The Last Excluded Case of Dirac's Map-Color Theorem for Choosability

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Abstract

In 1890, Heawood established the upper bound $H(\varepsilon) = \left\lfloor \frac{7+\sqrt{24\varepsilon+1}}{2} \right\rfloor$ on the chromatic number of every graph embedded on a surface of Euler genus $\varepsilon \geq 1$. Almost 80 years later, the bound was shown to be tight by Ringel and Youngs. These two results has became known under the name of the Map-Color Theorem. In 1956, Dirac refined this by showing that the upper bound $H(\varepsilon)$ is obtained only if a graph G contains $K_{H(\varepsilon)}$ as a subgraph with except of three surfaces. Albertson and Hutchinson settled these excluded cases in 1979. This result is nowadays known as Dirac's Map-Color Theorem.

Böhme, Mohar and Stiebitz extended Dirac's Map-Color Theorem to the case of choosability by showing that G is $(H(\varepsilon)-1)$-choosable unless G contains $K_{H(\varepsilon)}$ as a subgraph for $\varepsilon \geq 1$ and $\varepsilon \neq 3$. In the present paper, we settle the excluded case of $\varepsilon = 3$.

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1 Introduction

We study list colorings of graphs embedded on surfaces without boundary. Such surfaces are *orientable surfaces* Σ_g, the sphere with g handles, and *non-orientable surfaces* Π_h, the sphere with h cross-caps. The surface Π_1 is the projective plane, Π_2 is the Klein bottle, Σ_1 is the torus, etc. The *Euler genus* ε of the surface Σ_g is $2g$ and the Euler genus of the surface Π_h is h. The *Euler genus* of a graph is the smallest Euler genus of a surface on which the graph can be embedded.

Euler's formula for a graph G embedded on a surface of Euler genus ε states that $n - m + f \geq 2 - \varepsilon$ where n, m and f is the number of vertices, edges and faces of G, respectively. Moreover, the equality holds if and only if G is connected and every face of the embedding is a 2-cell. Therefore, the number of edges of an n-vertex simple graph G which can be embedded on the surface of Euler genus ε is at most $3n - 6 + 3\varepsilon$. For $\varepsilon \geq 1$, this implies that every graph embedded on a surface of Euler genus ε contains a vertex of degree at most $H(\varepsilon) - 1$ where $H(\varepsilon)$ is a so-called Heawood number defined as follows:

$$H(\varepsilon) = \left\lfloor \frac{7 + \sqrt{24\varepsilon + 1}}{2} \right\rfloor .$$

Hence, the chromatic number of a graph embedded on a surface of Euler genus ε is at most $H(\varepsilon)$. Let us remind that the chromatic number $\chi(G)$ of a graph G is the least number of colors needed to color the vertices of G so that no two adjacent vertices receive the same color. The above bound was conjectured to be tight by Heawood [15]. Indeed, Ringel [22] and Ringel and Youngs [24] showed that it is possible to embed the complete graph $K_{H(\varepsilon)}$ on each surface of Euler genus ε with an exception of the Klein bottle Π_2. This result became known as the Map-Color Theorem.

In 1956, Dirac extended the Map-Color Theorem by showing that the chromatic number of a graph G embedded on a surface of Euler genus $\varepsilon \geq 1$, $\varepsilon \neq 3$, is equal to $H(\varepsilon)$ if and only if G contains $K_{H(\varepsilon)}$ as a subgraph. Almost 25 years later, Albertson and Hutchinson [1] completed the three missing cases. Thus, we have the following theorem which is nowadays known under the name of Dirac's Map-Color Theorem:

Theorem 1 *Let G be a graph embedded on a surface of Euler genus $\varepsilon \geq 1$. If G does not contains $K_{H(\varepsilon)}$ as a subgraph, the chromatic number of G is at most $H(\varepsilon) - 1$.*

The chromatic number of a graph embedded on the Klein bottle Π_2 is at most six and there are 6-chromatic graphs which can be embedded on Π_2 and which do not contain K_6 as a subgraph [1, 13]. Let us remark that complete graphs $K_{H(\varepsilon)-1}$ (and in some cases also the join of the graphs $K_{H(\varepsilon)-4}$ and C_5) are the only critical $(H(\varepsilon)-1)$-colorable graphs embeddable on a surface of Euler genus ε [25]. We refer the reader for a more detailed introduction to embeddings of graphs on surfaces to [14, 21].

In this paper, we focus on list colorings of graphs embedded on surfaces. A *list assignment* is a function L which assigns each vertex $v \in V(G)$ a list $L(v)$ of available colors. For a given graph G and a given list assignment L, a coloring c of the vertices of G is called an *L-coloring* if $c(v) \in L(v)$ for every vertex $v \in V(G)$. If the size of the list $L(v)$ for every vertex $v \in V(G)$ is k, the list assignment is said to be a *list k-assignment*. The *choice number*, sometimes called also the *list chromatic number*, of a graph G is the smallest integer k such that the graph G can be colored from the lists of any list k-assignment. Such a graph G is said to be *k-choosable*. The choice number of a graph is clearly at least its chromatic number but the inequality might be strict. See the surveys [19, 27] for more details on this concept.

As in the case of the chromatic number, the choice number of a graph G embedded on a surface of Euler genus $\varepsilon \geq 1$ is at most $H(\varepsilon)$. The following extension of Theorem 1 was proved by Böhme, Mohar and Stiebitz [2]:

Theorem 2 *If G is a graph embedded on a surface of Euler genus $\varepsilon \geq 1$, $\varepsilon \neq 3$, then the choice number of G is at most $H(\varepsilon)$ and the equality holds if and only if G contains $K_{H(\varepsilon)}$ as a subgraph.*

As in the case of ordinary colorings, the cases $\varepsilon = 0, 1, 3$ turned out to need a special approach than the others. In the case of planar graphs, Thomassen [26] proved that the choice number of each planar graph is at most five and Voigt [28] constructed non-4-choosable planar graphs. The case of the projective plane required to be handled separately in [2] and the case of the surface Π_3 was left open. In this paper, we show that Theorem 1 holds also for the surface Π_3 (see Theorem 31). This completes the excluded case of Theorem 1. We remark that our result already found an application in coloring face hypergraphs of graphs embedded on the surface Π_3 [10].

We follow a standard graph theoretic notation (the reader is welcomed to see [6, 29] for missing definitions). Let us recall some less common notation which we use. If G is a graph and W is a subset of its vertices, then $G[W]$ denotes the subgraph of G induced by the vertices of W. Graphs which we consider need not to be simple graphs unless explicitly stated that they are simple, i.e., some vertices can be joined by parallel edges. Hence, we distinguish the *degree* $\deg(v)$ of a vertex v which is the number of edges incident with v and the *simple degree* of a vertex v which is the number of distinct vertices adjacent to v.

2 List colorings of graphs

In our considerations, we often work with list assignments in which the sizes of the lists are not the same but they are related to degrees of the vertices of a graph. So-called Gallai trees play a prominent role in this setting. A connected graph is said to be a *Gallai tree* if each of its blocks is a complete graph or an odd cycle. Let us remind that a *block* B of a graph G is its maximal 2-connected subgraph. A vertex of a block is said to be an *internal* vertex of B if it is not a cut vertex, i.e., B is the only block of G which contains it. A *Gallai forest* is a graph whose all components are Gallai trees. The following two theorems were independently proved by Borodin [3] and Erdős, Rubin and Taylor [11]:

Theorem 3 *Let G be a connected graph with a list assignment L. If $|L(v)| \geq \deg_G(v)$ for every vertex v of G and the inequality is strict for at least one vertex of G, then G has an L-coloring.*

Theorem 4 *Let G be a connected graph with a list assignment L such that $|L(v)| = \deg_G(v)$ for every vertex v. If G does not have an L-coloring, then G is a Gallai tree. Moreover, if G is 2-connected and it does not have an L-coloring, then the lists $L(v)$ of all the vertices v of G are the same.*

We remark that Theorems 3 and 4 have been extended to generalized colorings with respect to hereditary properties [4, 5], to list colorings of hypergraphs [17], the channel assignment problem [18, 20] and the list T-coloring [12].

A graph G is said to be *critical non-k-choosable* if it is not k-choosable and each proper subgraph of G is k-choosable. Note that such a graph G must have minimum degree at least k. We now state an extension of Dirac's inequality for the number of edges in color critical graphs [8] to list colorings which was proved by Kostochka and Stiebitz [16]:

Theorem 5 *If $G \neq K_{k+1}$ is a critical non-k-choosable graph of order n, then the number of edges of G is at least $(kn + k - 2)/2$.*

At the end of this section, we prove three specific lemmas for list assignments with only two kinds of lists which we later apply in Section 6:

Lemma 6 *Let G be a 6-colorable graph with a 6-list assignment L. Suppose that there exist two lists L_1 and L_2 such that the list $L(v)$ of every vertex $v \in V(G)$ is L_1 or L_2. Then, G has an L-coloring.*

Proof: Let $\alpha_1, \ldots, \alpha_6$ be the colors of the list L_1 and $\beta_1, \ldots, \beta_6$ the colors of the list L_2. Let k be further the number of colors which the lists L_1 and L_2 have in common. We may assume that $\alpha_1 = \beta_1, \ldots, \alpha_k = \beta_k$. Fix now a 6-coloring c_0 of G using the numbers $1, \ldots, 6$ as colors. We define an L-coloring c of G based on the coloring c_0:

$$c(v) = \begin{cases} \alpha_{c_0(v)} & \text{if } L(v) = L_1, \\ \beta_{c_0(v)} & \text{otherwise.} \end{cases}$$

Clearly, $c(v) \in L(v)$ for every vertex $v \in V(G)$. If two adjacent vertices were assigned the same color, then they would be colored with the same number by c_0. Hence, c is a desired L-coloring.

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The proof of Lemma 6 can be easily altered to a proof of each of the next two lemmas:

Lemma 7 *Let G be a 6-colorable graph with a 6-list assignment L. Suppose that there exist two lists L_1 and L_2 such that the list $L(v)$ of every vertex $v \in V(G)$ is L_1 or L_2. Let v_0 be a vertex of G and let $\gamma \in L(v_0)$. Then, G has an L-coloring c with $c(v_0) = \gamma$.*

Proof: Let us keep the notation used in the proof of Lemma 6 and assume without loss of generality that $\gamma \in L_1$ and $\gamma = \alpha_i$. Consider now a 6-coloring $c_0 : V(G) \to \{1, \ldots, 6\}$ of G with $c_0(v) = i$ and proceed as in the proof of Lemma 6.

■

In the previous lemma, we found a list coloring which assigns a prescribed color to a particular vertex. In the next lemma, we find a list coloring which avoids assigning a prescribed color to some of the vertices:

Lemma 8 *Let G be a 6-colorable graph with a 6-list assignment L. Suppose that there exist two lists L_1 and L_2 such that the list $L(v)$ of every vertex $v \in V(G)$ is L_1 or L_2. Let V_0 be a set consisting of at most five vertices of G and let γ be an arbitrary color. Then, G has an L-coloring c with $c(v) \neq \gamma$ for every $v \in V_0$.*

Proof: Let us keep the notation used in the proof of Lemma 6 and assume that $\gamma \in L_1$ and $\gamma = \alpha_i$. Similarly as in the proof of the previous lemma, consider first a 6-coloring $c_0 : V(G) \to \{1, \ldots, 6\}$ of G with $c_0(v) \neq i$ for all (at most five) vertices $v \in V_0$ and proceed next as in the proof of Lemma 6.

■

Let us remark that Lemmas 6–8 can be straightforwardly generalized to k-colorable graphs with k-list assignments.

3 Minimal non-6-choosable graphs on Π_3

In this section, we show that regarding our main result, Theorem 31, we can restrict our attention to triangulations of the surface Π_3 with minimum simple degree six and establish several properties which such triangulations do have. From now on, we allow triangulations to have parallel edges but we always forbid bigon faces. A *bigon* is a face whose boundary is formed by a single cycle of length two. If G is a triangulation with minimum simple degree six, its vertex v is said to be *small* if $\deg_G(v) = 6$ and it is called *big* otherwise. Note that no small vertex can be incident with two parallel edges; we use this fact later in the proofs without explicitly mentioning it. In a triangulation, we refer to the faces containing a vertex v as to the *neighborhood* of v. Subwalks of the boundary walk of the only non-triangular face of $G \setminus v$ are called *segments.*

Lemma 9 *Suppose that there is a non-6-choosable graph of Euler genus three which does not contain K_7 as a subgraph. Then, there exists a non-6-choosable triangulation of the surface Π_3 with minimum simple degree six which does not contain K_7 as a subgraph such that its small vertices induce a Gallai forest.*

Proof: Let G be a critical non-6-choosable graph of Euler genus at most three which does not contain K_7 as a subgraph such that the order n of G

is the smallest possible. In particular, G is a simple graph with minimum degree (at least) six. By Theorem 2, the graph G cannot be embedded on a surface of Euler genus two or less. Fix now an embedding of G on the surface Π_3. By Theorem 5, the graph G contains at least $3n + 2$ edges. By Euler's formula, the number of edges of an n-vertex simple graph embedded on the surface Π_3 is at most $3n + 3$ and the equality holds if and only if the graph is a triangulation. Note that this implies that the minimum degree of G is actually six.

In what follows, we first construct a non-6-choosable triangulation G' of Π_3 from the graph G. If the number of edges of G is $3n + 3$, the graph G itself is a triangulation of Π_3 and we set $G' = G$. In the rest, we deal with the case that the number of edges of G is $3n + 2$. Since the graph G cannot be embedded on a surface of Euler genus two or less, each face of the embedding of G on Π_3 is a 2-cell [30]. In addition, all the faces of the embedding of G are triangles except for a single quadrangular face by Euler's formula. Let $abcd$ be the 4-cycle which bounds the quadrangular face. Since G is a simple graph with minimum degree six, all the four vertices a, b, c and d are distinct. We now prove the following claim:

Claim 9.1 *Let $G + ac$ and $G + bd$ be the graphs obtained from G by adding edges ac and bd, respectively, to the interior of the face $abcd$. Then, $G + ac$ or $G + bd$ does not contain K_7 as a subgraph.*

Suppose that the claim is false. Let W_{ac} be the set of the vertices of a subgraph of $G + ac$ isomorphic to K_7 and W_{bd} the set of the vertices of a subgraph of $G + bd$ isomorphic to K_7. Since G does not contain K_7 as a subgraph, the vertices a and c must be contained in W_{ac} and the vertices b and d in W_{bd}. In addition, G contains neither an edge ac nor an edge bd. Otherwise, K_7 would be a subgraph of G. Finally, let $W = W_{ac} \cup W_{bd}$.

Let k denote the number of vertices contained in both the sets W_{ac} and W_{bd}, i.e., $k = |W_{ac} \cap W_{bd}|$. Observe that $|W| = |W_{ac} \cup W_{bd}| = 14 - k$ because each of the sets W_{ac} and W_{bd} contains exactly seven vertices. Consider the embedding of the graph $G[W]$ on Π_3 induced by the embedding of the graph G. Since this embedding of $G[W]$ contains at least one non-triangular face, namely the face $abcd$, the number of edges of $G[W]$ is at most $3(14-k)+2 = 44 - 3k$ by Euler's formula. In addition, the equality holds if and only if all the faces except for the face $abcd$ are triangular.

The number of edges of each of the graphs $G[W_{ac}]$ and $G[W_{bd}]$ is 20 because both of them are isomorphic to the graph K_7 without a single edge.

The number of edges of the graph $G[W]$ is thus at least $40 - m'$ where m' is the number of edges of the graph $G[W_{ac} \cap W_{bd}]$. Clearly, $m' \leq \binom{k}{2}$. Hence, the graph $G[W]$ contains at least $40 - \binom{k}{2}$ edges. This leads to an immediate contradiction for $k = 2, 3, 4, 5$. Thus, it remains to consider the cases $k = 0, 1, 6, 7$. We consider them separately.

If $k = 0$, then the sets W_{ac} and W_{bd} are disjoint. Hence, the edge set of the graph $G[W]$ consists of 40 edges of the graphs $G[W_{ac}]$ and $G[W_{bd}]$ and at least additional four edges forming the 4-cycle $abcd$. As noted above, the graph G cannot have more than 44 edges. Therefore, $G[W]$ has exactly 44 edges and each edge of $G[W]$ is either an edge of the 4-cycle $abcd$ or it is contained in one of the subgraphs $G[W_{ac}]$ and $G[W_{bd}]$. In addition, all the faces except for the face bounded by the 4-cycle $abcd$ are triangular. Consider now the face abv of $G[W]$ incident with the edge ab. Since the vertices a and c are not adjacent, we have $v \neq c$. The edge av must be contained in the subgraph $G[W_{ac}]$, and hence $v \in W_{ac}$. Similarly, we conclude that $v \in W_{bd}$. But this is impossible since the sets W_{ac} and W_{bd} are disjoint.

If $k = 1$, then at least two edges of the cycle $abcd$ are contained neither in the graph $G[W_{ac}]$ nor in the graph $G[W_{bd}]$. Hence, $G[W]$ contains at least 42 edges, namely 40 edges of the graphs $G[W_{ac}]$ and $G[W_{bd}]$, and at least two additional edges of the cycle $abcd$. But this is impossible because $G[W]$ can contain at most $44 - 3 \cdot 1 = 41$ edges by Euler's formula.

If $k = 7$, then $W_{ac} = W_{bd}$. Since the graph $G + ac$ contains a clique on the vertex set W_{ac} and $b, d \in W_{ac}$, we infer that the vertices b and d are joined by an edge in the graph G which we already argued not to be the case.

Let us consider the final case that $k = 6$. Since $|W_{ac} \cap W_{bd}| = 6$, at least one of the vertices b and d is contained in the set W_{ac}. If both b and d are contained in the set W_{ac}, then they are adjacent in G which is not the case. Similarly, the set W_{bd} contains precisely one of the vertices a and c. Since $G[W_{ac}] + ac$ and $G[W_{bd}] + bd$ are cliques, all the eight vertices of $G[W]$ are mutually adjacent except for the two pairs of vertices a, c and b, d. Insert now the edge ac inside the face bounded by the 4-cycle $abcd$. In this way, we obtain an embedding of K_8^- (the complete graph K_8 without an edge) on the surface Π_3 but Ringel [23] showed that such an embedding does not exist.

We excluded all the cases $k = 0, \ldots, 7$. Thus, $G + ac$ or $G + bd$ does not contain K_7 as a subgraph and so Claim 9.1 is established.

If G is a triangulation, we set G' to be the triangulation G itself. Otherwise, let G' be one of the triangulations $G + ac$ or $G + bd$ which does not contain K_7 as subgraph (at least one of them has this property by Claim 9.1). Note that G' may have a pair of parallel edges. The triangulation G' has obviously minimum simple degree at least six and it follows from Euler's formula that it is precisely six.

In the rest, we show that the small vertices of G' induce a Gallai forest. Assume the opposite. We show that G' is 6-choosable which contradicts the fact that G, which is a subgraph of G', is not 6-choosable. Fix a list 6-assignment L of G. Let H be a component of the subgraph induced by small vertices of G' which is not a Gallai tree. Since G is a critical non-6-choosable graph, the graph $G \setminus V(H)$ is 6-choosable. Color now its vertices by colors from the lists L. In particular, all the big vertices of G' are colored. For every $v \in V(H)$, let $L'(v)$ be a subset of $L(v)$ with the colors assigned to the big neighbors of v being removed. Since H consists solely of the small vertices, the size of a list $L'(v)$ is at least $\deg_H(v)$. By Theorem 4, the graph H has an L'-coloring. The L-coloring of the vertices of $G \setminus V(H)$ and the L'-coloring of H form an L-coloring of G — contradiction.

■

In the next lemma, we show that a triangulation of Π_3 with minimum simple degree six can contain only few big vertices:

Lemma 10 *If G is a triangulation with minimum simple degree six of the surface Π_3, then G contains at most six big vertices. In particular, each big vertex is adjacent to at least one small vertex. Moreover, if G contains precisely six big vertices, then the degree of each big vertex is seven.*

Proof: Let n be the number of vertices of the graph G. By Euler's formula, the number of edges of G is precisely $3n+3$. Hence, the sum of degrees of the vertices of G is precisely $6n+6$. Therefore, the triangulation G can contain at most six big vertices (recall that the minimum simple degree of G is six). In particular, each big vertex is adjacent to a small vertex. If G contains six big vertices, then the degree of each big vertex is seven.

■

In the following three lemmas, we study more specific properties of triangulations with minimum simple degree six:

Lemma 11 *Let G be a triangulation with minimum simple degree six of the surface Π_3. Suppose that G contains a small vertex v which is adjacent only* *to big vertices. Then, the graph G contains exactly six big vertices and each big vertex has degree seven.*

Proof: Since the minimum simple degree of G is six, all the six big neighbors of v must be distinct. The rest of the statement of the lemma now readily follows from Lemma 10.

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Lemma 12 *Let G be a triangulation with minimum simple degree six of the surface Π_3. Suppose that the small vertices of G induce a Gallai forest F. Let v be a big vertex of G and let $w_1w_2w_3$ be a segment contained in its neighborhood. If w_1 and w_3 are big vertices and w_2 is a small vertex, then the component H of F which contains the vertex w_2 is not isomorphic to K_5.*

Proof: Since the vertex w_2 of H is adjacent to three distinct big vertices in the triangulation G (recall that the simple degree of w_2 is six), namely the vertices v, w_1 and w_3, its degree in H is at most three. Hence, the Gallai tree H cannot be a clique of order five.

■

Lemma 13 *Let G be a triangulation with minimum simple degree six of the surface Π_3 which does not contain K_7 as a subgraph. If the small vertices induce a Gallai forest F in G, then each small vertex is adjacent to at least two big vertices. In particular, the maximum degree of F is at most four and no component of F is isomorphic to K_6.*

Proof: Let v be an arbitrary small vertex contained in a component H of F. By the assumption, H is a Gallai tree. If v is adjacent only to small vertices, then the vertex v and all its six small neighbors must be in the same block of H. Hence, the Gallai tree H must contain a clique of order seven. But this is impossible because G does not contain K_7 as a subgraph. Hence, each small vertex has at least one big neighbor.

Assume now for the sake of contradiction that v has a single big neighbor v'. In particular, $\deg_H(v) = 5$. Since G is a triangulation, then the vertex v is in the same block of H as its five small neighbors. Because each small vertex is adjacent to at least one big vertex, the maximum degree of H is at most five and H is 2-connected. Hence, the vertex v and its five neighbors

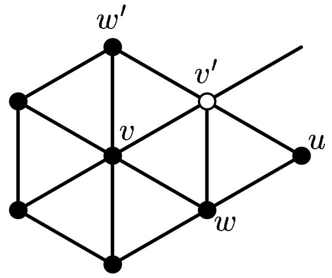

Figure 1: Notation used in the proof of Lemma 13.

are the only vertices of H and so H is isomorphic to K_6. In particular, each vertex of H is adjacent to exactly one big vertex and thus there are exactly six edges between the vertices of H and the big vertices of G.

Let w and w' be the neighbors of v such that the triangulation G contains the faces $vv'w$ and $vv'w'$ (cf. Figure 1). Each big vertex which is adjacent to a vertex of H must be adjacent to at least three vertices of H since G is a triangulation and each vertex of H is adjacent to exactly one big vertex. Hence, either there is a single big vertex adjacent to all the vertices of H, which is isomorphic to K_6, or there are two big vertices, each having exactly three neighbors in H. Since G does not contain K_7 as a subgraph, the former is impossible. Thus, the latter holds and v, w and w' are the only neighbors of the big vertex v' in H.

Let u be a common neighbor of w and v' different from v so that the triangulation G contains a face $v'wu$ (cf. Figure 1). Since the only neighbors of v' in H are the vertices v, w and w', we conclude that u is a big vertex. However, then the vertex w of H has two big neighbors u and v and so its degree in H is at most four — contradiction.

■

In the last lemma of this section, we show that if each component induced by small vertices in a triangulation consists of at most five vertices, then the minimum degree of a subgraph induced by the big vertices is at least two:

Lemma 14 *Let G be a triangulation with minimum simple degree six of the surface Π_3. If each component of the subgraph of G induced by the small vertices of G consists of at most five vertices, then each big vertex of G is incident with at least two edges joining it to other big vertices.*

Proof: Suppose that the claim is false, i.e., there is a big vertex v of G incident with at most one edge leading to another big vertex. Let k be the number of distinct small vertices adjacent to v. Since no small vertex is incident with parallel edges, $k \geq 6$. All the k small neighbors of v are contained in the same component H of the subgraph of G induced by the small vertices because G is a triangulation. Hence, the number of vertices of H is also at least six which contradicts the assumption of the lemma.

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4 Gallai trees in triangulations with minimum simple degree six

In the previous section, we have observed that we can restrict our attention to triangulations of the surface Π_3 in which small vertices induce a Gallai forest with maximum vertex degree at most four. In this section, we define a weight and an extended weight of a Gallai tree with maximum degree at most four. This concept is used in the next section to show that the small vertices of triangulations can induce only Gallai forests of a restricted type. This concept is defined and it can be used for all surfaces.

Fix a triangulation G of a surface such that the minimum simple degree of G is six. Let H be a component of the subgraph of G induced by the small vertices. Suppose that H is a Gallai tree. The *weight of H in the triangulation G*, denoted by $w_G(H)$, is equal to $|\partial_G H|$ where $\partial_G H$ is the set of edges between the vertices of H and the rest of G. Notation $\partial_G H$ is also used for other subgraphs H of G. It is easy to observe that the following equality holds:

$$w_G(H) = \sum_{v \in V(H)} (6 - \deg_H(v)). \tag{1}$$

In particular, the weight of the Gallai tree H does not depend on a considered triangulation G. Thus, we can define the *weight $w(H)$ of H*, independently of a triangulation G, as the sum in (1).

We say that a face f of a triangulation is *big* if exactly one vertex of f is small, i.e., exactly two vertices of f are big. The *extended weight $w_G^+(H)$ of H in the triangulation G* is equal to the weight of H increased by the number of big faces containing a vertex of H. Note that the extended weight of the Gallai tree H could depend on the triangulation G. We now define *the*

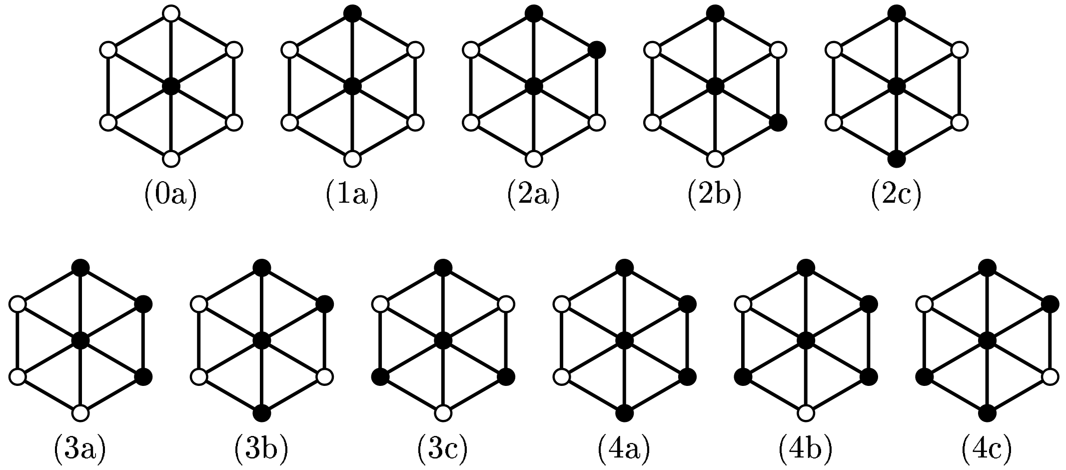

Figure 2: Possible neighborhoods (upto symmetry) of a small vertex of degree zero, one, two, three and four. The small vertices are depicted by full circles and the big vertices by empty ones. The types of neighborhoods are labeled by pairs consisting of the number of its small neighbors and a letter.

extended weight of H, denoted by $w^+(H)$, to be the minimum of the extended weights $w^+_G(H)$ for all triangulations G with minimum simple degree six which contain H and do not contain K_7 as a subgraph.

Next, we establish some lower bounds on the extended weight of Gallai trees with maximum vertex degree at most four. All our lower bounds will just depend on the structure of a Gallai tree, i.e., neither of them will be related to a considered triangulation. Our first lower bound (which will be later improved) is presented in the following proposition:

Proposition 15 *Let G be a triangulation with minimum simple degree six and let H be a component of the subgraph of G induced by the small vertices. Suppose that H is a Gallai tree with maximum degree at most four. The extended weight of H in G is at least $w^\oplus_1(H)$ where*

$$w^\oplus_1(H) := w(H) + \sum_{v \in V(H)} d^+_{\deg_H(v)}$$

with $d^+_0 = 6$, $d^+_1 = 4$, $d^+_2 = 2$ and $d^+_3 = d^+_4 = 0$.

Proof: Possible neighborhoods of a small vertex of degree zero, one, two, three and four are depicted in Figure 2. It is easy to verify that a vertex v of a Gallai tree H with $\deg_H(v) = k$ must be contained in at least d^+_k big faces.

The statement of the proposition now readily follows from the definition of the extended weight.

■

The label of a neighborhood in Figure 2 is said to be the *type* of the neighborhood of a small vertex v. This notion is to be used in the proof of the next proposition in which we improve the lower bound from Proposition 15 by realizing that certain types of neighborhoods cannot appear next to each other:

Proposition 16 *Let G be a triangulation with minimum vertex simple degree six and let H be a component of the subgraph induced by the small vertices. Suppose that H is a Gallai tree with maximum degree at most four. Then, the extended weight of H is at least $w_2^\oplus(H)$ where*

$$w_2^\oplus(H) := w_1^\oplus(H) + 2\ell_3^1 + \ell_3^2 + 4\ell_4^1 + 2\ell_4^2$$

with ℓ_j^i being the number of blocks of H which are cliques of order j, which contain precisely i cut-vertices in H and at least one of these cut-vertices has degree exactly four in H.

Proof: Let $d_0^+ = 6$, $d_1^+ = 4$, $d_2^+ = 2$ and $d_3^+ = d_4^+ = 0$ as defined in Proposition 15. Consider a fixed block B of the Gallai tree H which is a clique of order j, which contains exactly i cut-vertices and such that at least one cut-vertex v of B has degree four in H. The statement of the proposition is implied by the definitions of $w^+(H)$ and $w_1^\oplus(H)$ and by Claims 16.1–16.4 which follow:

Claim 16.1 *If $i = 1$ and $j = 3$, then the internal vertices of the block B are contained in at least $2d_2^+ + 2$ big faces.*

Since v is a cut-vertex and B is a clique of order four, the neighborhood of v must of type (4c). Then, the neighborhood of each of the remaining two vertices of B is of type (2a). Hence, the internal vertices of B are contained in at least $6 = 2d_2^+ + 2$ big faces.

Claim 16.2 *If $i = 2$ and $j = 3$, then the internal vertex of the block B is contained in at least $d_2^+ + 1$ big faces.*

The type of the neighborhood of v must again be (4c). Then, the neighborhood of the only internal vertex of B is of type (2a) and it is contained in exactly $3 = d_2^+ + 1$ big faces.

Claim 16.3 *If $i = 1$ and $j = 4$, then the internal vertices of the block B are contained in at least $3d_3^+ + 4$ big faces.*

Since v is a cut-vertex, its neighborhood must be of type (4b). Then, the neighborhood of none of the remaining three vertices of B is of type (3c) and at least one is of type (3a). Hence, the internal vertices of B are contained in at least $4 = 3d_3^+ + 4$ big faces.

Claim 16.4 *If $i = 2$ and $j = 4$, then the internal vertices of the block B are contained in at least $2d_3^+ + 2$ big faces.*

The type of the neighborhood of v must again be (4b). Then, the neighborhoods of the two internal vertices of B can be only of types (3a) and (3b). Hence, the internal vertices of B are contained in at least $2 = 2d_3^+ + 2$ big faces.

■

Finally, we define $w^\oplus(H)$ for a Gallai tree H with maximum degree at most four as follows:

$$w^\oplus(H) = \begin{cases} 16 & \text{if } H = K_4, \\ 13 & \text{if } H = K_5 \text{ and} \\ w_2^\oplus(H) & \text{otherwise.} \end{cases}$$

The weights $w(H)$ and the bounds $w^\oplus(H)$ of all Gallai trees H with maximum degree at most four and with $w^\oplus(H) \leq 32$ can be found in Figure 3 (it is straightforward to verify that all Gallai trees with this property are depicted in the figure; we avoid this verification in order to keep the paper short). In the next lemma, we show that $w^\oplus(H)$ is a lower bound on the extended weight of a Gallai tree H:

Lemma 17 *For each Gallai tree H with maximum degree at most four, the following inequality holds:*

$$w^\oplus(H) \leq w^+(H).$$

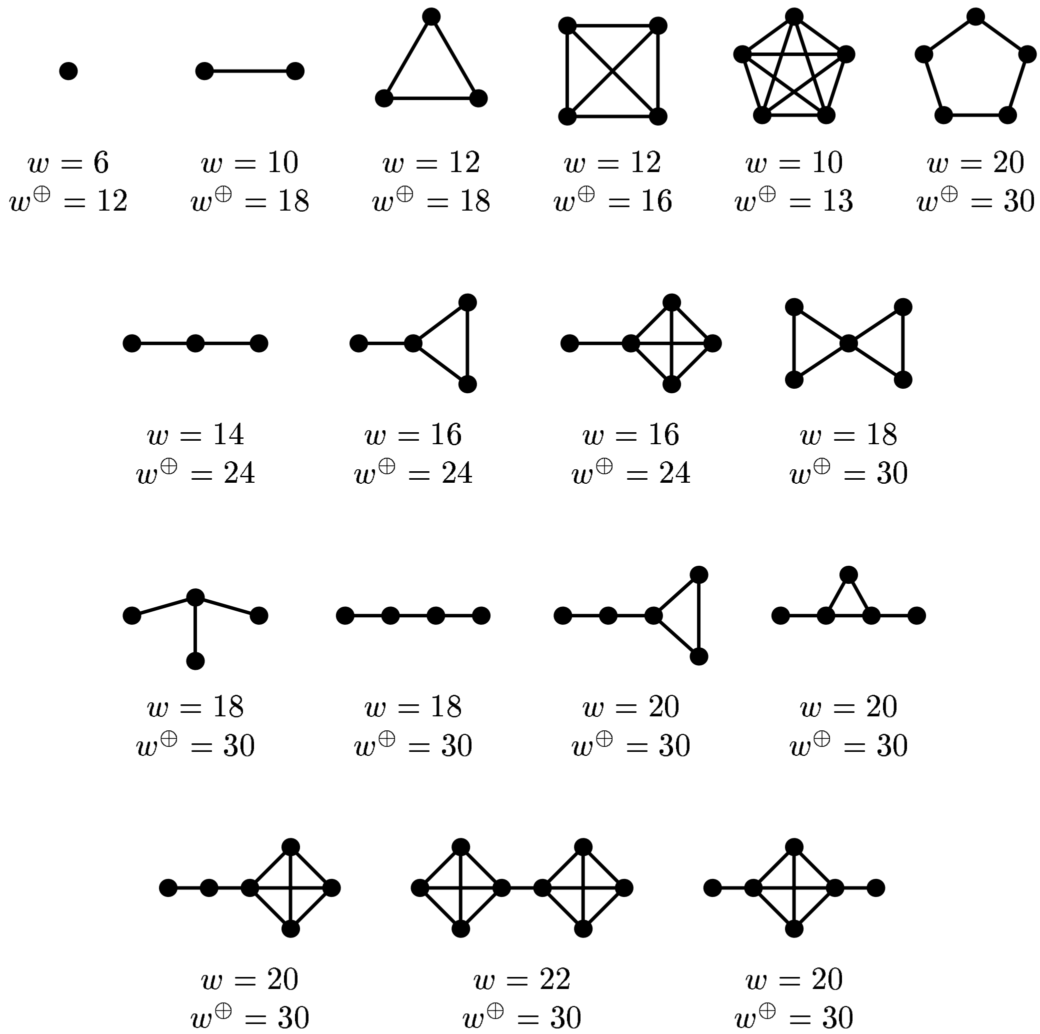

Figure 3: The weights $w(H)$ and the values of the bound $w^{\oplus}(H)$ of all Gallai trees H with maximum degree at most four and with $w^{\oplus}(H) \leq 32$.

Proof: Let G be an arbitrary triangulation with minimum simple vertex degree six which does not contain K_7 as a subgraph such that one of the components of the subgraph of G induced by the small vertices is isomorphic to H. We show that $w^{\oplus}(H) \leq w_G^{+}(H)$ which implies the statement of the lemma.

If the Gallai tree H is isomorphic to neither K_4 nor K_5, then $w^{\oplus}(H) = w_2^{\oplus}(H) \leq w_G^{+}(H)$ by Proposition 16. So, we can assume that H is a clique of order four or five. Let n be this order. Consider the (embedded) graph G' obtained from the triangulation G by removing the vertices of H and let f be the face of G' in which H was embedded. Note that since G is a triangulation, the face f is uniquely determined. The degree of each big vertex of G can be decreased by at most n because the minimum simple degree of G is six and only n small vertices of G were removed. The face f is incident with at least $7-n$ big vertices because each vertex of H is adjacent to six distinct vertices. If the face f is incident with precisely $7-n$ big vertices, then the n vertices of H are adjacent to the same $7-n$ big vertices and they altogether form a copy of K_7 in G. Therefore, f is incident with at least $8-n$ big vertices. Hence, the sum of the lengths of all the facial walks of f is at least $8-n$. Each edge of a facial walk of f is contained in a big face of G which contains a small vertex of H (recall that G is a triangulation of the surface and we removed only some of its small vertices). We can now conclude that $w_G^{+}(H) \geq w_G(H) + (8-n) = w(H) + 8 - n$. In particular, if $n = 4$, then $w_G^{+}(H) \geq 12 + 8 - 4 = 16$ and if $n = 5$, then $w_G^{+}(H) \geq 10 + 8 - 5 = 13$.

■

In the next lemma, we describe a relation between the number of big vertices and the weights and the extended weights of the components of a Gallai forest induced by small vertices:

Lemma 18 *Let G be a triangulation with minimum simple degree six of the surface Π_3 such that G does not contain K_7 as a subgraph. Suppose that the small vertices of G induce a Gallai forest F with maximum degree at most four which consists of k components $H_1, \ldots, H_k$. If b is the number of big vertices of G, then $w^{\oplus}(H_1) + \cdots + w^{\oplus}(H_k) \leq 6b + 6$. In addition, if each component H_i, $1 \leq i \leq k$, contains at most five vertices, then $w(H_1) + \cdots + w(H_k) \leq 4b + 6$.*

Proof: Let m_S be $|\partial_G F|$, i.e., the number of edges between the big vertices and the small vertices, and let m_B be the number of edges between the big vertices. By Euler's formula, the sum of degrees of the big vertices of G is $m_S + 2m_B = 6b + 6$.

The sum of $|\partial_G F|$ and the number of big faces of G is exactly $w^+_G(H_1) + \cdots + w^+_G(H_k)$, i.e., the sum of extended weights of the Gallai trees $H_1, \ldots, H_k$ in G. By Lemma 17, this sum is at least $w^\oplus(H_1) + \cdots + w^\oplus(H_k)$. On the other hand, the number of big faces is at most $2m_B$ because each big face is incident with an edge joining two big vertices and an edge joining two big vertices can be incident with at most two big faces. Therefore:

$$w^\oplus(H_1) + \cdots + w^\oplus(H_k) \leq w^+_G(H_1) + \cdots + w^+_G(H_k) \leq m_S + 2m_B = 6b + 6.$$

In order to prove the second part of the claim, assume that each Gallai tree H_i, $1 \leq i \leq k$, contains at most five vertices. Each big vertex is incident with at least two edges joining it to other big vertices by Lemma 14. Consequently, the number $|\partial_G F|$ of edges between the small vertices and the big vertices is at most $6b+6-2b = 4b+6$. Since $|\partial_G F|$ is equal to the sum of the weights of the Gallai trees $H_1, \ldots, H_k$, we conclude that $w(H_1)+\cdots+w(H_k) \leq 4b+6$.

■

5 Triangulations of the surface Π_3

As we have already noted, we can restrict our attention to triangulations of the surface Π_3 with minimum simple degree six in which the small vertices induce a Gallai forest. In this section, we study a possible structure of such triangulations and their Gallai forests.

Lemma 19 *Let G be a triangulation with minimum simple degree six of the surface Π_3 which does not contain K_7 as a subgraph. Suppose that the small vertices induce a Gallai forest F in G. Then, F has at most three components.*

Proof: By Lemma 13, the maximum degree of of F is at most four. Note that $w^\oplus(H) \geq 12$ for each component H of F (cf. Figure 3). The triangulation G contains at most six big vertices by Lemma 10. Therefore, we can infer from Lemma 18 that the sum of $w^\oplus(H)$ for all components H of F is at most 42. Hence, F can have at most three components.

■

In the next lemma, we describe a structure of Gallai forests with two components in triangulations which we are interested in:

Lemma 20 *Let G be a triangulation with minimum simple degree six of the surface Π_3 which does not contain K_7 as a subgraph. Suppose that the small vertices induce a Gallai forest F in G with two components H_1 and H_2. Then, at least one of the following holds:*

- *H_1 or H_2 is isomorphic to K_1,*
- *both H_1 and H_2 are cliques of order between two and five, or*
- *H_1 is a clique of order between two and five, H_2 contains a vertex of degree one (or vice versa) and G has precisely six big vertices.*

Proof: Each of the Gallai trees H_1 and H_2 has maximum degree at most four by Lemma 13. If H_1 or H_2 is isomorphic to K_1, then the lemma clearly holds. Let us assume in the rest that neither H_1 nor H_2 is isomorphic to K_1. Note that by Lemma 10 there are at most six big vertices. Hence, Lemma 18 implies the inequality $w^{\oplus}(H_1) + w^{\oplus}(H_2) \leq 42$. Since the extended weight of a Gallai tree with maximum degree at most four which is not a clique is at least 24 (cf. Figure 3), we conclude that at least one of H_1 and H_2 is a clique.

If both H_1 and H_2 are cliques, the forest F is of the desired form. Hence, assume that H_1 is a clique but H_2 is not. Since H_1 is a clique of order 2, 3, 4 or 5, we infer that $w^{\oplus}(H_1) \geq 13$ and hence $w^{\oplus}(H_2) \leq 29$. Thus, $w^{\oplus}(H_2) = 24$ and so H_2 must contain a vertex of degree one (cf. Figure 3). Since $w^{\oplus}(H_1)+w^{\oplus}(H_2) \geq 37$, there are exactly six big vertices by Lemmas 10 and 18. This completes the proof of the lemma.

■

Finally, we show that if a Gallai forest induced by the small vertices has exactly three components, then at least two of them are isomorphic to K_1:

Lemma 21 *Let G be a triangulation with minimum simple degree six of the surface Π_3 which does not contain K_7 as a subgraph. Suppose that the small vertices induce a Gallai forest F in G with three components H_1, H_2 and H_3 and that G contains a vertex with a simple degree at least seven. Then, at least two of H_1, H_2 and H_3 are isomorphic to K_1.*

Proof: The maximum degree of each of the Gallai trees H_1, H_2 and H_3 is at most four by Lemma 13 and thus each of $w^\oplus(H_1)$, $w^\oplus(H_2)$ and $w^\oplus(H_3)$ is at least twelve (cf. Figure 3). Since the number of big vertices of G is at most six by Lemma 10, we can infer from Lemma 18 that the sum of $w^\oplus(H_1)$, $w^\oplus(H_2)$ and $w^\oplus(H_3)$ can be at most 42. Therefore, each of $w^\oplus(H_1)$, $w^\oplus(H_2)$ and $w^\oplus(H_3)$ is at most $42 - 2 \cdot 12 = 18$. Hence, all the Gallai trees H_1, H_2 and H_3 must be cliques of order at most five (cf. Figure 3).

In the rest of the proof, we show in a series of claims that at least two of the cliques H_1, H_2 and H_3 are of order one. This will establish the lemma.

Claim 21.1 *If none of the cliques H_1, H_2 and H_3 is of order one, then all of them have order five.*

Assume that the order of each of the cliques H_1, H_2 and H_3 is distinct from one. Note first that the weights of the cliques K_2 and K_5 are equal to 10 and the weights of the cliques K_3 and K_4 are equal to 12. Since each of H_1, H_2 and H_3 has at most five vertices, the sum of their weights can be at most 30 by Lemmas 10 and 18. Hence, each of the cliques H_1, H_2 and H_3 is isomorphic to K_2 or K_5. Recall that $w^\oplus(K_2) = 18$ and $w^\oplus(K_5) = 13$. If at least one of the cliques is of order two, then the sum $w^\oplus(H_1) + w^\oplus(H_2) + w^\oplus(H_3)$ is at least $18 + 13 + 13 = 44 > 42$ — contradiction. Hence, all the cliques H_1, H_2 and H_3 are isomorphic to K_5.

Claim 21.2 *The order of at least one of the cliques H_1, H_2 and H_3 is not five.*

Assume for contradiction that orders of all the cliques H_1, H_2 and H_3 are five. Then, $w(H_1) + w(H_2) + w(H_3) = 30$. In other words, $|\partial_G F| = 30$. By Lemmas 10 and 18, the graph G contains exactly six big vertices and thus the degree of each big vertex is seven. Let further H_B be the subgraph of G induced by the big vertices. By Lemma 14, the degree of each vertex in H_B is at least two, i.e., each big vertex is adjacent to at most five small vertices. Since the number of edges between the small and big vertices is 30, each big vertex is adjacent to exactly five small vertices and thus the multigraph H_B is 2-regular, i.e., H_B is a union of cycles.

Since G is a triangulation and the small vertices induce a Gallai forest with three components, the embedding of H_B on Π_3 obtained from the triangulation G by removing the cliques H_1, H_2 and H_3 has at least three faces, namely the faces which originally contained embeddings of H_1, H_2 and H_3. Hence, H_B must consist of at least two disjoint cycles because it is 2-regular. The graph H_B cannot consist of more than three cycles because it has six vertices. If H_B consists of exactly three cycles, then it is formed by three cycles of length two. Since the degree of each big vertex in G is seven, its simple degree is six. This contradicts the assumption of the lemma that G contains a vertex with simple degree at least seven. Hence, we can conclude that H_B consists of exactly two cycles. Moreover, it consists of either two cycles of length three or a cycle of length two and a cycle of length four.

The embedding of H_B can have at most three faces. Recall that the subgraph induced by the small vertices has three components. So, the embedding of H_B has exactly three faces. Observe that the cliques H_1, H_2 and H_3 were drawn in different faces of H_B because G is a triangulation. Let f_i, $i = 1, 2, 3$, be the face of H_B in which the clique H_i was drawn.

If H_B consists of a cycle of length two and a cycle of length four, the boundary of one of the faces of H_B, say the face f_1, is formed by two big vertices b_1 and b_2 which are joined by two parallel edges. The vertices of the clique H_1 drawn in the face f_1 can be adjacent only to the vertices b_1 and b_2 and since the minimum simple degree of G is six, each vertex of H_1 is adjacent to both b_1 and b_2 (recall that H_1 is a clique of order five). Then, the vertices of H_1 together with the vertices b_1 and b_2 form a subgraph of G which is isomorphic to K_7, a contradiction. Hence, the graph H_B must consist of vertex-disjoint two cycles of length three.

Let $b_1b_2b_3$ and $b'_1b'_2b'_3$ be the two cycles of H_B. We can assume without loss of generality that the boundary of f_1 is formed by the 3-cycle $b_1b_2b_3$, the boundary of f_2 by the 3-cycles $b_1b_2b_3$ and $b'_1b'_2b'_3$ and the boundary of f_3 by the 3-cycle $b'_1b'_2b'_3$.

Let n_i^1, $i = 1, 2, 3$, be the number of neighbors of the vertex b_i in the clique H_1 and n_i^2 the number of neighbors of b_i in H_2. Observe that $n_i^1 + n_i^2 = 5$ for each $i = 1, 2, 3$ because the degree of b_i in G is seven. Since the face f_1 contained the clique H_1, the face f_2 contained the clique H_2 and G is a triangulation, each of the numbers n_i^1 and n_i^2 is non-zero. By Lemma 12, we have $n_i^1 \neq 1$ and $n_i^2 \neq 1$. Hence, each of them is either 2 or 3. In particular, $n_1^1 + n_2^1 + n_3^1 \leq 9$. But this is impossible because the sum $n_1^1 + n_2^1 + n_3^1$ should be equal to the weight $w(K_5) = 10$ of the clique H_1 — contradiction.

Claim 21.3 *At least one of the cliques H_1, H_2 and H_3 is isomorphic to K_1.*

The above claim directly follows from Claims 21.1 and 21.2.

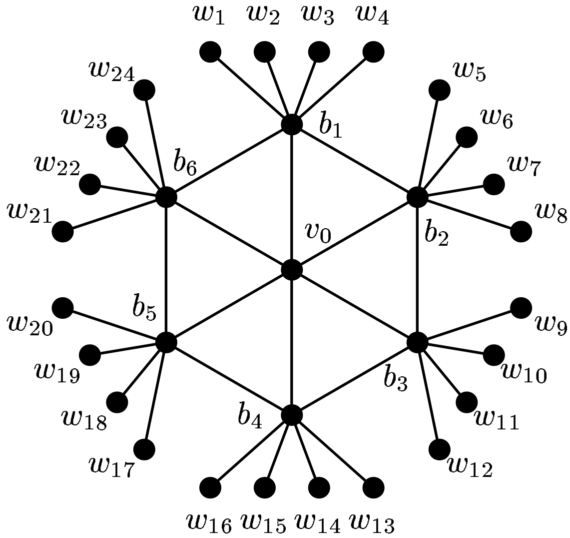

Figure 4: Notation used in Claim 21.4.

Claim 21.4 *At least two of the Gallai trees H_1, H_2 and H_3 are isomorphic to K_1.*

By Claim 21.3, we can assume that H_1 consists of a single small vertex v_0. Assume for contradiction that both H_2 and H_3 are cliques of order at least two. Then, the weights $w(H_2)$ and $w(H_3)$ are at least 10. Since G is a triangulation, the big neighbors $b_1, \ldots, b_6$ of the vertex v_0 form a 6-cycle C, say $C = b_1b_2b_3b_4b_5b_6$. Let $w_1, \ldots, w_{24}$ be the other neighbors of the big vertices so that $w_{4i-3}, w_{4i-2}, w_{4i-1}, w_{4i}$ are the neighbors of the big vertex b_i in the order depicted in Figure 4. Note that the vertices $w_1, \ldots, w_{24}$ are not necessarily all distinct, e.g., $w_1 = w_{24}$, and some of them could be neighbors of the vertex v_0.

In the rest of the proof, edges which join two big vertices and which are not included in the cycle C are called *diagonals*. The big vertices $b_1, \ldots, b_6$ are joined to the small vertices of H_2 and H_3 by precisely $w(H_2) + w(H_3)$ edges. Hence, besides the edges of the cycle C, there are $(24 - w(H_2) - w(H_3))/2$ diagonals. Hence, there are at most two diagonals. On the other hand, there is at least one diagonal: Otherwise, since G is a triangulation, all the vertices $w_1, \ldots, w_{24}$ are small and thus they are contained in the same component of the Gallai forest F. Then, F has only two components.

We first consider the case that there is exactly one diagonal. Hence, there are exactly two indices i and i', $1 \leq i < i' \leq 24$, such that w_i and

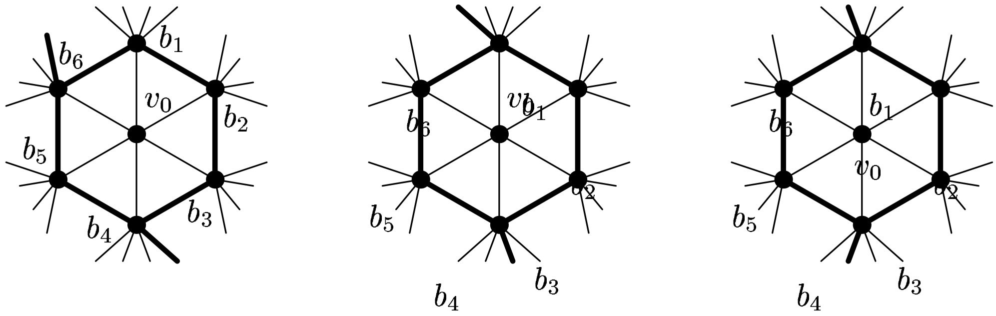

Figure 5: Possible configurations from the proof of Claim 21.4 in the case that there is a single diagonal. Edges joining two big vertices are drawn as bold.

$w_{i'}$ are big vertices. Then, the vertices $w_{i+1}, \ldots, w_{i'-1}$ are small vertices of the same Gallai tree, say H_2, and $w_{i'+1}, \ldots, w_{24}, w_1, \ldots, w_{i-1}$ are small vertices of the other Gallai tree H_3. In addition, the weight of one of H_2 and H_3 is 12 and the weight of the other one is 10. We may assume that $w(H_2) = 12$ and $w(H_3) = 10$. In particular, both H_2 and H_3 are cliques and $i' - i - 1 = w(H_2) = 12$ (modulo 24).

There are three possible configurations (upto symmetry) of the edges between the small vertices and the big ones in the neighborhood of the vertex v_0 which are depicted in Figure 5. Note that the vertex b_2 in each configuration is adjacent to four vertices of H_2 and thus H_2 is a clique of order at least four. Since the weight of H_2 is 12, the clique H_2 has order four. Similarly, the vertex b_5 in each configuration is adjacent to four vertices of H_3 and thus H_3 must be a clique of order five. The left and the middle configurations depicted in Figure 5 cannot appear in a triangulation: In order to see this, consider the face $b_1 b_6 w_1 = b_1 b_6 w_{24}$ where the vertex $w_1 = w_{24}$ should be simultaneously big and small. The right configuration cannot appear in G by Lemma 12 because the neighbor w_1 of the big vertex b_1 cannot be contained in the Gallai tree isomorphic to K_5.

Let us consider now the remaining case that there are two diagonals. Hence, each of H_2 and H_3 has weight 10, in particular, each of them is isomorphic to K_2 or K_5.

In this paragraph, we show that there cannot be two big vertices such that each of them has four neighbors in H_2 (the analogous statement also holds for H_3). Assume for contradiction that there are such two big vertices b_i and $b_{i'}$. Since the clique H_2 has at least four vertices, its order must be five. Recall now that $w(H_2) = 10$. By Lemma 12, the two edges which join the vertices of H_2 and the big vertices and which are incident neither with b_i nor $b_{i'}$ must be incident with the same big vertex. Let $b_{i''}$ be this big vertex. By symmetry, we can assume that $i = 1$. Since G is a triangulation, each of the big vertices b_6 and b_2 is adjacent to a vertex of H_2 (note that $w_{24} = w_1$ and $w_4 = w_5$). Thus, $\{i', i''\} = \{2, 6\}$. By symmetry, we can assume that $i'' = 6$ and $i' = 2$. Again, since G is a triangulation, the big vertex b_3 is adjacent to a vertex of H_2 (note that $w_8 = w_9$). But this is impossible because $b_i = b_1$, $b_{i'} = b_2$ and $b_{i''} = b_6$ are the only big vertices adjacent to a vertex of H_2.

Since there are only two diagonals, at least two big vertices b^2 and b^3 are adjacent to four small vertices different from the vertex v_0. As we have shown in the previous paragraph, the big vertices b^2 and b^3 cannot be adjacent to vertices of the same clique. Hence, we can assume that b^2 is adjacent to four vertices of the clique H_2 and b^3 is adjacent to four vertices of the clique H_3. Since each of the cliques H_2 and H_3 has at least four vertices, the order of both of them is five (recall that we showed that each of H_2 and H_3 is isomorphic to K_2 or K_5).

By Lemma 12, no big vertex has a single neighbor in H_2. Thus, each big vertex has either no neighbor in H_2 or it has at least two neighbors in H_2. Since there cannot be two big vertices with four neighbors in H_2, only the following two configurations can appear:

- There is a big vertex, namely the vertex b^2, adjacent to precisely four vertices of the clique H_2 and there are other two big vertices each adjacent to precisely three vertices of the clique H_2.

- There is a big vertex, namely the vertex b^2, adjacent to precisely four vertices of the clique H_2 and there are other three big vertices each adjacent to precisely two vertices of the clique H_2.

We show that the latter is impossible: Let b, b' and b'' be the three big vertices adjacent to two vertices of H_2. Since G is a triangulation, each of the vertices b, b' and b'' must be incident with at least one diagonal. If it is incident with just a single diagonal, then it is adjacent to precisely one vertex of H_3 which is impossible by Lemma 12. Hence, each of the vertices b, b' and b'' is incident with two diagonals and consequently, there must be at least three diagonals. But we assumed that there are only two diagonals. So, the big vertex b^2 is adjacent to four vertices of the clique H_2 and there are other two big vertices each adjacent to three vertices of the clique H_2. Similarly, the big vertex b^3 is adjacent to four vertices of the clique H_3 and

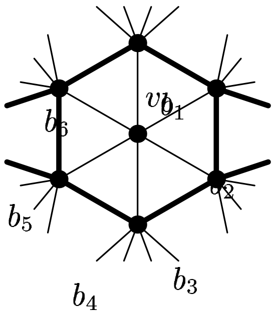

Figure 6: The only possible configuration from the proof of Claim 21.4 in the case that there are two diagonals. The diagonals are drawn as bold.

there are other two big vertices each adjacent to three vertices of the clique H_3.

Since G is a triangulation, the two big vertices adjacent to three vertices of H_2 must be neighbors of the vertex b^2 in the 6-cycle C. An analogous statement holds for H_3 and the vertex b^3. By symmetry, we can assume that the vertex b^2 is the big vertex b_1 and the two big vertices adjacent to precisely three vertices of H_2 are the big vertices b_6 and b_2. Then, b^3 must coincide with the big vertex b_4 and each of the vertices b_3 and b_5 is adjacent to three vertices of H_3. This configuration is depicted in Figure 6. Since G is a triangulation, the bold edges incident with the vertices b_2 and b_3 in Figure 6 must lead to the same big vertex, but there is no big vertex in G incident with two such edges — contradiction. This completes the proof of Claim 21.4. ∎

6 List colorings of triangulations of Π_3

As we have already seen, in order to prove our choosability result, it is enough to restrict our attention to triangulations of the surface Π_3 with minimum simple degree six in which small vertices induce a Gallai forest of a certain special type and which does not contain K_7 as a subgraph. In this section, we prove that graphs of this type embedded on the surface Π_3 are 6-choosable. In Section 7, we combine the results of Section 5 and the results of this section to conclude that if a graph embedded on the surface Π_3 does not contain K_7 as a subgraph, then it is 6-choosable.

In some of our proofs, we first color big vertices of the triangulation and then we try to extend this coloring to small vertices. The following four proposition will help us in this task:

Proposition 22 *Let G be a graph with minimum simple degree six, L a list 6-assignment of G and S the vertex set of a component of the subgraph of G induced by the small vertices. Suppose that some of the big vertices are precolored so that there is a big vertex b adjacent to a small vertex $s_0 \in S$ such that b is colored with a color $\alpha \notin L(s_0)$. Then, the precoloring of the big vertices can be extended to all the vertices of S.*

Proof: Let $L'(s)$ for each $s \in S$ be the list of the colors of $L(s)$ which are not used to color the big neighbors of s. Note that $|L'(s)| \geq \deg_{G[S]}(s)$ for each vertex $s \in S$ and $|L'(s_0)| > \deg_{G[S]}(s_0)$. Hence, there exists an L'-coloring of $G[S]$ by Theorem 3. This L'-coloring is the sought extension of the precoloring to S.

■

Similarly as Proposition 22, one can prove the following two propositions:

Proposition 23 *Let G be a graph with minimum simple degree six, L a list 6-assignment of G and S the vertex set of a component of the subgraph of G induced by the small vertices. Suppose that some of the big vertices are precolored so that there is a small vertex $s_0 \in S$ adjacent to two big vertices which are colored with the same color. Then, the precoloring of the big vertices can be extended to all the vertices of S.*

Proposition 24 *Let G be a graph with minimum simple degree six, L a list 6-assignment of G and S the vertex set of a component of the subgraph of G induced by the small vertices. Suppose that some of the big vertices are precolored so that there is a small vertex $s_0 \in S$ adjacent to a big vertex b so that b is not colored. Then, the precoloring of the big vertices can be extended to all the vertices of S.*

The last of our propositions requires a different proof:

Proposition 25 *Let G be a graph with minimum simple degree six, L a list 6-assignment of G and S the vertex set of a component of the subgraph of G induced by the small vertices. Suppose that $G[S]$ is a clique and there exist* *two vertices s_1 and s_2 of S and a color α such that $\alpha \in L(s_1)$, $\alpha \notin L(s_2)$ and no big neighbor of s_1 is colored with α. Then, the precoloring of the big vertices can be extended to all the vertices of S.*

Proof: Let $L'(s)$ for each $s \in S$ be the list of the colors of $L(s)$ not used to color all the big neighbors of s. Note that $|L'(s)| \geq \deg_{G[S]}(s)$ for each vertex $s \in S$ and $L'(s_1) \neq L'(s_2)$. Hence, there exists an L'-coloring of $G[S]$ by Theorem 4. This L'-coloring is the desired extension of the precoloring. ∎

Let us remark that Propositions 22–25 can be easily reformulated for list k-assignments for any $k \geq 1$. We keep them in the above form in order to make more clear their applications in the proofs of Lemmas 26–30.

In the rest of this section, we prove Lemmas 26–30 in which we deal with all types of triangulations of Π_3 which were described in Section 5. The course of the proofs of these lemmas is more or less the same: We fix a 6-list assignment L of a triangulation G of Π_3 and assume that G has no L-coloring. In the rest of each of the proofs, we proceed in a series of claims. We first show $L(s) = L(b)$ for most pairs of a small vertex s and a big vertex b which are adjacent. Then, we deduce that G contains precisely six big vertices and they can be grouped into three pairs so that the vertices of each pair have the same list. Based on the structure of the triangulation and the list assignment, we eventually find an L-coloring of G which contradicts our original assumption that there is no L-coloring. Although it might seem at the first sight that the proofs of the lemmas are essentially the same, the arguments used to establish the claims are different.

The first case which we consider is that small vertices induce a Gallai tree:

Lemma 26 *Let G be a triangulation with minimum simple degree six of the surface Π_3 such that G does not contain K_7 as a subgraph. If the vertices of degree six induce a Gallai tree in G, then G is 6-choosable.*

Proof: Suppose that the claim of the lemma is false. Fix a triangulation G, which has the properties described in the statement of the lemma, and a list 6-assignment L such that G has no L-coloring. Let S be the set of small vertices of G and B the set of big vertices of G. Note that $|B| \leq 6$ by Lemma 10.

Claim 26.1 *Let $s \in S$ and $b \in B$ be two adjacent vertices in G. Then, $L(s) = L(b)$.*

Assume the opposite and let $s \in S$ and $b \in B$ be two adjacent vertices with $L(s) \neq L(b)$. Color first the vertex b with a color $\alpha \in L(b) \setminus L(s)$. Then, color properly the remaining (at most five) big vertices by colors from their lists. This is possible because each vertex has a list of six available colors. By the choice of the color of the vertex b, the coloring of the big vertices can be extended to an L-coloring of G by Proposition 22 — contradiction.

Claim 26.2 *Let b be a big vertex. Then, there exists a big vertex $b' \neq b$ with $L(b) = L(b')$.*

Since the minimum simple degree of G is six and there are at most six big vertices, the vertex b is adjacent to a small vertex s. Since each small vertex is adjacent to at least two big vertices by Lemma 13, there exists a big vertex $b' \neq b$ which is adjacent to s. Then, $L(b) = L(s) = L(b')$ by Claim 26.1.

Claim 26.3 *There exist three big vertices whose lists are mutually distinct.*

Assume the opposite and let L_1 and L_2 be two lists such that the list of each big vertex is L_1 or L_2. Since each small vertex s is adjacent to a big vertex, the list of s must be L_1 or L_2 by Claim 26.1. Hence, the list of each vertex of G is L_1 or L_2. By Theorem 1, the triangulation G is 6-colorable. Therefore, G has an L-coloring by Lemma 6 — contradiction.

Claim 26.4 *The graph G contains precisely six big vertices. Moreover, there is an ordering of the big vertices b_1, b_2, b_3, b_4, b_5 and b_6 so that $L(b_1) = L(b_2)$, $L(b_3) = L(b_4)$ and $L(b_5) = L(b_6)$ and the lists of any other pair of the big vertices are distinct.*

The claim directly follows from Claims 26.2 and 26.3 and the fact that $|B| \leq 6$.

Claim 26.5 *Each small vertex is adjacent to precisely two big vertices. In particular, the graph $G[S]$ must be a clique of order five.*

Each small vertex is adjacent to at least two big vertices by Lemma 13. By Claims 26.1 and 26.4, it can be adjacent to at most two big vertices. Hence, each small vertex is adjacent to precisely two big vertices and it is adjacent to exactly four small vertices. Since the only 4-regular Gallai tree is K_5, the graph $G[S]$ must be a clique of order five.

Claim 26.6 *The graph G has an L-coloring.*

Let b_1, b_2, b_3, b_4, b_5 and b_6 be the big vertices ordered as in Claim 26.4. Let s_1 and s_3 be small neighbors of the vertices b_1 and b_3, respectively. Such vertices s_1 and s_3 exist because the minimum simple degree of G is six. By Claim 26.1, $L(s_1) = L(b_1)$ and $L(s_3) = L(b_3)$. So, $L(s_1) \neq L(s_3)$. Choose a color $\alpha \in L(s_3) \setminus L(s_1)$. Color properly the two big neighbors b_3 and b_4 of s_3 by colors from their lists different from the color α and the remaining big vertices by arbitrary colors from their lists. Since $G[S]$ is a clique of order five by Claim 26.5, no big neighbor of the vertex s_3 is colored with the color α and $\alpha \notin L(s_1)$, it follows that the coloring of the big vertices can be extended to an L-coloring of G by Proposition 25.

■

The second case which we consider is that small vertices induce a Gallai forest with two components such that at least one of the components is a single vertex:

Lemma 27 *Let G be a triangulation with minimum simple degree six of the surface Π_3 which does not contain K_7 as a subgraph. If the small vertices induce a Gallai forest in G with precisely two components such that at least one of them is isomorphic to K_1, then the graph G is 6-choosable.*

Proof: Suppose that the statement of the lemma is false. Fix a triangulation G of Π_3, which satisfy the assumptions of the lemma, and a list 6-assignment L such that G has no L-coloring. Let B be the set of big vertices of G and S the set of small vertices of G. Let s_0 be further an isolated vertex of $G[S]$ and let $S_0 = S \setminus \{s_0\}$. Finally, let $B_0 \subseteq B$ be the set of big vertices adjacent to at least one vertex of S_0. By Lemma 11, there are precisely six big vertices and thus each big vertex is adjacent to the vertex s_0. On the other hand, there are also two non-adjacent big vertices: Otherwise, $G[B]$ is a clique and so $G[B \cup \{s_0\}]$ is a clique of order seven.

Claim 27.1 *If $s \in S_0$ and $b \in B_0$ are adjacent vertices, then $L(s) = L(b)$.*

Assume the opposite and let $s \in S_0$ and $b \in B_0$ be two adjacent vertices with $L(s) \neq L(b)$. Color first the vertex b by a color $\alpha \in L(b) \setminus L(s)$. Let $b' \neq b$ be a vertex of $G[B]$ which is not adjacent to all the big vertices. Such a vertex exists as explained above. Color properly the vertex s_0 by a color from its list and then the remaining big vertices one by one by colors from their lists so that the vertex b' is colored as the last one. This is possible since when we color each of these vertices, at most five of its neighbors are previously colored. By the choice of the color of the vertex b, this coloring can be extended to an L-coloring of the vertices of S_0 by Proposition 22. So, we obtain an L-coloring of G — contradiction.

Claim 27.2 *There are three vertices of B_0 whose lists are mutually distinct.*

Assume the opposite and let L_1 and L_2 be two lists such that the list of each vertex of B_0 is L_1 or L_2. By Claim 27.1 and Lemma 13, the list of each vertex of S_0 is also L_1 or L_2.

We first consider the case that all the vertices of $S_0 \cup B_0$ have the same list, say L_0, i.e., $L_0 = L_1$ or $L_0 = L_2$. If $L(s_0) \neq L_0$, then color the vertices of $G[S_0 \cup B_0]$ by colors from their lists so that at least one big neighbor of s_0 is colored with a color $\alpha \in L_0 \setminus L(s_0)$. Lemma 7 implies that this is possible because $G[S_0 \cup B_0]$ is 6-colorable by Theorem 1 and all vertices of $S_0 \cup B_0$ has the same list L_0. Next, color properly the remaining big vertices by colors from their lists (note that the colored neighbors of each vertex from $B \setminus B_0$ are only big vertices and hence it is adjacent to at most five colored vertices). Since one of the neighbors of the vertex s_0 is colored with a color $\alpha \notin L(s_0)$, we can now color the vertex s_0 by a color from its list. Thus, we obtain an L-coloring of G — contradiction.

If $L(s_0) = L_0$, then s_0 has a big neighbor b with $L(b) \neq L(s_0)$: Otherwise, all the vertices of G have the same list and thus G has an L-coloring by Theorem 1. It follows from our assumption that the lists of all the vertices of $S_0 \cup B_0$ are the same that $b \notin B_0$. Fix a color $\alpha \in L(b) \setminus L(s_0)$. Since all the vertices of $S_0 \cup B_0$ have the same list L_0 and the graph $G[S_0 \cup B_0]$ is 6-colorable by Theorem 1, we can color the vertices of $G[S_0 \cup B_0]$ by colors from their lists. Afterwards, color the vertex b by α (recall that $\alpha \notin L_0$ and $L_0 = L(s_0)$). Next, color properly the remaining big vertices by colors from their lists. Note that the colored neighbors of each vertex from $B \setminus B_0$ are only big vertices and hence it is adjacent to at most five colored vertices. Finally, color the vertex s_0. This is possible because a neighbor of s_0 in G is colored with a color $\alpha \notin L(s_0)$. Thus, we obtain an L-coloring of G — contradiction.

The final case to consider is that the lists of all the vertices of $S_0 \cup B_0$ are not the same, in particular $L_1 \neq L_2$. Assume without loss of generality that $L(s_0) \neq L_1$. Let $\alpha \in L_1 \setminus L(s_0)$ and let $b_1 \in B_0$ be a big vertex with $L(b_1) = L_1$. The existence of a vertex b_1 follows from Claim 27.1. Fix a coloring of $G[S_0 \cup B_0]$ such that the color of the vertex b_1 is α. Such a coloring exists by Lemma 7. Next, color the remaining big vertices by colors from their lists. This is possible because the colored neighbors of each vertex from $B \setminus B_0$ are only big vertices and hence it is adjacent to at most five colored vertices. Since the vertex b_1, which is a neighbor of the vertex s_0, is colored with a color $\alpha \notin L(s_0)$, we can now color the vertex s_0. In this way, we obtain an L-coloring of G — contradiction.

Claim 27.3 *There is an ordering b_1, b_2, b_3, b_4, b_5 and b_6 of the six big vertices of G such that $L(b_1) = L(b_2)$, $L(b_3) = L(b_4)$ and $L(b_5) = L(b_6)$ and the lists of any other pair of the big vertices are distinct. Moreover, $B_0 = B$.*

By Claim 27.2, there are three vertices of B_0 whose lists are mutually distinct. Since each vertex of B_0 is adjacent to a vertex of S_0 (the set B_0 was defined to be the set of such big vertices) and each vertex of S_0 is adjacent to at least two big vertices by Lemma 13, it follows that there are exactly six big vertices and thus $B_0 = B$. The statement now readily follows from Claim 27.1.

Claim 27.4 *The graph G has an L-coloring.*

Let b_1, b_2, b_3, b_4, b_5 and b_6 be the big vertices ordered as in Claim 27.3. By Claims 27.1 and 27.3, each vertex of S_0 can be adjacent to at most two big vertices. Since it must be adjacent to at least two big vertices by Lemma 13, we conclude that the Gallai tree $G[S_0]$ is 4-regular and hence $G[S_0]$ is a clique of order five.

Since $G[B]$ is not a complete graph, there is a big vertex, say $b_3 \in B$, which is not adjacent to all the big vertices. Let s_1 and s_3 be vertices of S_0 adjacent to the big vertices b_1 and b_3, respectively. By Claim 27.1, $L(b_1) = L(s_1)$ and $L(b_3) = L(s_3)$. In particular, $L(s_1) \neq L(s_3)$.

Fix a color $\alpha \in L(s_1) \setminus L(s_3)$. Color first the vertices b_1 and b_2 by colors from their lists which are distinct from the color α. Next, color s_0 by a color from its list and then the remaining big vertices by colors from their lists so that the vertex b_3 is colored as the last one. Again, it is possible to color all the big vertices because when we color each of them at most five of its neighbors are already colored.

This coloring can be extended to the clique $G[S_0]$ by Proposition 25 because of the choice of the colors of the vertices b_1 and b_2 and the facts that

$\alpha \in L(s_1)$ and $\alpha \notin L(s_3)$. In this way, we obtain an L-coloring of G — contradiction.

■

The third case to consider is that the Gallai forest consists of two cliques:

Lemma 28 *Let G be a triangulation with minimum simple degree six of the surface Π_3 which does not contain K_7 as a subgraph. If the graph induced by small vertices consists of two components, each of them being a clique of order at least two, then the graph G is 6-choosable.*

Proof: Suppose that the lemma is false. Fix a triangulation G with the properties from the statement of the lemma and a list 6-assignment L of G such that there is no L-coloring of G. Let B be the set of big vertices of G and let S_1 and S_2 be the vertex sets of the two cliques of the subgraph of G induced by the small vertices. Note that the orders of both the cliques $G[S_1]$ and $G[S_2]$ are at most five by Lemma 13.

Claim 28.1 *Let b_1 be an arbitrary big vertex of G. It is possible to order the big vertices $b_1, b_2, \ldots, b_{|B|}$ so that each big vertex b_i is adjacent to at most four big vertices b_j with $j < i$.*

The claim is clear if $|B| \leq 5$ or if $|B| = 6$ and $G[B]$ contains two non-adjacent vertices. Assume for contradiction that $|B| = 6$ and all the vertices of $G[B]$ are mutually adjacent. Since the degree of each vertex of B is seven by Lemma 10, each of them is adjacent to at most two small vertices. Hence, there are at most 12 edges between the big vertices and the small vertices. However, since the weights of $G[S_1]$ and $G[S_2]$ are at least ten, there are at least ten edges between S_1 and B as well as between S_2 and B — contradiction.

Claim 28.2 *Let $s \in S_1 \cup S_2$ and $b \in B$ be two adjacent vertices in G. Then, $L(s) = L(b)$.*

Assume the opposite and let $s_1 \in S_1$ and $b \in B$ be two adjacent vertices with $L(s_1) \neq L(b)$ (the case that such a small vertex is contained in S_2 is symmetric). Fix a color $\alpha \in L(b) \setminus L(s_1)$. For every vertex $s \in S_2$, let $L'(s) = L(s) \setminus \{\alpha\}$ if s is adjacent to the vertex b and $L'(s) = L(s)$ otherwise.

Let us consider first the case that there are two small vertices s^1 and s^2 of S_2 with $L'(s^1) \neq L'(s^2)$. We can assume without loss of generality that $|L'(s^1)| \geq |L'(s^2)|$. Fix a color $\beta \in L'(s^1) \setminus L'(s^2)$. Color the vertex b by α and the remaining big vertices properly by arbitrary colors from their lists so that each of them is colored with a color different from β. This is clearly possible: Just color the big vertices in the order from Claim 28.1 with $b_1 = b$. The coloring of the big vertices can be extended to S_1 by Proposition 22. Afterwards, it can be extended to S_2 by Proposition 25 (note that $\beta \in L'(s^1)$ and no neighbor of s^1 is colored with β and $\beta \notin L'(s^2)$). This yields an L-coloring of G — contradiction.

Next, we consider the case that there is a vertex $v \in S_2$ adjacent to b with $\alpha \notin L(v)$. In this case, we color the big vertices arbitrarily so that the vertex b is colored with α. However, the precoloring of the big vertices can be extended to both S_1 and S_2 by Proposition 22 — contradiction.

In the rest, we assume that all the lists $L'(v)$ of the vertices of $v \in S_2$ are the same and the list of each vertex of S_2 adjacent to b contains α. Hence, either no vertex of S_2 is adjacent to the vertex b or all the vertices of S_2 are adjacent to the vertex b and the lists of all of them contain the color α. Let $N_B(s)$ be further the set of big neighbors of a vertex $s \in S_2$. Recall that, by our assumption, the vertex b is contained either in all the sets $N_B(s)$, $s \in S_2$, or in no set $N_B(s)$, $s \in S_2$.

We now consider the case that there exist two small vertices s^1 and s^2 of S_2 with $N_B(s^1) \neq N_B(s^2)$. Color the vertex b by α and the remaining big vertices by colors from their lists so that no two big vertices are colored with the same color (recall that there are at most six big vertices). The coloring of the big vertices can be extended to S_1 by Proposition 22. For each small vertex $s \in S_2$, let $L''(s)$ be the subset of the list $L(s)$ which contains the colors of $L(s)$ not assigned to the big neighbors of s. Note that $|L''(s)| \geq \deg_{G[S_2]}(s)$ for every vertex $s \in S_2$. In addition, $L''(s^1) \neq L''(s^2)$ because $L'(s^1) = L'(s^2)$, $N_B(s^1) \neq N_B(s^2)$ and all the big vertices are colored with mutually distinct colors. By Theorem 4, there exists an L''-coloring of $G[S_2]$. The L''-coloring of $G[S_2]$ and the coloring of the big vertices and S_1 form an L-coloring of G — contradiction.

The final case is that the lists $L'(s)$ of all the small vertices $s \in S_2$ are the same list, say L_0, and the small vertices of S_2 are adjacent to the same set N_B of big vertices, i.e., $N_B(s) = N_B$ for every $s \in S_2$. Note that $|N_B| = 7 - |S_2|$. Let b_1 and b_2 be two non-adjacent vertices of N_B. Such two vertices b_1 and b_2 exist: Otherwise, the graph $G[N_B \cup S_2]$ would be a clique of order seven.

In this paragraph, we carefully color at least two of the vertices b, b_1 and b_2. Afterwards, we extend this precoloring to an L-coloring of G. Note that b may coincide with b_1 and b_2. Color first the vertex b by the color α. If $b_1 = b$, then $|L_0| = 5$, in particular $|L_0| < |L(b_2)|$, and we color the vertex b_2 by a color $\beta \in L(b_2) \setminus L_0$. If $b_2 = b$, we proceed analogously. Assume now that the vertex b is neither b_1 nor b_2. Color the vertex b_1 by a color $\beta \in L(b_1) \setminus (L_0 \cup \{\alpha\})$. This is possible unless b_1 is adjacent to b and α is the only color contained in $L(b_1) \setminus L_0$. If b_1 was not colored, then color b_2 by a color $\beta \in L(b_2) \setminus (L_0 \cup \{\alpha\})$. We can color the vertex b_2 in this way unless b_2 is adjacent to b and α is the only color contained in $L(b_2) \setminus L_0$. If both b_1 and b_2 are not colored, then both the lists $L(b_1)$ and $L(b_2)$ contain at least five colors (possibly distinct) common with the list L_0. Hence, there is a color $\beta \in L(b_1) \cap L(b_2) \cap L_0$ and we can color both b_1 and b_2 by β. Some of the big vertices are now colored so that each vertex of S_2 is adjacent to a vertex colored with a color not contained in L_0 or to two vertices colored with the same color.

Extend now the obtained coloring to all the big vertices (recall again that there are at most six big vertices). The coloring can be further extended to the vertices of S_1 by Proposition 22 (the vertex b is colored with α) and to the vertices of S_2 by Propositions 22 or 23 because each vertex of S_2 is adjacent to a vertex colored with a color not contained in the list L_0 or to two vertices colored with the same color, respectively. This yields an L-coloring of G — contradiction.

Claim 28.3 *Let b be an arbitrary big vertex. Then, there exists a big vertex $b' \neq b$ with $L(b) = L(b')$.*

Since the minimum simple degree of G is six and there are at most six big vertices, the vertex b is adjacent to a small vertex s. And, since each small vertex is adjacent to at least two big vertices by Lemma 13, there exists another big vertex $b' \neq b$ which is adjacent to s. Then, $L(b) = L(b')$ by Claim 28.2.

Claim 28.4 *There exist three big vertices whose lists are mutually distinct.*

Assume the opposite and let L_1 and L_2 be two lists such that the list of each big vertex is L_1 or L_2. Since each small vertex s is adjacent to a big vertex, the list of the vertex s must be L_1 or L_2 by Claim 28.2. Hence, the list of each vertex of G is L_1 or L_2. By Theorem 1, G is 6-colorable. Therefore, G has an L-coloring by Lemma 6 — contradiction.

Claim 28.5 *There are exactly six big vertices. In addition, there is an ordering* b_1, b_2, b_3, b_4, b_5 *and* b_6 *of the big vertices with* $L(b_1) = L(b_2)$, $L(b_3) = L(b_4)$ *and* $L(b_5) = L(b_6)$ *and the lists of any other pair of the big vertices are distinct.*

The above claim directly follows from Claims 28.3 and 28.4 and Lemma 10.

Claim 28.6 *Each small vertex is adjacent to precisely two big vertices. In particular, both* $G[S_1]$ *and* $G[S_2]$ *are isomorphic to* K_5*.*

Each small vertex is adjacent to at least two big vertices by Lemma 13. By Claims 28.2 and 28.5, it can be adjacent to at most two big vertices. Hence, each small vertex is adjacent to precisely two big vertices and so it is adjacent to precisely four small vertices. Since K_5 is the only 4-regular Gallai tree, it follows that both $G[S_1]$ and $G[S_2]$ must be cliques of order five.

Claim 28.7 *There are two vertices of* S_1 *whose lists are different. Similarly, there are two vertices of* S_2 *whose lists are different.*

Let b_1, b_2, b_3, b_4, b_5 and b_6 be the big vertices of G ordered as in Claim 28.5. Since G contains six big vertices, each big vertex has degree seven. Suppose that the claim is false, i.e., the lists of all the vertices of S_1 are the same, say the list $L(b_1)$. Since all the five vertices of the clique $G[S_1]$ are adjacent to both b_1 and b_2 by Claim 28.2, the vertices b_1 and b_2 are not adjacent: Otherwise, $G[S_1 \cup \{b_1, b_2\}]$ would be a clique of order seven. By Lemma 14, the five vertices of S_1 are the only small vertices which are adjacent to the vertices b_1 and b_2. Fix a color $\alpha \in L(b_1)$. Color the graph $G[S_2 \cup \{b_3, b_4, b_5, b_6\}]$ so that none of the vertices b_3, b_4, b_5 and b_6 is colored with the color α. This is possible by Lemma 8 because the list of each vertex of $G[S_2 \cup \{b_3, b_4, b_5, b_6\}]$ is $L(b_3)$ or $L(b_5)$ and $G[S_2 \cup \{b_3, b_4, b_5, b_6\}]$ is 6-colorable by Theorem 1. Color now both b_1 and b_2 by the color α and the five vertices of S_2 properly by the remaining five colors from their lists. In this way, we obtain an L-coloring of G — contradiction. An analogue argument yields the second part of the claim.

Claim 28.8 *There are three vertices of* S_1 *whose lists are mutually distinct. Similarly, there are three vertices of* S_2 *whose lists are mutually distinct.*

Let b_1, b_2, b_3, b_4, b_5 and b_6 be the big vertices ordered as in Claim 28.5. Assume that the claim is false, e.g., that the list of each vertex of S_1 is either $L(b_1)$ or $L(b_3)$. Since the vertex b_5 is adjacent to at least one small vertex and it is adjacent to no vertex of S_1, it must be adjacent to a vertex of S_2. By Claim 28.7, there are two vertices s^1 and s^2 of S_2 whose lists $L(s^1)$ and $L(s^2)$ are different. By symmetry, we can assume that $L(s^1) = L(b_1)$ and $L(s^2) = L(b_5)$.

Fix a color $\alpha \in L(s^1) \setminus L(s^2)$. Color now the vertices of $G[S_1 \cup \{b_1, b_2, b_3, b_4\}]$ so that the color of each of the vertices b_1, b_2, b_3 and b_4 is different from α. This is possible by Lemma 8 because $G[S_1 \cup \{b_1, b_2, b_3, b_4\}]$ is 6-colorable by Theorem 1. Color now the remaining two big vertices b_5 and b_6. Note that neither b_5 nor b_6 is adjacent to a vertex of S_1 by Claim 28.2 (recall that $\alpha \notin L(s^2)$). The coloring of the big vertices can be extended to $G[S_2]$ by Proposition 25 because of $\alpha \in L(s^1)$, $\alpha \notin L(s^2)$ and the choice of the colors of the big vertices — contradiction.

Claim 28.9 *The graph G has an L-coloring.*

Let b_1, b_2, b_3, b_4, b_5 and b_6 be the big vertices ordered as in Claim 28.5. By Claims 28.2 and 28.8, the set S_1 contains two vertices s_1^1 and s_1^2 with $L(s_1^1) = L(b_1)$ and $L(s_1^2) = L(b_3)$. Similarly, S_2 contains two vertices s_2^1 and s_2^2 such that $L(s_2^1) = L(b_1)$ and $L(s_2^2) = L(b_3)$. Fix a color $\alpha \in L(b_1) \setminus L(b_3)$. Color now the vertices b_1 and b_2 by colors from their lists which are different from the color α. Afterwards, color the remaining four big vertices. The coloring of the big vertices can be extended to both $G[S_1]$ and $G[S_2]$ by Proposition 25 because of $\alpha \in L(s_1^1)$, $\alpha \in L(s_2^1)$, $\alpha \notin L(s_1^2)$, $\alpha \notin L(s_2^2)$ and the choice of the colors of the big vertices. In this way, we eventually obtain an L-coloring of G.

■

We now consider the fourth possible type of the triangulation:

Lemma 29 *Let G be a triangulation with minimum simple degree six of the surface Π_3 which does not contain K_7 as a subgraph. Suppose that G contains six big vertices and that the small vertices induce a Gallai forest in G with two components whose vertex sets are S_1 and S_2. In addition, suppose that $G[S_1]$ is isomorphic to K_2, K_3, K_4 or K_5 and that the minimum degree of $G[S_2]$ is one. Then, the graph G is 6-choosable.*

Proof: Suppose that the statement of the lemma is false. Fix a triangulation G, which has the properties described in the statement of the lemma, and a list 6-assignment L such that G has no L-coloring. Let B be the set of the six big vertices of G and let s_0 be a vertex of degree one in $G[S_2]$.

Claim 29.1 *There are two big vertices b^1 and b^2 adjacent to the vertex s_0 such that each of them has at least two neighbors in S_1.*

We distinguish four cases regarding to the order of the clique $G[S_1]$. If $G[S_1]$ is a clique of order two, then at least four big vertices are adjacent to both the vertices of S_1. Since only one big vertex is not adjacent to the vertex s_0, there are at least three big vertices adjacent to s_0 and simultaneously adjacent to both the vertices of S_1.

If $G[S_1]$ is a clique of order three, then there are twelve edges between the vertices of S_1 and B (recall that $w(K_3) = 12$). Since $|S_1| = 3$, at most three of these edges can lead to the big vertex which is not adjacent to the vertex s_0. Hence, at least nine of these edges join the big neighbors of s_0 and the vertices of S_1. If four out of the five big neighbors of the vertex s_0 has at most one neighbor in S_1, then there exists a big neighbor of s_0 adjacent to at least five vertices of S_1 which is clearly impossible (recall that $|S_1| = 3$). Therefore, there are two big neighbors of s_0 with at least two neighbors in S_1.

The third case is that the order of $G[S_1]$ is four. Hence, there are again twelve edges between the vertices of S_1 and the vertices of B. At most four of these edges (recall that the order of S_1 is four) can lead to the big vertex b_0 which is not adjacent to s_0 and thus at least eight of them lead to the big neighbors of s_0. Hence, the claim is true unless all the four vertices of S_1 are adjacent to b_0, all the four vertices of S_1 are adjacent to a big neighbor b_1 of the vertex s_0 and each of the remaining big vertices b_2, b_3, b_4 and b_5 has exactly one neighbor in S_1. In that case, the neighborhoods of the vertices of S_1 contain the segments $b_0b_2b_1$, $b_0b_3b_1$, $b_0b_4b_1$ and $b_0b_5b_1$ because G is a triangulation. In particular, the vertex b_0 is adjacent to all the four vertices of S_1 and to the big vertices b_2, b_3, b_4 and b_5. This is impossible because the degree of b_0 is seven by Lemma 10.

The final case to consider is that $G[S_1]$ is a clique of order five. Each big vertex adjacent to a vertex of S_1 is adjacent to at least two vertices of S_1 by Lemma 12. Hence, it is enough to show that at least two big neighbors of s_0 are also adjacent to a vertex of S_1. Each vertex of S_1 must be adjacent to at least one big neighbor of the vertex s_0 because there is only one big vertex non-adjacent to the vertex s_0. On the other hand, each big neighbor of s_0 can be adjacent to at most four vertices of S_0 since its degree in G is seven by Lemma 10. Thus, there are at least two big neighbors of s_0 which are adjacent to a vertex of S_1 as desired. This completes the proof of Claim 29.1.

Claim 29.2 *If $s \in S_1$ and $b \in B$ are adjacent vertices, then $L(s) = L(b)$.*

Assume the opposite for the sake of contradiction and let $s \in S_1$ and $b \in B$ be two adjacent vertices with $L(s) \neq L(b)$. Let b' be further a big vertex different from b which is simultaneously adjacent to s_0 and to at least two vertices of S_1. Such a vertex exists by Claim 29.1. Color the vertex b by a color $\alpha \in L(b) \setminus L(s)$ and the remaining big vertices except the vertex b' properly by arbitrary colors from their lists. We can extend the coloring of the big vertices to S_2 by Proposition 24 because the vertex b' is not colored. Color now b' by a color from its list. This is possible because b' has at most five colored neighbors (recall that b' is adjacent to at least two vertices of S_1). Finally, we can extend this coloring to $G[S_1]$ by Proposition 22 because of the choice of the color of the vertex b. Thus, we obtain an L-coloring of G — contradiction.

Claim 29.3 *Suppose that there exist two lists L_1 and L_2 such that the list of each vertex of S_1 is L_1 or L_2. Then, $L(s) = L(b)$ for every adjacent vertices $s \in S_2$ and $b \in B$.*

Assume the opposite and let $s \in S_2$ and $b \in B$ be two adjacent vertices with $L(s) \neq L(b)$. Let B_1 be the set of big vertices adjacent to a vertex of S_1. By Claim 29.2, the list of each vertex of $S_1 \cup B_1$ is L_1 or L_2. In the rest, we consider two cases regarding whether the vertex b is contained in the set B_1 or not.

If $b \in B_1$, then consider a coloring of $G[S_1 \cup B_1]$ which assigns the vertex b a color $\alpha \in L(b) \setminus L(s)$. Such a coloring exists by Lemma 7. Color now the remaining big vertices properly by arbitrary colors from their lists. This is possible because each big vertex contained in $B \setminus B_1$ is adjacent only to the vertices of $B \cup S_2$ and thus each big vertex from $B \setminus B_1$ is adjacent to at most five colored (big) vertices.

If $b \notin B_1$, then consider a coloring of $G[S_1 \cup B_1]$ such that no vertex of B_1 is colored with a color $\alpha \in L(b) \setminus L(s)$. Such a coloring exists by Lemma 8 because $|B_1| < |B| = 6$. Color now the vertex b by α and the remaining big vertices properly by arbitrary colors from their lists.

In both the cases considered above, the coloring can be extended to S_2 by Proposition 22 because of the choice of the color of the vertex b — contradiction.

Claim 29.4 *There exist three vertices of S_1 with mutually distinct lists.*

Suppose that the claim is false, i.e., there exist two lists L_1 and L_2 such that the list of each vertex of S_1 is L_1 or L_2. Then, the lists of all the five big vertices adjacent to the vertex s_0 are the same by Claim 29.3. Let b be the big vertex which is not adjacent to s_0. Since the minimum degree of G is six, the vertex b is adjacent to a small vertex s. By Lemma 13, the vertex s is adjacent to at least two big vertices and they must have the same list either by Claim 29.2 or by Claim 29.3. Hence, all the big vertices have the same list. By Claims 29.2 and 29.3, the lists of all the vertices are the same. Then, G has an L-coloring by Theorem 1 — contradiction.

Claim 29.5 *There exists an ordering b_1, b_2, b_3, b_4, b_5 and b_6 of the big vertices of G so that $L(b_1) = L(b_2)$, $L(b_3) = L(b_4)$ and $L(b_5) = L(b_6)$ and the lists of any other pair of the big vertices are distinct.*

Let s, s' and s'' be three vertices of $G[S_1]$ with mutually distinct lists. They exist by Claim 29.4. Each of the vertices s, s' and s'' is adjacent to at least two big vertices by Lemma 13 and its big neighbors must have the same list by Claim 29.2. The claim now readily follows.

Claim 29.6 *The graph G has an L-coloring.*

Let b_1, b_2, b_3, b_4, b_5 and b_6 be the big vertices ordered as in Claim 29.5 and let s_1, s_2 and s_3 be three vertices of $G[S_1]$ with mutually distinct lists (they exist by Claim 29.4). By symmetry, we can assume that $L(s_1) = L(b_1)$, $L(s_2) = L(b_3)$ and $L(s_3) = L(b_5)$ and that the vertex s_0 is adjacent to the vertices b_1, b_2, b_3, b_4 and b_5. By symmetry, we can assume that $L(s_0) \neq L(b_i)$ for $i \in \{1, 2, 5\}$.

Fix colors $\alpha \in L(b_i) \setminus L(s_0)$ and $\beta \in L(b_3) \setminus L(b_5)$. Color the vertex b_i by the color α and the vertices b_3 and b_4 properly by colors from their lists which are different from the color β. Finally, color properly the remaining big vertices by arbitrary colors from their lists. The coloring of the big vertices can be extended to S_1 by Proposition 25 and to S_2 by Proposition 22. In this way, we construct an L-coloring of G — contradiction.

■

The final case to consider is that the Gallai forest induced by small vertices consist of three components:

Lemma 30 *Let G be a triangulation with minimum simple degree six of the surface Π_3 which does not contain K_7 as a subgraph. If the small vertices induce a Gallai forest in G with three components such that at least two of the components are isomorphic to K_1, then the graph G is 6-choosable.*

Proof: Suppose that the lemma is false. Fix a triangulation G, which satisfies the assumptions of the lemma, and a list 6-assignment L such that G has no L-coloring. Let B be the set of big vertices of G and let S be the set of small vertices of G. Note that $|B| = 6$ and each big vertex has degree seven by Lemma 11. By the assumption of the lemma, $G[S]$ contains two isolated vertices s_1 and s_2. Let S_0 be the set of the remaining small vertices. Note that each big vertex is adjacent to both s_1 and s_2. Since G is a triangulation, the graph $G[B]$ contains a cycle of length six (consider e.g., a cycle around the vertex s_1).

Claim 30.1 *There are at least four big vertices with a neighbor in the set S_0.*

If $G[S_0]$ is isomorphic to K_1, then each big vertex is adjacent to the only vertex of S_0 and the claim obviously holds. Assume in the rest that $G[S_0]$ is not isomorphic to K_1. Each big vertex is adjacent to both the small vertices s_1 and s_2 and to at least two other big vertices since $G[B]$ contains a 6-cycle. Hence, each big vertex can be adjacent to at most three vertices of S_0. By Lemma 13, $G[S_0]$ is a Gallai tree with maximum degree at most four. Therefore, the weight of $G[S_0]$ is at least 10 (cf. Figure 3). So, there are at least ten edges joining the vertices of S_0 to the big vertices. Since each big vertex is adjacent to at most three vertices of S_0, then there are at least four big vertices with a neighbor from S_0.

Claim 30.2 *There exists a big vertex b_1 adjacent to the small vertex s_1 with $L(s_1) \neq L(b_1)$. Similarly, there exists a big vertex b_2 adjacent to the small vertex s_2 with $L(s_2) \neq L(b_2)$.*

Assume the opposite, e.g., that all the big vertices adjacent to s_1 have the list $L(s_1)$ (the other part of the claim is symmetric). Since s_1 is adjacent to all the big vertices, all the big vertices have the same list. Let b and b' be two non-adjacent big vertices (they exist because the vertex s_1 is adjacent to all the six big vertices and G does not contain K_7 as a subgraph). Note that $L(b) = L(b')$. Let b_0 be a big vertex different from the vertices b and b' which has a neighbor in S_0 (such a vertex b_0 exists by Claim 30.1).

Color now the vertices b and b' by the same color $\alpha \in L(b) = L(b')$. Extend this coloring to S_0; such an extension exists by Proposition 24 (recall that b_0 is yet uncolored). Color properly the remaining four big vertices by colors from their lists. Note that this is possible since each big vertex has degree seven and it is adjacent to both the vertices s_1 and s_2 which are not colored. Finally, color properly the vertices s_1 and s_2 from their lists. We can do this because the vertices s_1 and s_2 have degree six and two of their neighbors, namely the vertices b and b', are colored with the same color. Therefore, we obtain an L-coloring of G — contradiction.

Claim 30.3 *There exists exactly one big vertex b_1 adjacent to the small vertex s_1 with $L(s_1) \neq L(b_1)$. Similarly, there exists exactly one big vertex b_2 adjacent to the small vertex s_2 with $L(s_2) \neq L(b_2)$.*

Assume for contradiction that the vertex s_1 is adjacent to at least two big vertices with their lists different from the list $L(s_1)$ (the other part of the claim is symmetric). Let b_2 be a big vertex adjacent to the small vertex s_2 with $L(s_2) \neq L(b_2)$ (such a big vertex exists by Claim 30.2). By our assumption, there exists a big vertex $b_1 \neq b_2$ adjacent to the small vertex s_1 with $L(s_1) \neq L(b_1)$. By Claim 30.1, there exists a big vertex b_0 which is different from the vertices b_1 and b_2 and which is adjacent to a vertex of S_0.

Color the vertex b_2 by a color $\alpha_2 \in L(b_2) \setminus L(s_2)$. Let $\alpha_1 \in L(b_1) \setminus L(s_1)$. If $\alpha_1 \neq \alpha_2$, then color the vertex b_1 by the color α_1. If $\alpha_1 = \alpha_2$, color the vertex b_1 by any color from its list different from α_1. Note that both s_1 and s_2 are adjacent to a big vertex colored with the color not contained in the lists $L(s_1)$ and $L(s_2)$, respectively. By Proposition 24 (note that the vertex b_0 is still not colored), we can extend this coloring to the vertices of S_0.

Color now properly the remaining four big vertices by colors from their lists. This is possible since each big vertex has degree seven and it is adjacent to both the vertices s_1 and s_2 which are not colored. Finally, color properly the vertices s_1 and s_2 from their lists. Note that the vertices s_1 and s_2 have degree six and each of them has a neighbor colored with a color not contained in its list. In this way, we obtain an L-coloring of G — contradiction.

Claim 30.4 *The graph G has an L-coloring.*

By Claim 30.3, the vertices s_1 and s_2 are adjacent to exactly one big vertex with a list different from $L(s_1)$ and $L(s_2)$, respectively. Hence, $L(s_1) = L(s_2)$. Let b be now the unique big vertex with $L(b) \neq L(s_1)$. Let b_0 be a big vertex different from the vertex b which is adjacent to a vertex of S_0 (such a vertex b_0 exists by Claim 30.1).

Color the vertex b with the color $\alpha \in L(b) \setminus L(s_1) = L(b) \setminus L(s_2)$. By Proposition 24 (the vertex b_0 is still not colored), we can extend this coloring to the vertices of S_0. Color properly the remaining five big vertices by colors from their lists. Note that this is possible since each big vertex has degree seven and it is adjacent to both the vertices s_1 and s_2 which are not colored yet. Finally, color properly the vertices s_1 and s_2 from their lists. We can do this because both the vertices s_1 and s_2 are adjacent to the vertex b colored with the color $\alpha \notin L(s_1)$ and $L(s_1) = L(s_2)$. In this way, we obtain a proper L-coloring of G — contradiction.

∎

7 Dirac's Map-Color Theorem for Choosability for the surface Π_3

We are now ready to prove the main theorem of this paper:

Theorem 31 *Let G be a graph of Euler genus three which does not contain K_7 as a subgraph. Then, G is 6-choosable.*

Proof: By Lemma 9, it is enough to prove the theorem for triangulations of the surface Π_3 with minimum simple degree six which do not contain K_7 as a subgraph and in which the small vertices induce Gallai forests. Let us consider an arbitrary triangulation G of Π_3 with these properties. By Theorem 4, G contains a vertex with simple degree at least seven because G is a triangulation, in particular, it is 2-connected.

Let F be the Gallai forest induced by the small vertices in G. By Lemma 19, the number k of the components of F is at most three. If $k = 0$, the graph G contains no small vertices and the number of its vertices is at most six. Hence, G is 6-choosable. If $k = 1$, then G is 6-choosable by Lemma 26. If $k = 3$, then two of the components of F are isomorphic to K_1 by Lemma 21 and so G is 6-choosable by Lemma 30. It remains to consider the case that F consists of precisely two components, say H_1 and H_2. By Lemma 20, at least one of the following holds:

- H_1 or H_2 is isomorphic to K_1,
- both H_1 and H_2 are cliques of order between two and five, or
- H_1 is a clique of order between two and five, H_2 contains a vertex of degree one (or vice versa) and G contains precisely six big vertices.

In the first case, G is 6-choosable by Lemma 27, in the second case, it is 6-choosable by Lemma 28 and in the last case, it is 6-choosable by Lemma 29. This completes the proof of Theorem 31.

■

We can combine Theorems 2 and 31 to get the following:

Theorem 32 *If G is a graph embedded on a surface of Euler genus $\varepsilon \geq 1$, then the choice number of G is at most $H(\varepsilon)$ and the equality holds if and only if G contains $K_{H(\varepsilon)}$ as a subgraph.*

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