Rainbow Ramsey Theory

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Abstract

This paper presents an overview of the current state in research directions in the rainbow Ramsey theory. We list results, problems, and conjectures related to existence of rainbow arithmetic progressions in $[n]$ and $\mathbb{N}$. A general perspective on other rainbow Ramsey type problems is given.

1 Introduction

Ramsey theory can be described as the study of unavoidable regularity in large structures. In the words of T. Motzkin, "complete disorder is impossible" [16]. In [22], we started a new trend, which can be categorized as the *rainbow* Ramsey theory. We are interested in the existence of rainbow/heterochromatic structures in a colored universe, under certain density conditions on the coloring. The general goal is to show that *complete disorder is unavoidable as well.*

Previous work regarding the existence of rainbow structures in a colored universe has been done in the context of *canonical* Ramsey theory (see [11, 10, 9, 33, 31, 25, 27, 26, 35] and references therein). However, the canonical theorems prove the existence of *either* a monochromatic structure *or* a rainbow

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structure. The results obtained in [22, 23, 5, 14] are not "either-or"–type statements and, thus, are the first results in the literature guaranteeing solely the existence of rainbow structures in colored sets of integers. In a sense, the conjectures and theorems we describe below can be thought of as the first rainbow counterparts of classical theorems in Ramsey theory, such as van der Waerden's, Rado's and Szemerédi's theorems [44, 43, 17, 24]. It is curious to note that rainbow Ramsey problems have received great attention in the context of graph theory (see [12, 8, 2, 4, 36, 13, 6, 29, 20, 3, 28, 21] and references therein).

2 Rainbow arithmetic progressions in $[n]$ and $\mathbb{N}$

In 1916, Schur [39] proved that for every k, if n is sufficiently large, then every k-coloring of $[n] := \{1, \ldots, n\}$ contains a monochromatic solution of the equation $x + y = z$. More than seven decades later, Alekseev and Savchev [1] considered what Bill Sands calls an *un-Schur* problem [18]. They proved that for every equinumerous 3-coloring of $[3n]$ (i.e., a coloring in which different color classes have the same cardinality), the equation $x + y = z$ has a solution with x, y and z belonging to different color classes. Such solutions will be called *rainbow* solutions. E. and G. Szekeres asked whether the condition of equal cardinalities for three color classes can be weakened [42]. Indeed, Schönheim [38] proved that for every 3-coloring of $[n]$, such that every color class has cardinality greater than $n/4$, the equation $x + y = z$ has rainbow solutions. Moreover, he showed that $n/4$ is optimal.

Inspired by the problem above, the second author posed the following conjecture at the open problem session of the 2001 MIT Combinatorics Seminar [22], which was subsequently proved by the authors in [23].

Theorem 1 (Conjectured in [22], proved in [23].) For every equinumerous 3-coloring of $[3n]$, there exists a rainbow $AP(3)$, that is, a solution to the equation $x + y = 2z$ in which x, y, and z are colored with three different colors.

In [22], Fox [1], Mahdian, and the authors proved the following *infinite* version

[1] then Licht

of Theorem 1.

Theorem 2 [22] Let $c : \mathbb{N} \mapsto \{R, G, B\}$ be a 3-coloring of the set of natural numbers with colors Red, Green, and Blue, satisfying the following density condition

$$\limsup_{n\to\infty} \left(\min(\mathcal{R}_c(n), \mathcal{G}_c(n), \mathcal{B}_c(n)) - n/6\right) = +\infty,$$

where $\mathcal{R}_c(n)$ is the number of integers less than or equal to n that are colored red, and $\mathcal{G}_c(n)$ and $\mathcal{B}_c(n)$ are defined similarly. Then c contains a rainbow $AP(3)$.

Basically Theorem 2 states that every 3-coloring of the set of natural numbers with the upper density of each color greater than $1/6$ contains a rainbow $AP(3)$.
Based on the computer evidence and the intuitive belief that the *finite* version of Theorem 2 should be true as well, in [22], we posed as a conjecture the following stronger form of Theorem 1, which has been recently confirmed by Axenovich and Fon-Der-Flaass [5].

Theorem 3 (Conjectured in [22], proved in [5].) For every $n \geq 3$, every partition of $[n]$ into three color classes $\mathcal{R}$, $\mathcal{G}$, and $\mathcal{B}$ with $\min(|\mathcal{R}|, |\mathcal{G}|, |\mathcal{B}|) > r(n)$, where

$$r(n) := \begin{cases} \lfloor (n+2)/6 \rfloor & \text{if } n \not\equiv 2 \pmod 6 \\ (n+4)/6 & \text{if } n \equiv 2 \pmod 6 \end{cases} \tag{1}$$

contains a rainbow $AP(3)$.

The following coloring of $\mathbb{N}$:

$$c(i) := \begin{cases} B & \text{if } i \equiv 1 \pmod 6 \\ G & \text{if } i \equiv 4 \pmod 6 \\ R & \text{otherwise} \end{cases}$$

contains no rainbow $AP(3)$ and $\min(\mathcal{R}_c(n), \mathcal{G}_c(n), \mathcal{B}_c(n)) = \lfloor (n+2)/6 \rfloor$, hence showing that Theorem 2 is the best possible. Clearly, for $n \not\equiv 2 \pmod 6$, this coloring shows that Theorem 3 is tight as well. As for the remaining case (when $n = 6k+2$ for an integer k), we define a coloring c as follows:

$$c(i) := \begin{cases} B & \text{if } i \le 2k+1 \text{ and } i \text{ is odd} \\ G & \text{if } i \ge 4k+2 \text{ and } i \text{ is even} \\ R & \text{otherwise.} \end{cases}$$

Since every blue number is at most $2k+1$, and every green number is at least $4k+2$, a blue and a green number cannot be the first and the second, or the second and the third terms of an arithmetic progression with all terms in $[n]$. Also, since blue numbers are odd and green numbers are even, a blue and a green cannot be the first and the third terms of an arithmetic progression. Therefore, c does not contain any rainbow $AP(3)$. It is not difficult to see that c contains no rainbow $AP(3)$ and $\min(\mathcal{R}_c(n), \mathcal{G}_c(n), \mathcal{B}_c(n)) = k+1 = (n+4)/6$.

The existing proofs of Theorems 1, 2, and 3 use a fact that every rainbow-free coloring contains a *dominant color*, that is, a color x such that for every two consecutive numbers that are colored with different colors, one of them is colored with x. The rest is to show that under certain density conditions the dominant color is not excessively dominant, so a rainbow $AP(3)$ exists.

One way to generalize Theorems 1 and 3 is to increase the number of colors and the length of a rainbow AP.

Axenovich and Fon-Der-Flaass came up with a construction for $k \ge 5$, that no matter how large the smallest color class is, there is a coloring with no rainbow $AP(k)$. Their construction is as follows [5].

Let $n = 2mk$, $k \ge 5$. We subdivide $[n]$ into k consecutive intervals of length $2m$ each, say $A_1, \dots, A_k$ and let $t = \lfloor k/2 \rfloor$. Then,

$$c(i) = \begin{cases} j-1 & \text{if } i \in A_j \text{ and } j \ne t, j \ne t+2 \\ t-1 & \text{if } i \in A_t \cup A_{t+2} \text{ and } i \text{ is even,} \\ t+1 & \text{if } i \in A_t \cup A_{t+2} \text{ and } i \text{ is odd.} \end{cases}$$

It is easy to see that the above coloring does not contain any rainbow $AP(k)$ and the size of each color class is n/k. For example, the coloring c in the case $n = 60$, $k = 5$, $m = 6$, $t = 2$, is as follows.

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However, the case $k = 4$ is still unresolved.

Problem 1 *Is it true that for sufficiently large values of n every equinumerous 4-coloring of $[4n]$ contains a rainbow $AP(4)$?*

Axenovich and Fon-Der-Flaass [5] provide a coloring c of $[n]$, where $n = 10m+1$ with the smallest color class of size $(n-1)/5$ and no rainbow $AP(4)$. This improves the previously known coloring [22], where the smallest color class had size $\lfloor \frac{n+2}{6} \rfloor$. Let $[n] = A_1 \cup \ldots \cup A_5$, where $A_1, \ldots, A_5$ are consecutive intervals of lengths $2m, 2m, 2m+1, 2m, 2m$ respectively. Then

$$c(i) = \begin{cases} 0 & \text{if } i \in A_1 \cup A_2 \text{ and } i \text{ is odd,} \\ 3 & \text{if } i \in A_4 \cup A_5 \text{ and } i \text{ is even,} \\ 1 & \text{if } i \in A_1 \text{ and } i \text{ is even,} \\ 1 & \text{if } i \in A_5 \text{ and } i \text{ is odd,} \\ 2 & \text{otherwise.} \end{cases}$$

When $n = 61$, $m = 6$, the coloring c is as follows.

0101010101010202020202022222222222222323232323232313131313131

Sicherman [41] has found equinumerous colorings of $[n]$ for $n \leq 60$ without rainbow $AP(4)$. A few of such colorings for $n = 60$ are shown below.

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000000000000012101202121313122312333233122212331313133312221

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One is tempted to also generalize Theorem 2 and conjecture that any partition $\mathbb{N} = C_1 \cup C_2 \cup \ldots \cup C_k$ of the positive integers into k equinumerous color classes, contains a rainbow $AP(k)$. However, it is easy to verify that the following equinumerous colorings of $\mathbb{N}$ do not contain any rainbow $AP(5)$, and hence the generalization is not true for $k = 5, 6$ [22].

$$c_5(i) := \begin{cases} 0 & \text{if } i \equiv 1,3 \pmod{10} \\ 1 & \text{if } i \equiv 2,5 \pmod{10} \\ 2 & \text{if } i \equiv 4,8 \pmod{10} \\ 3 & \text{if } i \equiv 6,7 \pmod{10} \\ 4 & \text{if } i \equiv 9,0 \pmod{10} \end{cases}$$

$$c_6(i) := \begin{cases} 0 & \text{if } i \equiv 1,3 \pmod{12} \\ 1 & \text{if } i \equiv 2,4 \pmod{12} \\ 2 & \text{if } i \equiv 5,7 \pmod{12} \\ 3 & \text{if } i \equiv 6,8 \pmod{12} \\ 4 & \text{if } i \equiv 9,11 \pmod{12} \\ 5 & \text{if } i \equiv 10,0 \pmod{12} \end{cases}$$

Cs. Sándor [37] generalized this approach and showed the existence of equinumerous colorings of $\mathbb{N}$ in $k \geq 10$ colors that do not contain a rainbow $AP(k)$. His proof is a clever counting argument. One imitates the colorings c_5 and c_6 given above, i.e., shows the existence of an equinumerous coloring of $\mathbb{Z}_{2k} = A_1 \cup A_2 \cup \ldots \cup A_k$ without a rainbow $AP(k)$ and then extends the coloring modulo $2k$ to all of $\mathbb{N}$.

The number of all equinumerous colorings $\mathbb{Z}_{2k} = A_1 \cup A_2 \cup \ldots \cup A_k$ is $\frac{(2k)!}{2^k}$. Let d denote the difference of an $AP(k)$ in $\mathbb{Z}_{2k}$. Then, clearly, $(d, 2k)$, the greatest common divisor of d and $2k$ is at most 2. If $(d, 2k) = 1$, we have $\phi(2k)$ choices for d and for $(d, 2k) = 2$, we have $\phi(k)$ choices for d, where $\phi(n)$ is the standard Euler's function. There are also $2k$ choices for the initial term of an $AP(k)$. Additionally, notice that every equinumerous coloring of $\mathbb{Z}_{2k}$ with a rainbow $AP(k)$ contains at least two rainbow $AP(k)$'s. Indeed, if $a_1, a_2, \ldots, a_k$ form an $AP(k)$ in $\mathbb{Z}_{2k}$, then $\{b_1, b_2, \ldots, b_k\} := \mathbb{Z}_{2k} \setminus \{a_1, a_2, \ldots, a_k\}$ form one as well. Since the number of possible colorings of $\{a_1, a_2, \ldots, a_k\}$ and $\{b_1, b_2, \ldots, b_k\}$ is $(k!)^2$, we conclude that the number of equinumerous colorings that *do* contain a rainbow $AP(k)$ is at most

$$\frac{1}{2} 2k(\phi(2k) + \phi(k))(k!)^2,$$

which is strictly less than $\frac{(2k)!}{2^k}$ for $k \geq 10$. This completes the proof.

If the number of colors is infinite, the following proposition shows that one cannot guarantee even the existence of a rainbow $AP(3)$ with the assumption that each color has a positive density.

Proposition 1 *[22] There is a coloring of $\mathbb{N}$ with infinitely many colors, with each color having positive density such that there is no rainbow $AP(3)$.*

Proof: For each $x \in \mathbb{N}$, let $c(x)$ be the largest integer k such that x is divisible by 3^k. It is easy to see that the color k has density $2 \cdot 3^{-k-1} > 0$ in this coloring. Also, if $c(x) \neq c(y)$, it is not difficult to see that $c(2y-x) = c(2x-y) = c((x+y)/2) = \min(c(x), c(y))$. Therefore, if two elements of an arithmetic progression are colored with two different colors, the third term must be colored with one of those two colors. Thus, there is no rainbow $AP(3)$ in c. $\square$

3 Rainbow arithmetic progressions in $\mathbb{Z}_n$ and $\mathbb{Z}_p$

An interesting corollary of Theorem 2 is a *modular* variant, which states that if $\mathbb{Z}_n$ is colored with 3 colors such that the size of every color class is greater than $n/6$, then there exist x, y and z, each of a different color with $x+y \equiv 2z \pmod{n}$.

It turns out that in this case $n/6$ is not the best possible. If for any integer n, we let $m(n)$ be the largest integer m for which there is a rainbow-free 3-coloring c of $\mathbb{Z}_n$ with $|\mathcal{R}|, |\mathcal{G}|, |\mathcal{B}| \geq m$, then we can show that $m(n) \leq \min(n/6, n/q)$, where q is the smallest prime factor of n [22]. Additionally, for every integer n that is not a power of 2, $m(n) \geq \lfloor \frac{n}{2r} \rfloor$, where r denotes the smallest odd prime factor of n.

Finally, Licht, Mahdian, and the authors characterize the set of natural numbers n for which $m(n) = 0$.

Theorem 4 [22] For every integer n, there is a rainbow-free 3-coloring of $\mathbb{Z}_n$ with non-empty color classes if and only if n does not satisfy any of the following conditions:

(a) n is a power of 2.

(b) n is a prime and $\mathrm{ord}_n(2) = n-1$ (that is, 2 is a generator of $\mathbb{Z}_n$).

(c) n is a prime, $\mathrm{ord}_n(2) = (n-1)/2$, and $(n-1)/2$ is an odd number.

Computing the exact value of $m(n)$ for every n remains a challenge. We state the following conjecture:

Conjecture 1 *Let n be an integer which is not a power of 2. Let $p_1 \leq p_2 \leq \ldots$ be the odd prime factors of n which have 2 as a generator or* $\mathrm{ord}_2(p_i) = (p_i - 1)/2$ *is odd. Let $q_1 \leq q_2 \leq \ldots$ be the remaining odd prime factors of n. Then $m(n) = \lfloor \frac{n}{\min\{2p_1, q_1\}} \rfloor$.*

Strong inverse theorems from additive number theory have proved to be useful tools in Ramsey theory. For example, Gowers' proof of Szemerédi's theorem relies on the theorem of Freiman [30, 15]. Likewise, in [22], we used a recent theorem of Hamidoune and Rødseth [19], generalizing the classical Vosper's theorem [45], to prove that *almost* every coloring of $\mathbb{Z}_p$ with three colors has rainbow solutions for *almost* all linear equations in three variables in $\mathbb{Z}_p$. Moreover, we classified all the exceptions.

Theorem 5 [22] Let $a, b, c, e \in \mathbb{Z}_p$, with $abc \not\equiv 0 \pmod p$. Then every coloring of $\mathbb{Z}_p = \mathcal{R} \cup \mathcal{B} \cup \mathcal{G}$ with $|\mathcal{R}|, |\mathcal{B}|, |\mathcal{G}| \geq 4$, contains a rainbow solution of $ax + by + cz \equiv e \pmod p$ with the only exception being the case when $a = b = c =: t$ and every color class is an arithmetic progression with the same common difference d, so that $d^{-1}\mathcal{R} = \{i\}_{i=a_1}^{a_2-1}$, $d^{-1}\mathcal{B} = \{i\}_{i=a_2}^{a_3-1}$ and $d^{-1}\mathcal{G} = \{i\}_{i=a_3}^{a_1-1}$, where $(a_1 + a_2 + a_3) \equiv t^{-1}e + 1$ or $t^{-1}e + 2 \pmod p$.

4 Other generalizations and directions for future research

There are many more directions and generalizations one can consider, such as searching for rainbow counterparts of other classical theorems in Ramsey theory [17], or proving the existence of more than one rainbow AP. Some results in these directions were obtained in [22] and [14].

One natural direction is generalizing the problems above for rainbow solutions of any linear equation, imitating Rado's theorem about the monochromatic analogue [34]. The following theorem considering the Sidon equation, a classical object in additive number theory [7, 32], is an example of this.

Theorem 6 [14] Every coloring of $[n]$ in four colors: red, blue, green and yellow, such that

$$\min\{|\mathcal{R}|, |\mathcal{B}|, |\mathcal{G}|, |\mathcal{Y}|\} > \frac{n+1}{6}$$

contains a rainbow solution of $x + y = z + w$.

It is curious to note that the minimal "density" for the color classes is $\frac{1}{6}$ again.

The question of *rainbow partition regularity* is an interesting one. We can generalize our discussion from rainbow solutions to one equation to rainbow solutions to a system of equations. A matrix A with integer entries is called *partition regular* if whenever the natural numbers are finitely colored there is a monochromatic vector x with $Ax = 0$. Richard Rado's thesis 70 years ago classified the partition regular matrices [34]. We say a vector is *rainbow* if every entry of the vector is colored differently. A matrix A with rational entries is called *rainbow partition k-regular* if for all n and every equinumerous k-coloring of $[kn]$ there exists a rainbow vector x such that $Ax = 0$. We say that A is *rainbow regular* if there exists k_1 such that A is rainbow partition k-regular for all $k \geq k_1$. For example, Theorem 6 shows that the following matrix is rainbow partition 4-regular:

$$A = \begin{pmatrix} 1 & 1 & -1 & -1 \end{pmatrix}$$

We let the *rainbow number* of A, denoted by $r(A)$, be the least k for which A is rainbow partition k-regular. It is not difficult to see that for all $1 \times n$ matrices A with nonzero entries, A is rainbow regular if and only if not all the entries in A are of the same sign.

In general, we conjecture the following characterization of the rainbow regularity.

Conjecture 2 *Matrix A with integer entries is rainbow regular if and only if there exists two independent vectors x_1, x_2 with positive integer entries, such that $Ax_1 = Ax_2 = 0$.*

Furthermore, it would be interesting to study the rainbow number $r(A)$.

A search for a rainbow counterpart of the Hales-Jewett theorem, though an exciting possibility, led us to some negative results [22]. First, let us recall some standard terminology [17].

The n-cube over t elements C_t^n is defined by

$$C_t^n = \{(x_1, \ldots, x_n) : x_i \in \{0, 1, \ldots, t-1\}\}.$$

A *combinatorial line* in C_t^n is then a sequence of points, $\mathbf{x_0}, \ldots, \mathbf{x_{t-1}}$, with $\mathbf{x_i} = (x_{i1}, \ldots, x_{in})$ so that in each coordinate j, $1 \le j \le n$, either

$$x_{0j} = x_{1j} = \ldots = x_{t-1,j}$$

or

$$x_{sj} = s \qquad \text{for } 0 \le s \le t-1,$$

and the latter occurs for at least one j.

Theorem 7 (Hales-Jewett Theorem) *For all r, t there exists $N = HJ(r,t)$ so that for $n \ge N$, the following holds: If the vertices of C_t^n are r-colored, then there exists a monochromatic combinatorial line.*

This motivates the following question: Is it true that for every equinumerous t-coloring of C_t^n there exists a rainbow combinatorial line? The following coloring shows that the answer is negative even for small values of t and n. A 3-coloring of C_3^3 defined by

$$C_1 = \{000, 002, 020, 200, 220, 022, 202, 222, 001\},$$

$$C_2 = \{011, 021, 101, 201, 111, 221, 010, 210, 012\}, \text{and}$$

$$C_3 = \{100, 110, 120, 121, 211, 102, 112, 122, 212\}$$

(parentheses and commas being removed for clarity), has no rainbow combinatorial lines. Indeed, suppose that $\mathbf{x_0}$, $\mathbf{x_1}$, $\mathbf{x_2}$ is a rainbow combinatorial line. Suppose that $\mathbf{x_0}$ is colored by C_1. Then $x_{0,1} \in \{0, 2\}$. Assume $x_{0,1} = 0$. Then, either $x_{1,1} = x_{2,1} = 0$ or $x_{1,1} = 1$, $x_{2,1} = 2$. In the former case neither $\mathbf{x_1}$ nor $\mathbf{x_2}$ is colored by C_3, which contradicts with $\mathbf{x_0}$, $\mathbf{x_1}$, $\mathbf{x_2}$ being rainbow. In the latter case, suppose that $\mathbf{x_1}$ is colored by C_2. Then $\mathbf{x_2}$ is colored by C_3. Hence, $\mathbf{x_1} \in \{101, 111\}$ and $\mathbf{x_2} \in \{212, 211\}$. It follows that either $\mathbf{x_1} = 111$ and $\mathbf{x_2} = 211$ or $\mathbf{x_1} = 111$ and $\mathbf{x_2} = 212$. Then $\mathbf{x_0} = 011$ or $\mathbf{x_0} = 010$. This contradicts with the assumption that $\mathbf{x_0}$ is colored by C_1. Other cases are handled similarly.

Yet another direction is limiting our attention to equinumerous colorings and letting the number of colors be different from the desired length of a rainbow AP. Let T_k denote the minimal number $t \in \mathbb{N}$ such that there is a rainbow $AP(k)$ in every equinumerous t-coloring of $[tn]$ for every $n \in \mathbb{N}$. Then, $T(3) = 3$ by Theorem 3. It is also clear that $T_k = r(A)$, where A is the following $(m-1) \times (m+1)$ matrix.

$$\begin{pmatrix} 1 & 1 & -1 & 0 & \dots & 0 & 0 \\ 1 & 0 & 1 & -1 & \dots & 0 & 0 \\ 1 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & \dots & 1 & -1 \end{pmatrix}$$

In [22], Licht, Mahdian, and the authors prove the following lower and upper bounds on T_k.

Proposition 2 [22] For every $k \geq 3$, $\lfloor \frac{k^2}{4} \rfloor < T_k \leq \frac{k(k-1)^2}{2}$.

Proof: The upper bound follows from a relatively simple counting argument. As for the lower bound, we will exhibit colorings c_1 and c_2, showing that $T_{2k+1} > k^2 + k$ and $T_{2k} > k^2$.

Let a j-*block* B_j ($j \in \mathbb{N}$) be the sequence $12 \dots j12 \dots j$, where the *left half* and the *right half* of the block are naturally defined. For $a \in \mathbb{Z}$, let $B_j + a$ be the sequence $(a+1)(a+2)\dots(a+j)(a+1)(a+2)\dots(a+j)$.

We define the coloring c_1 of $[2k^2 + 2k]$ in the following way (bars denoting endpoints of the blocks):

$$\Big| B_k^- \Big| \dots \Big| B_j^- \Big| \dots \Big| B_2^- \Big| B_1^- \Big| B_1^+ \Big| B_2^+ \Big| \dots \Big| B_i^+ \Big| \dots \Big| B_k^+ \Big|,$$

where $B_j^- = B_j - \binom{j+1}{2}$ and $B_i^+ = B_i + \binom{i}{2}$. Note that c_1 uses each of the $k^2 + k$ colors exactly twice.

The coloring c_2 of $[2k^2]$ is defined similarly:

$$\Big| B_{k-1}^- \Big| \dots \Big| B_j^- \Big| \dots \Big| B_2^- \Big| B_1^- \Big| B_1^+ \Big| B_2^+ \Big| \dots \Big| B_i^+ \Big| \dots \Big| B_k^+ \Big|,$$

thus using each of the k^2 colors exactly twice.

Next, we show that $[2k^2 + 2k]$, colored by c_1, does not contain a rainbow $AP(2k+1)$. The key observation is that a rainbow AP with common difference d cannot contain elements from opposite halves of any block B_j, where

d divides j. Fix a longest rainbow AP $\mathcal{A}$ and let d denote its common difference. If $d > k$, then the length of $\mathcal{A}$ is $\leq 2k$. If $d \leq k$, then $\mathcal{A}$ is one of the following three types:

1. $\mathcal{A}$ is contained in $\left|B_d^-\middle|B_{d-1}^-\middle|\ldots\middle|B_2^-\middle|B_1^-\middle|B_1^+\middle|B_2^+\middle|\ldots\middle|B_{d-1}^+\middle|B_d^+\right|$.

   Then $\mathcal{A}$ does not intersect either the left half of B_d^- or the right half of B_d^+. Hence, the length of $\mathcal{A}$ is at most $2d \leq 2k$.

2. $\mathcal{A}$ is contained in $\left|B_{(j+1)d}^-\middle|B_{(j+1)d-1}^-\middle|\ldots\middle|B_{jd}^-\right|$ or in $\left|B_{jd}^+\middle|B_{jd+1}^+\middle|\ldots\middle|B_{(j+1)d}^+\right|$, where $(j+1)d \leq k$.

   Assume that the first case occurs. Then $\mathcal{A}$ does not intersect either the left half of $B_{(j+1)d}^-$ or the right half of B_{jd}^-. Hence, the length of $\mathcal{A}$ is at most $\frac{1}{d}(jd + 2(jd+1) + 2(jd+2) + \ldots + 2(jd+d-1) + (jd+d) \leq 2(j+1)d \leq 2k$.

3. $\mathcal{A}$ is contained in $\left|B_{jd+x}^-\middle|B_{jd+x-1}^-\middle|\ldots\middle|B_{jd}^-\right|$ or in $\left|B_{jd}^+\middle|B_{jd+1}^+\middle|\ldots\middle|B_{jd+x}^+\right|$, where $jd + x < k$. Assume that the first case occurs. Then $\mathcal{A}$ does not intersect the right half of B_{jd}^-. Hence, the length of $\mathcal{A}$ is at most $\frac{1}{d}(jd + 2(jd+1) + 2(jd+2) + \ldots + 2(jd+x-1) + 2(jd+x)) \leq \frac{1}{d}(jd + 2jd(d-1) + d(x-1)) < 2(jd+x) < 2k$.

Similarly, one shows that $[2k^2]$, colored by c_2, does not contain a rainbow $AP(2k)$. $\square$

We conjectured that the asymptotic order of growth of T_k is quadratic in k.

Conjecture 3 [22] For all $k \geq 3$, $T_k = \Theta(k^2)$.

It is easy to show that the maximal number of rainbow $AP(3)$s over all equinumerous 3-colorings of $[3n]$ is $\lfloor 3n^2/2 \rfloor$, this being achieved for the unique 3-coloring with color classes $R = \{n|n \equiv 0 \pmod 3\}$, $B = \{n|n \equiv 1 \pmod 3\}$ and $G = \{n|n \equiv 2 \pmod 3\}$. It seems very difficult to characterize those equinumerous 3-colorings (in general, k-colorings) that minimize the number of rainbow $AP(3)$'s. Letting $f_k(n)$ denote the minimal number of rainbow $AP(k)$s, over all equinumerous k-colorings of $[kn]$, we pose the following conjecture.

Conjecture 4 [22] $f_3(n) = \Omega(n)$.

The following interpretation of Theorem 1 could be a possible approach to Conjecture 4.

Let $V = [3n]$ and let c be an equinumerous 3-coloring of V. Let K^c be the union of three copies of K_n so that K^c represents the coloring c. Let $\mathcal{F}$ be the family of all 2-paths with the vertex set $\{a, a+d, a+2d\} \subseteq V$. By Theorem 1, there is at least one $G \in \mathcal{F}$ with a placement in K^c, i.e., there is an injection from $V(G)$ to $V = V(K^c)$ that is a homomorphism from G to the complement of K^c. $f_3(n)$ is the minimal number of the elements of $\mathcal{F}$ with a placement in K^c over all equinumerous 3-colorings c of $[3n]$.

If we define $g_k(n)$ as the minimal number of rainbow $AP(k)$s, over all equinumerous k-colorings of $\mathbb{Z}_{kn}$, then a straightforward counting argument shows that $g_3(n) \geq n$, when n is odd.

Finally, the further generalization of Vosper's theorem, due to Serra and Zémor [40], may lead to a generalization of Theorem 5 for more than 3 color classes.

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