Open caps and cups in planar point sets*

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Abstract

A *configuration* is a finite set of points in the plane such that no 3 points lie on a line and no 2 points have the same x-coordinate. Let X be a configuration and let $p_1, \ldots, p_k$ be k points of X ordered according to the increasing x-coordinate. For $i = 1, \ldots, k-1$, let s_i be the slope of the line $p_i p_{i+1}$. The set $P = \{p_1, \ldots, p_k\}$ is a *k-cap* or a *k-cup*, if the sequence $s_1, s_2, \ldots, s_{k-1}$ is decreasing or increasing, respectively (see Fig. 1). The set $P = \{p_1, \ldots, p_k\}$ is *open (in X)*, if no point $p \in X$ with $x(p_1) < x(p) < x(p_k)$ lies above the polygonal line $p_1 p_2 \ldots p_k$. We prove that for every $k, l \geq 2$ there is an integer $f(k, l)$ such that any configuration of size $\geq f(k, l)$ contains an open k-cap or an open l-cup. This can be seen as a generalization of the Erdős-Szekeres theorem. It implies results on empty polygons in k-convex configurations proved by Károlyi *et al.* [5], Kun and Lippner [7], and Valtr [11] (a configuration is *k-convex*, if it determines no triangle with more than k points in the interior). Another immediate corollary is that for any $k, l \geq 2$ any sufficiently large configuration with no (open) k-cap contains an empty l-gon. We give double–exponential lower and upper bounds on $f(k, l)$.

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1 Introduction

A *configuration* is a finite set of points in the plane such that no 3 points lie on a line and no 2 points have the same x-coordinate. Let X be a configuration. We say that a subset of X is *in convex position*, if it is the vertex set of a convex polygon. Let $p_1, \ldots, p_k$ be k points of X ordered according to the increasing x-coordinate (i.e., $x(p_1) < x(p_2) < \cdots < x(p_k)$). For $i = 1, \ldots, k-1$, let s_i be the slope of the line $p_i p_{i+1}$, i.e. $s_i = (y(p_{i+1}) - y(p_i))/(x(p_{i+1}) - x(p_i))$. The set $P = \{p_1, \ldots, p_k\}$ is a *k-cap* or a *k-cup*, if the sequence $s_1, s_2, \ldots, s_{k-1}$ is decreasing or increasing, respectively (see Fig. 1). The set

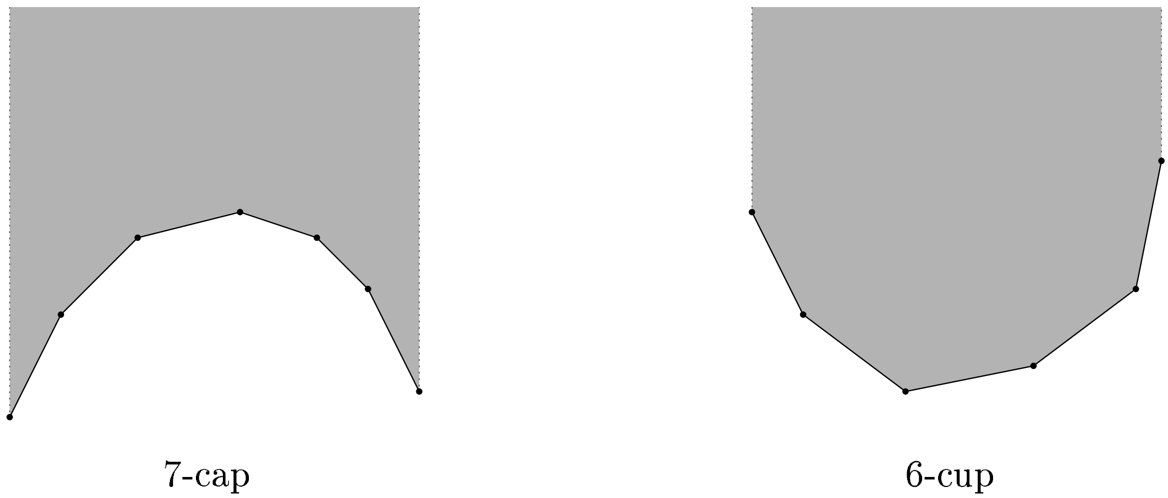

Figure 1: A 7-cap and a 6-cup; each of them is open if the unbounded shaded region contains no points of X.

$P = \{p_1, \ldots, p_k\}$ is *open (in X)*, if no point $p \in X$ with $x(p_1) < x(p) < x(p_k)$ lies above the polygonal line $p_1 p_2 \ldots p_k$.

In 1935 Erdős and Szekeres proved the following classical result:

Theorem 1 (Erdős and Szekeres) *For any $n \geq 3$, there is an integer $F(n)$ such that any set of at least $F(n)$ points in general position in the plane contains n points in convex position.*

Erdős and Szekeres [2] proved Theorem 1 by proving that any sufficiently large configuration contains a k-cap or an l-cup. Here we show the following generalization of this result:

Theorem 2 *For any $k, l \geq 2$, there is a (least) integer $f(k,l)$ such that any configuration of size $\geq f(k,l)$ contains an open k-cap or an open l-cup.*

Theorem 2 is a "generic" result with some interesting corollaries mentioned below. The function $f(k,l)$ is double–exponential, as shown in Theorem 3 below.

If X is a configuration, then a subset $P \subseteq X$ in convex position is called an *empty polygon (in X)*, if the interior of $\operatorname{conv} P$ contains no point of X. Answering a question of Erdős [1], Horton [4] constructed arbitrarily large configurations with no empty 7-gon. Harborth [3] showed that any configuration with more than 9 points contains an empty pentagon. It is a challenging open problem to prove or disprove that any sufficiently large configuration contains an empty hexagon. This problem is one of the motivations of our paper. Our results show some structural properties of configurations containing no empty polygons with many vertices. Thus, they might help in investigations of the empty–hexagon problem.

We say that a configuration X is *k-convex*, if the interior of every triangle determined by X contains at most k points of X. Theorem 2 implies the following result which shows that large k-convex configurations contain large empty polygons:

Corollary 1 (Valtr [11], Kun and Lippner [7]) *For any $k \geq 1$ and $l \geq 3$, there is a (least) integer $N(k,l)$ such that any k-convex configuration of size $\geq N(k,l)$ contains an empty l-gon.*

Corollary 1 in the special case $k = 1$ was first proved by Károlyi *et al.* [5]. In case $k = 1$, $N(1,l)$ is exponential in l [5] and its exact value is determined in [6]. For general k and l, our proof of Corollary 1 gives a double–exponential upper bound $N(k,l) \leq 2^{\binom{k+l}{k+2}-1}+1$, which is slightly better than the previous bound $N(k,l) \leq (k+2)^{(k+2)^l-1} \leq 2^{(k+2)^{l+1}}$ of Kun and Lippner [7]. No lower bound on $N(k,l)$ better than exponential in $k+l$ is known.

We asked in [11] if also any large configuration with no k-cap contains a large empty polygon. This is answered affirmatively by the following direct consequence of Theorem 2:

Corollary 2 *For any $k, l \geq 2$, there is a least integer $m(k,l)$ such that any configuration of size $\geq m(k,l)$ with no open k-cap contains an empty l-gon.*

A k-*moon* is a k-point configuration M with a specific point $a(M) \in M$ (called *the apex of* M) such that any 4-point subset of M is in convex position if and only if it does not contain $a(M)$. The combinatorial structure of a k-moon is unique for each $k \geq 3$ (see Fig. 2). A k-moon M is *empty (in X)*,

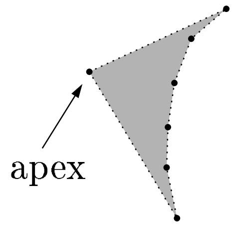

Figure 2: A 7-moon; it is empty in X if the shaded region contains no points of X.

if the interior of the region $\operatorname{conv} M \setminus \operatorname{conv}(M \setminus \{a(M)\})$ contains no point of X (see Fig. 2). Theorem 2 is equivalent to the following corollary:

Corollary 3 *For any $k, l \geq 3$, there is a least integer $z(k,l)$ such that any point p in any configuration X of size $\geq z(k,l)$ is the apex of an empty k-moon in X or it is one of the vertices of an empty l-gon in X.*

We obtain double-exponential lower and upper bounds on $f(k,l)$ and on two related functions:

Theorem 3 *Let $k, l \geq 2$ and let $f(k,l)$ be the number given by Theorem 2. Then:*

(i) If $f'(k,l)$ denotes the minimum integer such that any configuration of size $\geq f'(k,l)$ contains a k-cap or an open l-cup, then

$$2^{\binom{\lfloor k/2 \rfloor + \lfloor l/2 \rfloor - 2}{\lfloor k/2 \rfloor - 1}} \leq f'(k,l) \leq f(k,l) \leq 2^{\binom{k+l-2}{k-1}-1},$$

(ii) if $f''(k,l)$ denotes the minimum integer such that any configuration of size $\geq f''(k,l)$ contains an open k-cap or an l-cup, then

$$2^{\binom{\lfloor k/2 \rfloor + \lfloor (l-1)/2 \rfloor - 2}{\lfloor k/2 \rfloor - 1}} \leq f''(k,l) \leq f(k,l).$$

If neither the k-cap nor the l-cup are required to be open, then the extremal function is exponential: If $f'''(k,l)$ denotes the minimum integer such that any configuration of size $\geq f'''(k,l)$ contains a k-cap or an l-cup, then

$f'''(k,l) = \binom{k+l-4}{k-2} + 1$ (this was shown by Erdős and Szekeres [2] about 70 years ago).

Our proof of Theorem 2 uses some of the ideas of Kun and Lippner [7]. It has also some similarities with one of the original proofs of the Erdős–Szekeres theorem (Theorem 1).

OPEN PROBLEMS. Corollaries 1–3 give the (affirmative) answer to the following problem for some special types of subconfigurations:

Problem 1 *Which types of subconfigurations are contained in any sufficiently large configuration with no empty l-gon ($l \geq 6$ fixed)?*

If the sought type of subconfiguration is the empty hexagon then for each $l \geq 6$, Problem 1 is equivalent to the well–known empty–hexagon problem [1].

Problem 2 (empty–hexagon problem) *Does any sufficiently large configuration contain an empty hexagon?*

Since the so-called Horton sets (e.g. see [9, 10, 8]) contain no empty 7-gons, Problem 1 can have an affirmative answer for $l > 6$ only for types of configurations contained in all sufficiently large Horton sets.

2 Proof of Theorem 2

We write $C = c_1c_2\dots c_k$, if $C = \{c_1, c_2, \dots, c_k\}$ is a k-cap (or a k-cup) with $x(c_1) < x(c_2) < \dots < x(c_k)$. If $x(p) < x(q)$ then we say that p lies *to the left of* q and q lies *to the right of* p. If $x(p) < x(q) < x(r)$ then we say that q lies *between* p *and* r.

The *left strip* of an l-cup $D = d_1d_2\dots d_l$ is the vertical strip $L(D) = \{p \in \mathbb{R}^2 : x(d_1) < x(p) < x(d_2)\}$ (see Fig. 3). A *right strip* of a k-cap $C = c_1c_2\dots c_k$ is any vertical strip $R(C,w) = \{p \in \mathbb{R}^2 : x(c_k) < x(p) < x(c_k) + w\}$, where $w \in \mathbb{R}^+$. We further define $R^-(C,w)$ as the set of points of $R(C,w)$ lying below the line $c_{k-1}c_k$ (see Fig. 3).

For $\varepsilon > 0$ and for a configuration X, we say that an l-cup D, $D \subseteq X$, is *ε-good (in X)*, if it is open and $|X \cap L(D)| \geq \varepsilon|X| - 1$ (see Fig. 4). We say that a k-cap C, $C \subseteq X$, is *ε-good (in X)*, if it is open and there is a $w > 0$ such that $|X \cap R(C,w)| = |X \cap R^-(C,w)| \geq \varepsilon|X| - 1$ (see Fig. 4).

For $k, l \geq 2$, we recursively define a parameter $\varepsilon(k,l) \in (0, 1/2]$ as follows:

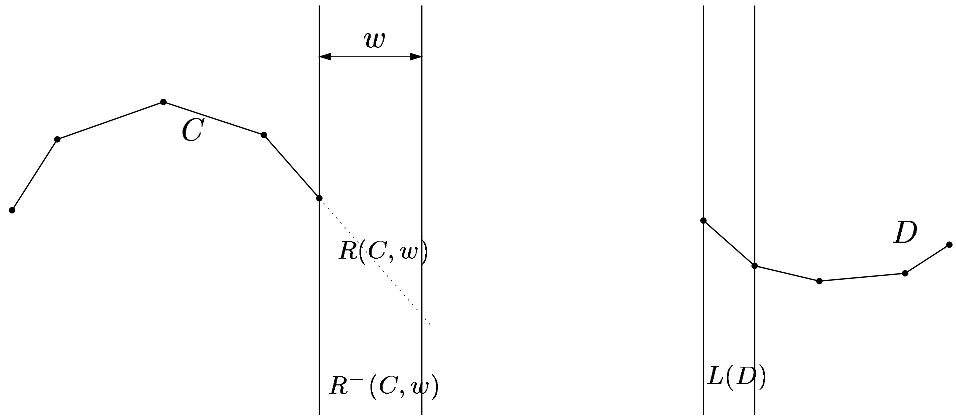

Figure 3: The regions $R(C,w), R^-(C,w)$ and $L(D)$.

(i) $\varepsilon(i,2) = \varepsilon(2,i) = 1/i$ for $i \geq 2$,

(ii) $\varepsilon(k,l) = \varepsilon(k-1,l) \cdot \varepsilon(k,l-1)/2$ for $k,l \geq 3$.

We prove Theorem 2 by proving the following statement:

($*$) For any $k,l \geq 2$, any configuration X of size at least $1/\varepsilon(k,l)$ contains an $\varepsilon(k,l)$-good k-cap or an $\varepsilon(k,l)$-good l-cup.

First we verify ($*$) for $k=2, l \geq 2$ and for $l=2, k \geq 2$, and then continue by induction on $k+l$. The case $m=l$ in the following lemma gives ($*$) for $k=2, l \geq 2$.

Lemma 1 *Let X be a configuration of size at least $l \geq 2$ with no $1/l$-good 2-cap. Then for each $m = 2,3,\ldots,l$, the set X contains a $1/l$-good m-cup $D_m = d_1 d_2 \ldots d_m$ such that fewer than $m|X|/l$ points of X lie to the left of d_m.*

Proof. First let $m=2$. Let d_1 be the leftmost point of X, and let X_0 be the set of $\lceil 2|X|/l \rceil$ leftmost points of X. Further, let d_2 be the point viewed from d_1 as the highest point of $X_0 \setminus \{d_1\}$. Thus, $d_2 \in X_0 \setminus \{d_1\}$ and no point of X_0 lies above the line $d_1 d_2$ (see Fig. 5). Since the open 2-cap $d_1 d_2$ is not $1/l$-good in X, fewer than $|X|/l - 1$ points of X_0 lie to the right of d_2. It follows that at least $|X|/l - 1$ points of X_0 lie between d_1 and d_2 (recall that the size of X_0 is $\lceil 2|X|/l \rceil$). Thus, the 2-cup $D_2 = d_1 d_2$ is $1/l$-good in X and fewer than $2|X|/l$ points of X lie to the left of d_2. This finishes the case $m=2$.

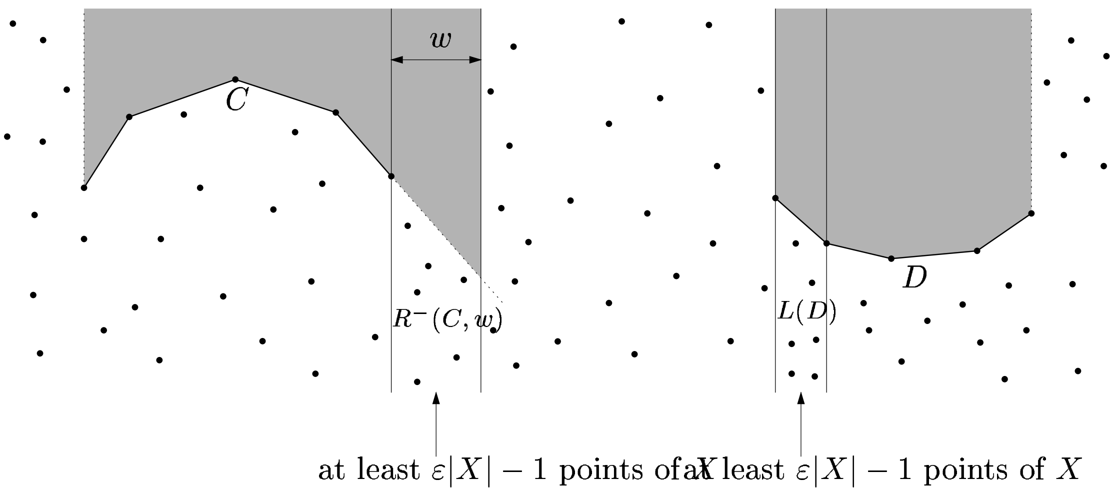

Figure 4: An ε-good 5-cap and an ε-good 5-cup (the shaded regions contain no point of X).

We further proceed by induction on m. Let $2 \leq m < l$ and let $D_m = d_1d_2\dots d_m$ be a $1/l$-good m-cup $D_m = d_1d_2\dots d_m$ such that fewer than $m|X|/l$ points of X lie to the left of d_m. Let d_{m+1} be the leftmost point of X lying to the right of d_m and above the line $d_{m-1}d_m$ (see Fig. 6). Such a point d_{m+1} exists, since otherwise the 2-cap $d_{m-1}d_m$ would be $1/l$-good (at least $|X| - m|X|/l - 1 \geq |X|/l - 1$ points of X lie to the right of d_m). For the same reason fewer than $|X|/l - 1$ points of X lie between d_m and d_{m+1}. It follows that fewer than $m|X|/l + 1 + (|X|/l - 1) = (m+1)|X|/l$ points of X lie to the left of d_{m+1}. Since the $(m+1)$-cup $D_{m+1} = d_1d_2\dots d_{m+1}$ is $1/l$-good, this finishes the inductive step from m to $m+1$. $\square$

We now prove $(*)$ for $l = 2, k \geq 2$. Let $|X| \geq k$, and let $p_1, p_2, \dots, p_t$, $x(p_1) < x(p_2) < \dots < x(p_t)$, be the vertices of the upper envelope of $\operatorname{conv} X$ (see Fig. 7). If X contains $\geq |X|/k - 1$ points p with $x(p_i) < x(p) < x(p_{i+1})$ for some i, then the 2-cup p_ip_{i+1} is $1/k$-good. Otherwise $t > k$ and there are more than $|X|/k - 1$ points of X to the right of p_k. Thus, $p_1p_2\dots p_k$ is a $1/k$-good k-cap in this case.

It remains to derive $(*)$ for $k, l \geq 3$ from $(*)$ for all $k', l' \geq 2, k' + l' < k + l$. Suppose that $k, l \geq 3$ and that $|X| \geq 1/\varepsilon(k,l)$. We want to prove that X contains an $\varepsilon(k,l)$-good k-cap or an $\varepsilon(k,l)$-good l-cup. By the inductive hypothesis, X contains an $\varepsilon(k-1,l)$-good $(k-1)$-cap or an $\varepsilon(k-1,l)$-good l-cup. In the second case we are done, since $\varepsilon(k-1,l) > \varepsilon(k,l)$. Thus,

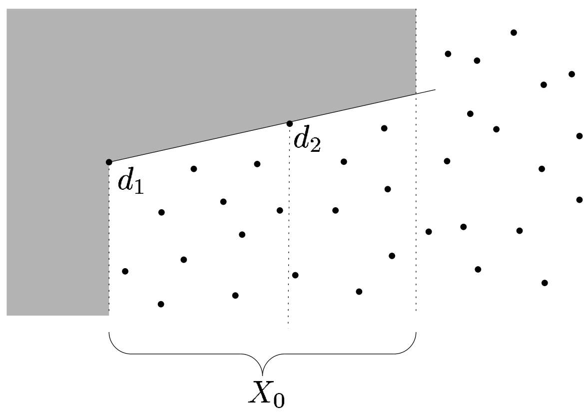

Figure 5: The points d_1, d_2 (the shaded region contains no point of X).

assume that X contains an $\varepsilon(k-1,l)$-good $(k-1)$-cap $C = c_1c_2\dots c_{k-1}$. Let X_0 be the set of $\lceil \varepsilon(k-1,l)|X|-1\rceil$ leftmost points of X lying to the right of c_{k-1}. Since C is $\varepsilon(k-1,l)$-good, all points of X_0 lie under the line $c_{k-2}c_{k-1}$.

Since $\lceil \alpha - 1\rceil \geq (\lceil \alpha/4\rceil - 1) + (\lceil \alpha/4\rceil - 2) + \lceil \alpha/2\rceil$ holds[1] for any $\alpha \in \mathbb{R}$, we can partition the set X_0 into three subsets R, S, T of sizes

$$\begin{aligned}
|R| &= \left\lceil \frac{\varepsilon(k-1,l)}{4}|X| \right\rceil - 1 > 2\cdot\varepsilon(k,l-1)\cdot\frac{\varepsilon(k-1,l)}{4}|X| - 1 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \varepsilon(k,l)|X| - 1, \\
|S| &= \left\lceil \frac{\varepsilon(k-1,l)}{4}|X| \right\rceil - 2 > \varepsilon(k,l)|X| - 2, \\
|T| &\geq \left\lceil \frac{\varepsilon(k-1,l)}{2}|X| \right\rceil
\end{aligned}$$

(in the estimate of $|R|$ we used that $2\cdot\varepsilon(k,l-1) < 1$ for any $k,l \geq 3$). The partition is done "from left to right" so that $x(r) < x(s) < x(t)$ holds for any $r \in R, s \in S, t \in T$.

Since

$$|T| \geq \frac{\varepsilon(k-1,l)}{2}|X| \geq \frac{\varepsilon(k-1,l)}{2}\cdot\frac{1}{\varepsilon(k,l)} = \frac{1}{\varepsilon(k,l-1)},$$

the set T contains an $\varepsilon(k,l-1)$-good k-cap or an $\varepsilon(k,l-1)$-good $(l-1)$-cup. In the first case we are done since $|T| \geq \frac{\varepsilon(k-1,l)}{2}|X|$ and therefore any k-cap

[1]The inequality holds with equality for $\alpha = 4m+q, m \in \mathbf{Z}, q \in (0,1]$. Otherwise the left-hand side is larger than the right-hand side.

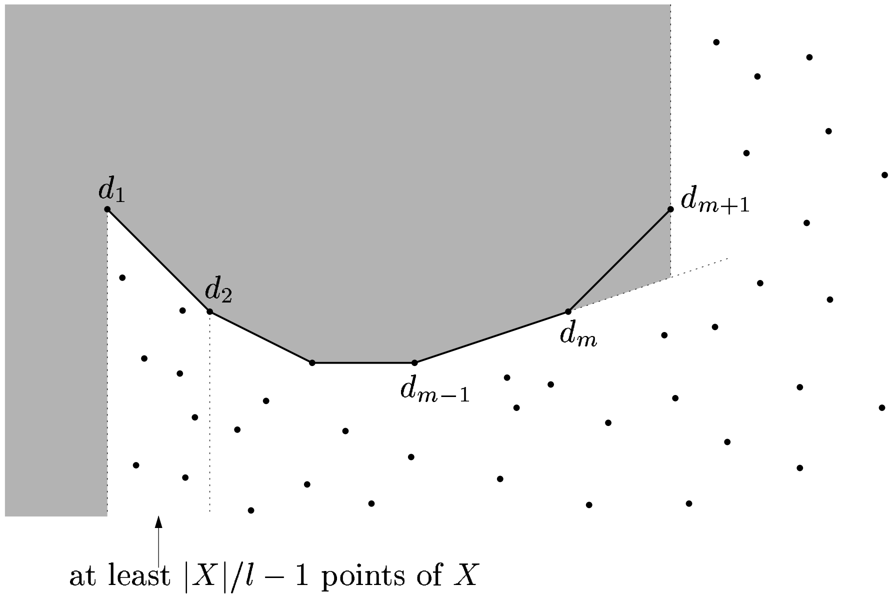

Figure 6: The $(m+1)$-cup $D_{m+1} = d_1 d_2 \ldots d_{m+1}$.

which is $\varepsilon(k, l-1)$-good in T is $\varepsilon(k, l-1) \cdot \frac{\varepsilon(k-1,l)}{2} = \varepsilon(k, l)$-good in X. So we may suppose that T contains an $\varepsilon(k, l-1)$-good $(l-1)$-cup $D = d_1 d_2 \ldots d_{l-1}$. Since $|T| \geq \frac{\varepsilon(k-1,l)}{2}|X|$, D is $\varepsilon(k, l-1) \cdot \frac{\varepsilon(k-1,l)}{2} = \varepsilon(k, l)$-good in X.

Let U be the set of points p of X_0 with $x(p) \leq x(d_1)$. Let a be the point viewed from c_{k-1} as the highest point of U (see Fig. 8). In other words, $a \in U$ and no point of U lies above the line $c_{k-1}a$. We now distinguish three cases.

Case 1: a lies above the line $c_{k-1}d_2$ (see Fig. 8). In this case, $c_1 c_2 \ldots c_{k-1} a$ is an $\varepsilon(k, l)$-good k-cap in X (recall that D is $\varepsilon(k, l)$-good in X and thus $|L(D) \cap X| \geq \varepsilon(k, l)|X| - 1$).

Case 2: a lies below the line $c_{k-1}d_2$ and in R (see Fig. 9). In this case, $c_1 c_2 \ldots c_{k-1} a$ is an $\varepsilon(k, l)$-good k-cap in X ($S \cup \{d_1\}$ lies entirely below the line $c_{k-1}a$).

Case 3: a lies below the line $c_{k-1}d_2$ and in $U \setminus R$ (see Fig. 10). In this case, $c_{k-1} a d_2 d_3 \ldots d_{l-1}$ is an $\varepsilon(k, l)$-good l-cup in X (R lies entirely below the segment $c_{k-1}a$).

This concludes the proof of Theorem 2. $\square$

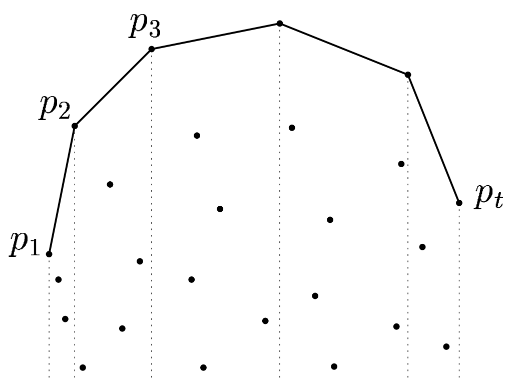

Figure 7: The upper envelope of X.

3 Proof of Corollaries 1–3

Proof of Corollary 1. Let X be a configuration of size at least $f(k+3, l-1)+1$. Let p be a vertex of conv X, and let l be a line touching conv X at the point p. We transform X to a set X' by a projective transformation P which sends the point p to the "point" $(0, \infty)$ and the line l to the line at infinity. The configuration X' has at least $f(k+3, l-1)$ points in the real plane, thus it contains an open $(k+3)$-cap C or an open $(l-1)$-cup D. The first case cannot happen, since $P^{-1}(C) \cup \{p\}$ would be an (empty) $(k+4)$-moon and we would get a contradiction with the k-convexity of X. In the second case the set $P^{-1}(D) \cup \{p\}$ forms an empty l-gon. Thus, $N(k, l) \leq f(k+3, l-1)+1$. $\square$

Proof of Corollary 2. Since any open l-cup is an empty l-gon, Theorem 2 immediately gives $m(k, l) \leq f(k, l)$. $\square$

Proof of Corollary 3. If a configuration X has at least $2 \cdot f(k-1, l-1)$ points and $p \in X$ is any point in X, then p is a vertex of the convex hull of at least $f(k-1, l-1)+1$ points of X. We further continue analogously as in the proof of Corollary 1, obtaining $z(k, l) \leq 2 \cdot f(k-1, l-1)$. $\square$

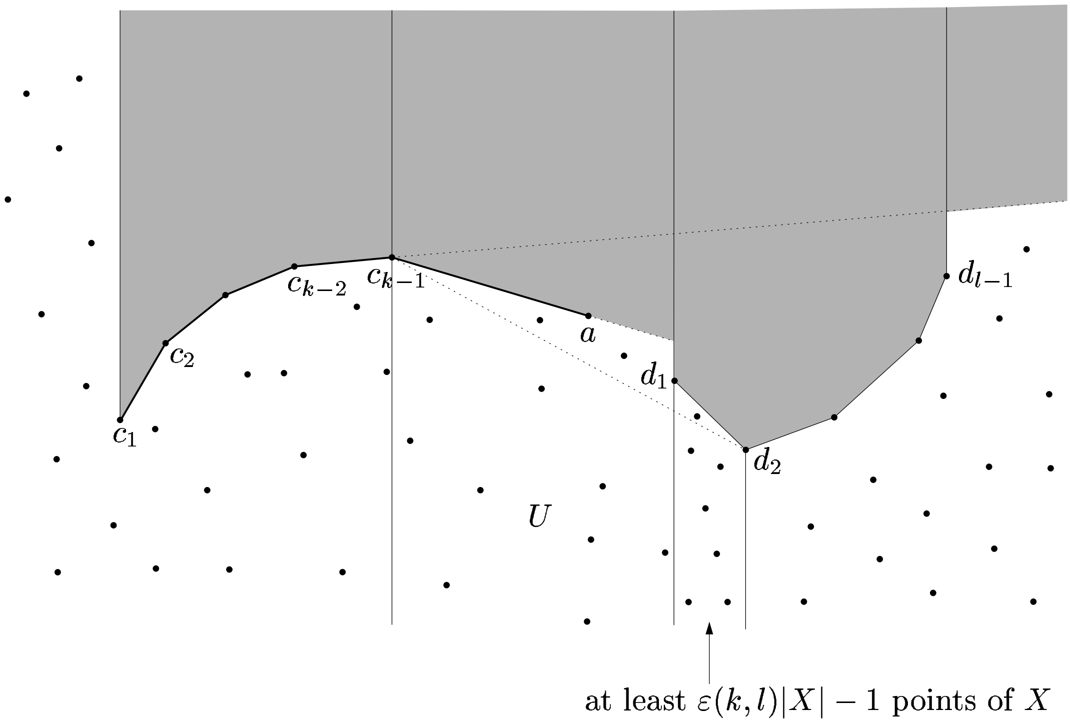

Figure 8: Case 1 (the shaded regions contain no points of X).

4 Proof of Theorem 3

Proof of Theorem 3. The inequalities $f'(k,l) \leq f(k,l), f''(k,l) \leq f(k,l)$ are trivial, and the proof of Theorem 2 gives the upper bound in (i):

$$f(k,l) \leq \frac{1}{\varepsilon(k,l)} \leq 2^{\binom{k+l-2}{k-1}-1}$$

(the second inequality can be easily proved by induction from the definition of the coefficients $\varepsilon(k,l)$). It remains to prove the lower bounds.

For $k,l \geq 2$ even, we recursively construct a configuration $S_{k,l}$ of size $2^{\binom{k/2+l/2-2}{k/2-1}}-1$ with no k-cap and no open l-cup. This gives the lower bound in (i). Slight changes in the construction will then give the lower bound in (ii).

If $k=2$ or $l=2$ then $S_{k,l}=\{(0,0)\}$. We continue by induction on $k+l$. Let $k,l \geq 4$ be even, and suppose that the configurations $S_{k-2,l}, S_{k,l-2}$ have already been constructed. We take the configuration $S_{k-2,l}$ and partition the plane into $|S_{k-2,l}|+1$ regions by the vertical lines throught each point of $S_{k-2,l}$. In each region we place a small (shrunk) copy of $S_{k,l-2}$. We denote the copies

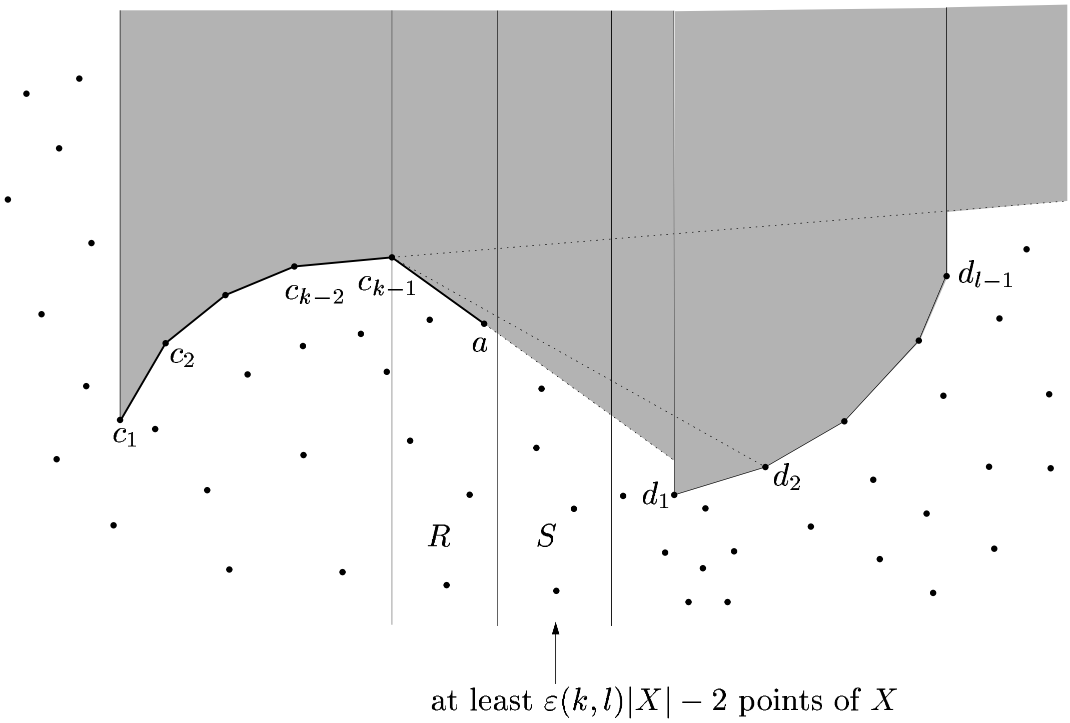

Figure 9: Case 2.

of $S_{k,l-2}$ by $S^0, S^1, \ldots, S^{|S_{k-2,l}|}$. In each S^i we shrink the y-coordinates. Then we rotate and vertically shift each S^i so that any subset of $S^0 \cup \ldots \cup S^{|S_{k-2,l}|}$ intersecting each S^i in at most two points is a t-cup for some t. Finally, we shift $S^0 \cup \ldots \cup S^{|S_{k-2,l}|}$ vertically so far downwards that the following two conditions hold in the set $S_{k,l} := S_{k-2,l} \cup (S^0 \cup \ldots \cup S^{|S_{k-2,l}|})$ (see Fig. 11):

(C1) If $C = c_1c_2 \ldots c_t$ is a t-cap in $S_{k,l}$, then either it is fully contained in one of the configurations S^i or $c_2c_3 \ldots c_{t-1}$ is a $(t-2)$-cap in $S_{k-2,l}$,

(C2) if $D = d_1d_2 \ldots d_t$ is an open t-cup in $S_{k,l}$, then either it is fully contained in $S_{k-2,l}$ or $d_2d_3 \ldots d_{t-1}$ is an open $(t-2)$-cup in one of the configurations S^i.

It follows from (C1) and (C2) that $S_{k,l}$ contains no k-cap and no open l-cup. Its size is

$$|S_{k,l}| = (|S_{k-2,l}|+1)(|S_{k,l-2}|+1)-1 = 2^{\binom{k/2+l/2-3}{k/2-2}} \cdot 2^{\binom{k/2+l/2-3}{k/2-1}} - 1 = 2^{\binom{k/2+l/2-2}{k/2-1}} - 1.$$

This gives the lower bound in (i).

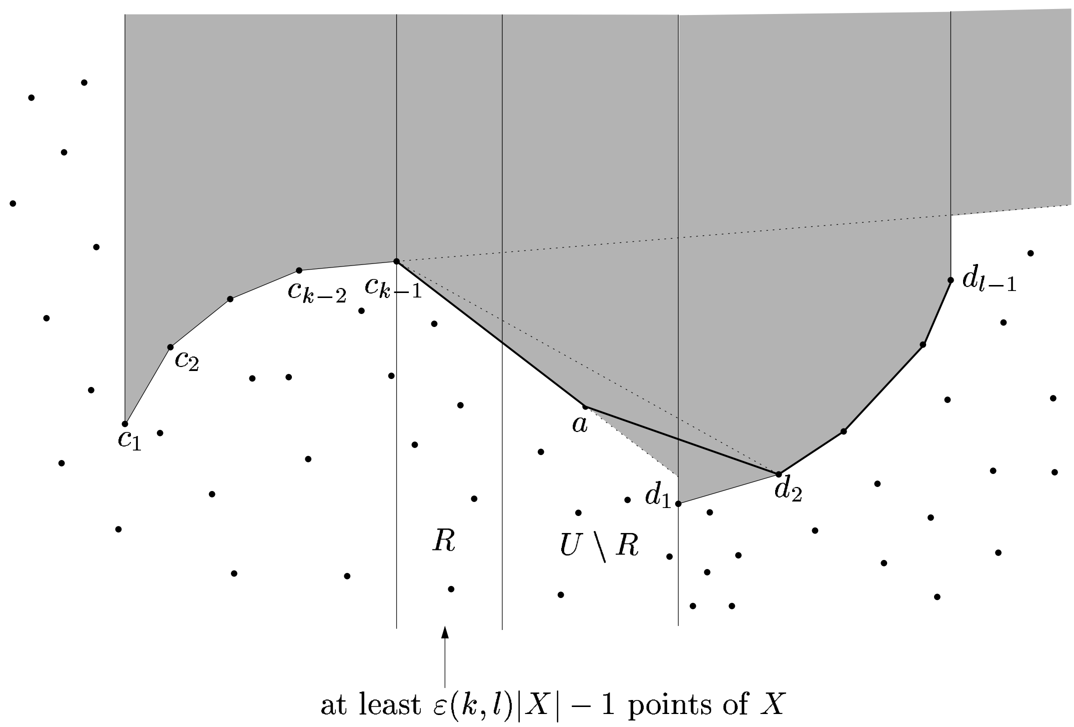

Figure 10: Case 3.

To get the lower bound in (ii), we slightly change the recursive step in the construction. For $k, l \geq 2$ even, we now construct a configuration $S_{k,l}$ of size $2^{\binom{k/2+l/2-2}{k/2-1}}-1$ with no open k-cap and no $(l+1)$-cup. Again $S_{k,l} = \{(0,0)\}$, if $\min\{k,l\} = 2$. For $k, l \geq 4$ even, the recursive step indicated in Fig. 12 starts similarly as above but the rotations and vertical shifts of the sets S^i are done so that the following holds for the set $S_{k,l} := S_{k-2,l} \cup (S^0 \cup \ldots \cup S^{|S_{k-2,l}|})$:

(C1′) If $C = c_1c_2\ldots c_t$ is an open t-cap in $S_{k,l}$, then either it is fully contained in one of the configurations S^i or $c_2c_3\ldots c_{t-1}$ is an open $(t-2)$-cap in $S_{k-2,l}$, and

(C2′) if $D = d_1d_2\ldots d_t$ is a t-cup in $S_{k,l}$ and $t \geq 5$, then either it is fully contained in $S_{k-2,l}$ or $d_2d_3\ldots d_{t-1}$ is a $(t-2)$-cup in one of the configurations S^i.

Then the set $S_{k,l}$ of size $2^{\binom{k/2+l/2-2}{k/2-1}}-1$ indeed contains no open k-cap and no $(l+1)$-cup. The lower bound in (ii) follows. □

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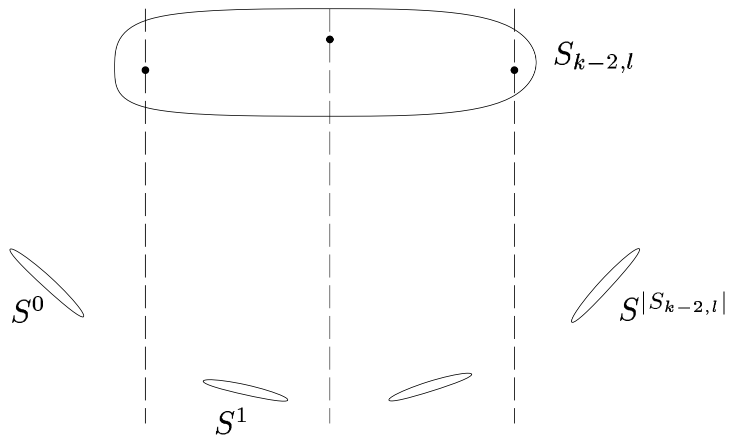

Figure 11: The configuration $S_{k,l}$ giving the lower bound in Theorem 3(i).

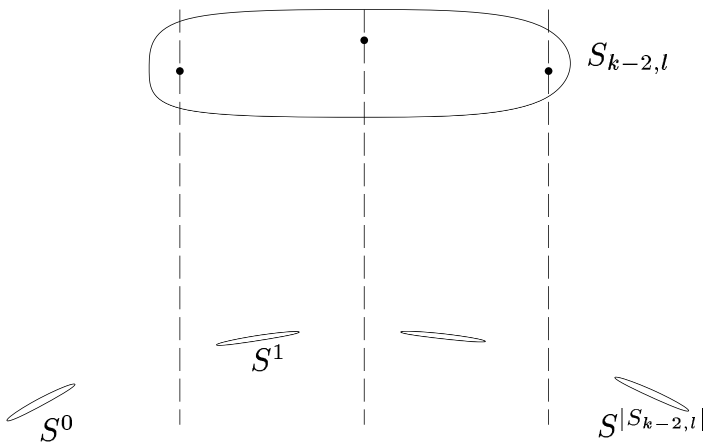

Figure 12: The configuration $S_{k,l}$ giving the lower bound in Theorem 3(ii).

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