, and in [17] it was shown that deciding if a cubic graph is $(1, 3)$-subcolorable is NP-complete. Theorem 2 and Fact 1 imply that a similar result holds if we restrict ourselves to line graphs.

**Corollary 5** Recognizing vertex $(3, 3)$-subcolorable graphs is NP-complete, even for line graphs (of triangle-free graphs) with maximum degree 4.

### 3.2 NP-hardness of edge 3-Subcoloring

This section deals with the proof that Edge 3-SUBCOLORING is NP-complete. Given a graph $G$ we construct a graph $H$ as follows. Take three copies $G_1, G_2, G_3$ of $G$, take a triangle $(v_1, v_2, v_3)$ and, for each $i = 1, 2, 3$, connect $v_i$ with all vertices in $G_i$. Finally, take two vertices $x, y$ and connect $x$ to $v_1, v_2, v_3$ and $y$. See Fig. 5.
Figure 5: The graph $H$ obtained from the copies $G_1, G_2, G_3$ of the given graph $G$

In the graph $H$, the set of edges between $v_i$ and all vertices of $G_i$ is called the $(v_i, G_i)$-star. We now point out that $\chi'_s(G) \leq 2$ if and only if $\chi'_s(H) \leq 3$.

Assume that $\chi'_s(G) \leq 2$ and consider an edge 2-subcoloring of $G$ with colors $c_1, c_2$. In $H$, color the three copies $G_i$ with this coloring and the $(v_i, G_i)$-stars with the third color $c_3$. Color the triangle $(v_1, v_2, v_3)$ with color $c_1$ and the 4-star at $x$ with color $c_2$. This yields an edge 3-subcoloring for $H$.

Assume that $\chi'_s(H) \leq 3$ and consider an edge 3-subcoloring of $H$ with colors $c_1, c_2, c_3$.

**Claim.** One of the $(v_i, G_i)$-stars, $i = 1, 2, 3$, is monochromatic.

**Proof of the Claim.** Assume that the claim is false. Then no two edges of the triangle $(v_1, v_2, v_3)$ have the same color; For, if $(v_1, v_2)$ and $(v_2, v_3)$ are colored with $c_1$, say, then the $(v_1, G_1)$-star and the $(v_3, G_3)$-star must be colored with $c_2$ and $c_3$. This implies that the edge $(v_1, v_3)$ is also colored with $c_1$ and the $(v_2, G_2)$-star is also colored with $c_2$ and $c_3$. Now, the edges $(x, v_i)$, $i = 1, 2, 3$, must have color $c_2$ or $c_3$ and there exists a $P_4$ with color $c_2$ or $c_3$, a contradiction.

Thus, let without loss of generality that $(v_1, v_2)$ is colored with $c_1$, $(v_2, v_3)$ with $c_2$ and $(v_1, v_3)$ with $c_3$. If $c_1$ does not appear in the $(v_3, G_3)$-star then, by assumption, the $(v_3, G_3)$-star is colored with $c_2$ and $c_3$. Hence $c_2$ cannot appear in the $(v_2, G_2)$-star; otherwise there is a $P_4$ colored with $c_2$. Therefore, the $(v_2, G_2)$-star is colored with $c_1$ and $c_3$, implying $c_1$ cannot appear in the $(v_1, G_1)$-star; otherwise there is a $P_4$ colored with $c_1$. Thus, the $(v_1, G_1)$-star is colored with $c_2$ and $c_3$. But then there exists a $P_4$ colored with $c_3$. This contradiction shows that $c_1$ must appear in the $(v_3, G_3)$-star, and by symmetry, $c_2$ must appear in the $(v_1, G_1)$-star and $c_3$ must appear in the $(v_2, G_2)$-star.

In particular, for each $v_i$, each color is appeared in the star at $v_i$ minus
the edge \((v_i, x)\).

Thus, the edge \((x, y)\) is colored differently with each of the edges \((v_i, x)\), \(i = 1, 2, 3\); otherwise there is a monochromatic \(P_4\). Now, as only two colors are available for the edges \((x, v_1), (x, v_2)\) and \((x, v_3)\), at least two of these edges must have the same color. By symmetry, let \((x, v_1)\) and \((x, v_2)\) have color \(c\), say. Then \(c = c_1\); otherwise there is a monochromatic \(P_4\) in the \(K_4\) induced by the \(v_i\) and \(x\). Hence both the \((v_1, G_1)\)-star and the \((v_2, G_2)\)-star are colored with \(c_2\) and \(c_3\). But, since \(c_2\) or \(c_3\) appears in the \((v_3, G_3)\)-star, there exists a monochromatic \(P_4\). This last contradiction proves the claim.

By the claim we may assume that the \((v_1, G_1)\)-star is colored with \(c_1\). If \(G_1\) has less than three vertices, clearly \(\chi'_4(G) \leq 2\). If \(G_1\) has at least three vertices, then \(c_1\) cannot appear in \(G_1\); otherwise there is a \(P_4\) colored with \(c_1\). Thus, the restriction of the edge 3-subcoloring of \(H\) on \(G_1\) is an edge 2-subcoloring for \(G\).

From the reduction above and Theorem 2 we obtain

**Theorem 3** Edge 3-Subcolorability is \(\text{NP-complete} \). □

### 3.3 Graphs of bounded treewidth

We note first, that for a fixed \(k\) the **Edge \(k\)-Subcolorability** problem can be expressed in Monadic Second Order Logic (MSOL), hence the existence of a linear-time algorithm for graphs of bounded treewidth [10].

Since this general method in inapplicable due to hidden constants, we follow the usual scheme for constructing linear-time algorithms for graphs of bounded treewidth, cf. [6, 7] and outline the main aspects of the dynamic programming algorithm.

A **nice tree decomposition** of width at most \(t\) of a graph \(G\) is a rooted tree \(T\), where nodes \(X_i\) of \(T\) represent subsets of vertices of \(G\) according to the following rules:

- For each edge \((u, v) \in E(G)\) there exists a node \(X_i\) such that \(\{u, v\} \subseteq X_i\).
- For each vertex \(u \in V(G)\) the nodes \(X_i\) containing \(u\) induce a connected subtree in \(T\).
- The size of each node \(|X_i| \leq t + 1\).
- Each node \(X_i\) has at most two children, and
it is called a leaf node if it has no children and $|X_i| = 1$,

- or $X_i$ has one child $X_j$, then $X_i$ is either an introduce node if $X_i = X_j \cup \{u\}$ for some vertex $u \in X_j$, or a forget node when $X_i = X_j \setminus \{u\}$ for some $u \in X_j$,

- or $X_i$ has two children $X_j, X_j'$, then it is called a join node and it holds that $X_i = X_j = X_j'$.

**Theorem 4** For any fixed $k$ and $t$, the Edge $k$-Subcoloring problem can be solved in linear time for graphs of treewidth at most $t$.

**Proof:** For the dynamic programming we compute with each node $X_i$ of $T$ a table $\text{Tab}_i$ of the following contents: Each entry $(\phi, r) \in \text{Tab}_i$ consists of an edge $k$-subcoloring $\phi$ of the graph $G_i$, the subgraph of $G$ induced by the vertex set $X_i$ and of a ranking $r : X_i \times \{1, \ldots, k\} \to \{0, 1, 2, 3, 4\}$ of the following meaning: For any edge $k$-subcoloring $\psi$ extending $\phi$ to the subgraph of $G$ induced by the union of descendants of $X_i$, we define

- $r(u, c) = 0$ if no edge of color $c$ in $\psi$ contains the vertex $u$.

- $r(u, c) = 1$ if $u$ is incident with exactly one edge $(u, v)$ of color $c$, and $v$ is incident to no other edge of color $c$ in $\psi$.

- $r(u, c) = 2$ if $u$ is incident with exactly one edge $(u, v)$ of color $c$, but $v$ is incident also with other edges of this color in $\psi$.

- $r(u, c) = 3$ if $u$ belongs to a monochromatic triangle of color $c$ in $\psi$.

- Finally $r(u, c) = 4$ if $u$ is the center of a monochromatic star $K_{1,k}$, $k \geq 2$ of color $c$ in $\psi$.

The evaluation of $\text{Tab}_i$ proceeds follows:

1. If $X_i$ is a leaf node, we let $\text{Tab}_i = (\emptyset, r(u, c) = 0)$, for $\{u\} = X_i$, $1 \leq c \leq k$.

2. If $X_i$ is a forget node with child $X_j$, then we store in $\text{Tab}_i$ all pairs $(\phi, r)$ where both $\phi$ and $r$ are restrictions to the subgraph induced by $X_i$ (or to the subgraph induced by $X_i$) of some $(\phi', r') \in \text{Tab}_j$. 
3. If $X_i$ is an introduce node with child $X_j$ and $\{u\} = X_i \setminus X_j$, we consider all entries $(\phi', r') \in X_j$, and all possible extensions $\phi$ of $\phi'$. Here a pair $(\phi, r)$ is feasible if for every color $c$

3a) either $u$ is incident with no edge of color $c$, then $r(u, c) = 0$,
3b) or $u$ is incident with only one edge $(u, v)$ of color $c$, then $r(u, c) = r(v, c) = 1$ if $r'(v, c) = 0$, or $r(u, c) = 2$ and $r(v, c) = 4$ if $r'(v, c) \in \{1, 4\}$,
3c) or $u$ is incident with two edges $(u, v), (u, w)$ of color $c$, where $\phi(v, w) = c$ and $r'(v, c) = r'(w, c) = 1$, then we let $r(u, c) = r(v, c) = r(w, c) = 3$,
3d) or finally $u$ is incident with $q \geq 2$ edges $(u, v_s)$ of color $c$, where $r'(v_s, c) = 0$ for $1 \leq s \leq q$. Then we let $r(u, c) = 4$ and $r(v_s, c) = 2$ for $1 \leq s \leq q$.

If not specified above we let $r(v, c) = r'(v, c)$ for all other $v \in X_j$, and store all feasible pairs $(\phi, r)$ in $\text{Tab}_i$.

4. If $X_i$ is a join node with children $X_j, X_j'$ we will keep in $\text{Tab}_i$ all pairs $(\phi, r)$ for which exists $(\phi', r') \in \text{Tab}_j$ and $(\phi'', r'') \in \text{Tab}_{j'}$ such that $\phi = \phi' = \phi''$ and moreover for each color $c$ and for every vertex $u \in X_i$ at least one of the following cases apply:

4a) either $r(u, c) = \max \{r'(u, c), r''(u, c)\}$, when $\min \{r'(u, c), r''(u, c)\} = 0$,
4b) or $r(u, c) = \max \{r'(u, c), r''(u, c)\}$ if there exists unique vertex $v \in X_i$, such that $(u, v)$ is of color $c$, and $r'(u, c), r''(u, c) \in \{1, 2\}$.
4c) or $r(u, c) = 3$ if
4ca) either there are $v, w \in X_i$ such that $u, v, w$ form a monochromatic triangle in $\phi$ (then $r(u, c) = r'(u, c) = \ldots = r''(w, c)$).
4cb) or there is a vertex $v \in X_i$ such that $\phi(u, v) = c$ and $r'(u, c) = r'(v, c) \neq r''(u, c) = r''(v, c)$ where $\{r'(u, c), r''(u, c)\} = \{1, 3\}$
4d) or finally $r(u, c) = 4$ if
4da) either $\max \{r'(u, c), r''(u, c)\} = 4$ and $\min \{r'(u, c), r''(u, c)\} \in \{1, 4\}$
4db) or $r'(u, c) = r''(u, c) = 1$ and no edge of color $c$ is incident with $u$ in $\phi$. 

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To argue the correctness of these rules consider any edge $k$-subcoloring $\psi$ of the subgraph induced by the subgraph of $G$ induced by the union of descendants of some node $X_i$. We show that for $\psi$ there exists a record $(\phi, r) \in \text{Tab}_i$, if the table entries were evaluated recursively according to the above rules. For leaf and forget nodes the statement is correct straightforwardly.

Let $X_i$ be an introduce node for the vertex $u$. If $u$ has no edge incident of color $c$ in $\psi$, we get $r(u, c) = 0$.

If in $\psi$ $u$ is incident exactly with one edge $(u, v)$ of color $c$, then the value of $r(u, c)$ depends, whether $v$ is incident with some other edges or not. If $v$ does not appear on $X_i$, we keep the value $r'(c, u)$ according the rule 3a). Else, the edge $(u, v)$ is colored also by $\phi$ and the value of $r(u, c)$ depends whether $v$ has been before incident with an edge of color $c$ by the rule 3b). Note that in this case $v$ cannot be a member of a monochromatic triangle nor a leaf of a monochromatic star of color $c$.

If in $\psi$ the vertex $u$ become a part of a monochromatic triangle $u, v, w$, then $(v, w)$ is the only edge of color $c$ in $\psi$ incident with $v$ (or $w$ resp.). This is the only case how a monochromatic triangle may appear and is captured by the case 3c).

Finally the case 3d) shows how a monochromatic star may appear with a new vertex, clearly all $q$ leaves of this star cannot be incident with any other edge of the same color.

Assume now that $X_i$ is a join node. We again distinguish five cases according to the presence of edges colored in $\psi$ by a color $c$ around a vertex $u$. If there is no such an edge, we get $r(u, c) = 0$ and the same must hold also by on the children nodes (hence rule 4a).

If $u$ is incident with exactly one edge $(u, v)$ of color $c$, and this is also the only edge of color $c$ incident with $v$, either $v \in X_i$ and the case 4b) apply or $v \not\in X_i$ and we follow 4a). (Here $v$ appears either in the subtree rooted either in $X_j$, or in the subtree below $X_j$, but not in both. Clearly $\phi$ cannot contain any other edge incident with $u$ of color $c$.)

Similarly, if the edge $(u, v)$ belongs to a monochromatic star of color $c$ in $\psi$, either the center $v$ and also no other edge of the star appears on $X_i$ — case 4a), or $v$ belongs to $X_i$ and the star also appears in some of the two children of $X_i$ — case 4b).

If $u$ is a member of a triangle in $\psi$, then either the entire triangle appears in only one subtrees below $X_j$ or $X_{j'}$ — case 4a), or it completely lies in $X_i$ — case 4ca) or possibly only one edge $(u, v)$ of the triangle appears in $X_i$. 

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This last case is captured in 4cb), here the edge \((u, v)\) must be recognized and the only edge of color \(c\) incident with \(u\) and \(v\) in one of the subtrees, while in the other subtree we must find the remaining vertex of the triangle.

Finally, if \(u\) is a center of a monochromatic star of color \(c\), either the star is completely placed in one of the subtrees and case 4a) apply. Or the star appears as a union of a star with another star of with a new edge — case 4da) or the star is formed out of two edges, each coming from different subtrees below \(X_j\) and \(X_{j'}\); and we get the case 4db).

The dynamic programming algorithm evaluates the tables \(\text{Tab}_r\) in a bottom-up manner. An edge \(k\)-subcoloring of the entire graph \(G\) exists if and only if the table \(\text{Tab}_r\) for the root node \(X_r\) is nonempty. For each node, the table may contain at most \(k^{O(t^2)} \cdot 5^{(t+1)k}\) entries, each of length \(O(t^2 \log k + kt)\). Both these values are bounded by a constant, since the treewidth \(t\) and the number of colors \(k\) are also bounded by a constant.

The evaluation of each table can be performed in time depending on \(k\) and \(t\) only hence the entire complexity of the dynamic programming algorithm depends only on the number of nodes of the nice tree decomposition \(T\). As it is mentioned in [7] a nice decomposition of width at most \(t\) containing at most \(O(|V(G)|)\) nodes exists for any graph of treewidth at most \(t\), and can be found in linear time [8] \(\square\)

If we restrict the rankings \(r\) only to values \(\{0, 1, 2, 4\}\), the dynamic programming will check for the existence of a edge \(k\)-subcoloring without monochromatic triangles, i.e. for a star partition with at most \(k\) subsets. Hence we can conclude that

**Corollary 6** For any fixed parameters \(k\) and \(t\), the test whether \(\chi^*(G) \leq k\) can be performed in time linear in \(|V(G)|\) for any graph of treewidth at most \(t\).

### 4 Conclusion

The concept of edge subcoloring of graphs is introduced for the first time in this paper, motivated by the study of vertex subcolorings.

Among many interesting open questions we pose the following.

1. What is the exact value for \(\chi'_s(K_n)\)?
   (Known for \(n \leq 10\); see section 2)
2. What is the computational complexity of \textsc{edge 2-subcolorability} for planar graphs?
(Note that for all planar graphs $G$, $\chi'_s(G) \leq \chi^*(G) \leq 5$; cf. [16]. Moreover the complexity of finding the ordinary chromatic index is not yet determined for planar graphs. It is widely expected to be a nontrivial problem since already the fact that any bridgeless cubic planar graph has chromatic index $3$ is equivalent to the four color theorem, see e.g. [13].)

3. What is the computational complexity of \textsc{edge k-subcolorability} for fixed $k \geq 4$?
(We have proved that \textsc{edge 2-subcolorability} is \textsc{NP}-complete for triangle-free graphs with maximum degree $3$; see Theorem 2 and that \textsc{edge 3-subcolorability} is \textsc{NP}-complete; see Theorem 3)

\textbf{References}


