A Ramsey property of planar graphs*

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Abstract

We prove the following colored version of the well-known result of Wagner and Fáry. Suppose that the line segments of the plane are partitioned into finitely many classes $C_1, \ldots, C_k$. Then for some C_i, every planar graph G has a crossing-free drawing in the plane

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such that all its edges are drawn by segments belonging to C_i. If there is only one C_i with this property, then in fact this C_i contains a drawing of any graph (with possible crossings). Furthermore, all this is true also if we distinguish topologically non-isomorphic drawings of a graph. We also give a generalization of these results to so-called pseudoplanar graphs.

1 Introduction

In their papers, Erdős, Graham, Montgomery, Rothschild, Spencer, and Straus [3, 4, 5] have initiated a study in the field called *Euclidean Ramsey theory*. The general question is as follows. Let V be a set of points in $\mathbb{R}^n$. Is it true that for every k-coloring of all the points in $\mathbb{R}^n$ there is a monochromatic V' congruent to V? Analogous questions for colorings of pairs of points in $\mathbb{R}^n$ were also studied [1, 4, 9]. In this case, the general question is as follows. Let G be a graph drawn by straight-line segments in $\mathbb{R}^n$ (i.e. G is a collection of points of $\mathbb{R}^n$ representing vertices and of some of the line segments between the point-pairs representing edges). Is it true that for every k-coloring of all the segments in $\mathbb{R}^n$ there is a monochromatic H congruent to G? It is easy to see that the only non-trivial questions occur when the edges in G have the same length. This case was well-characterized by Cantwell in [1].

In the plane, where G is a so-called geometric graph (defined below), instead of congruence we can require a weaker relation, *the (combinatorial) isomorphism*. In this case several positive and negative Ramsey-type results are known, see e.g. [2, 10, 11, 12, 14, 15].

We say that a set of points in the plane is *in general position* if no three points in the set lie on a common line. A *geometric graph* is a pair $G = (V, E)$ where V is a finite set of points in general position in the plane and E is a subset of the set of line segments connecting points of V. A *planar geometric graph* is a geometric graph with no pair of crossing edges (two edges sharing a vertex are not considered as crossing edges). In other words, a planar geometric graph is a planar graph drawn in the plane by straight segments with no crossings.

Two geometric graphs $G = (V, E), H = (V', E')$ are said to be *(combinatorially) isomorphic*, $G \sim H$, if there exists a bijection $f : V \to V'$ satisfying the following three conditions:

(i) $v_1v_2 \in E$ if and only if $f(v_1)f(v_2) \in E'$,

(ii) two edges (segments) $v_1v_2, v_3v_4 \in E$ cross if and only if the edges $f(v_1)f(v_2), f(v_3)f(v_4) \in E'$ cross,

(iii) if two edges $v_1v_2, v_3v_4 \in E$ cross then v_3 lies to the left of the oriented line v_1v_2 if and only if $f(v_3)$ lies to the left of the oriented line $f(v_1)f(v_2)$.

Two finite planar point sets X and X' in general position are said to be *combinatorially equivalent* (or: *of the same order type*) if there exists a bijection $f : X \to X'$ satisfying that a point $a \in X$ lies to the left of the oriented line bc ($b, c \in X$) if and only if the point $f(a)$ lies to the left of the oriented line $f(b)f(c)$.

It is easy to see that two complete geometric graphs G, H are isomorphic if and only if the sets $V(G)$ and $V(H)$ are combinatorially equivalent.

Geometric graphs have received a lot of attention in recent years (see e.g. Chapter 14 in the book [17] or [16]). In this paper we study Ramsey-type questions motivated by the results of [14, 15]. In [15] it was proved that there is a finite planar point set X in general position and a coloring of the segments of the plane with two colors such that no set combinatorially equivalent to X induces a monochromatic complete geometric graph.

Here is the key concept of this paper: Let $G_1, G_2, \ldots, G_k$ be geometric graphs. The notation $T(G_1, G_2, \ldots, G_k)$ means that for every k-coloring of all segments in the plane there is an index i ($1 \le i \le k$) and a geometric graph G_i' isomorphic to G_i such that all the edges of G_i' have the i-th color.

Now the above result of [15] can be formulated in the following way:

Theorem 1 ([15]) *There is a (complete) geometric graph G such that $T(G, G)$ does not hold.*

In the same paper it was proved that if $|V(G)| \le 4$ then $T(G, G)$ holds. Here we prove the following Ramsey-type result:

Theorem 2 *Let $G_1, \ldots, G_k$ be k planar geometric graphs and let H be any geometric graph. Then $T(G_1, \ldots, G_k, H)$ holds.*

A theorem of Wagner [19] and Fáry [6] (see e.g. [18]) says that every planar graph has a planar drawing with edges represented by pairwise non-crossing straight-line segments. The following corollary of Theorem 2, mentioned in the abstract, is a colored version of the theorem of Wagner and Fáry:

Corollary 1 *Suppose that the line segments of the plane are partitioned into finitely many classes* $C_1, \ldots, C_k$. *Then for at least one class* C_i *the following holds. For every planar graph* G *there is an isomorphic planar geometric graph with all edges (segments) belonging to* C_i. *If the class* C_i *is unique with this property then in fact it contains the edges of some drawing of any geometric graph* G.

In fact, we can prove Theorem 2 and Corollary 1 in somewhat stronger forms. We say that a geometric graph G is *pseudoplanar*, if it can be obtained from a planar geometric graph by a finite number of applications of the following operations:

(R) removal of an edge or vertex,

(D) duplication of a vertex u in the graph (i.e., addition of a new vertex u' such that no line trought two vertices in the graph separates u' from u, and addition of an edge from u' to u and to all neighbors of u).

Theorem 3 *Theorem 2 and Corollary 1 hold also for pseudoplanar graphs, i.e. they hold also if we replace each word "planar" by "pseudoplanar" in the statement.*

For example all geometric graphs on at most 5 vertices are pseudoplanar; this is illustrated in Figure 1.

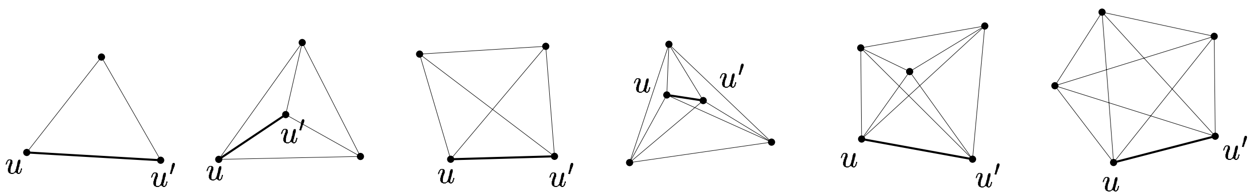

Figure 1: The vertices u, u' used for obtaining the six combinatorially different complete geometric graphs on 3, 4, or 5 vertices by an application of operation (D) on a smaller complete geometric graph.

However, there are geometric graphs on 6 vertices which are not pseudoplanar, see Fig. 2.

Figure 2: Two non–pseudoplanar geometric graphs on 6 vertices.

2 Ramsey property of planar graphs

In this section we prove Theorem 2. In our proof we look at the coloring of the point pairs of a huge grid. An $N \times N$ *grid*, denoted by $L(N \times N)$, is the set of N^2 points in the plane with integer coordinates $1, 2, \ldots, N$. We will use the following theorem of Fürstenberg [7] (i.e. the density version of the Gallai–Witt Theorem [8]) and three simple claims:

Theorem 4 ([7]) *For any $c \in \mathbb{R}^+$ and $t \in \mathbb{N}$, there is an $N = N(c, t)$ such that if S is a subset of $L(N \times N)$ and $|S| \geq cN^2$ then S contains t^2 points which form a scaled and shifted copy of the grid $L(t \times t)$.*

The following claim says that for any line l, in a huge grid one can find a non-degenerate affine image of a large grid, such that any point pair of this image determines a line almost parallel to l.

Claim 1 *Let l be a line, $m \geq 2$ a natural number, and $\delta > 0$. Then there exists an $M = M(m, l, \delta)$ with the following property: There exists an affine image A of the $m \times m$ grid such that A is a subset of the $M \times M$ grid, it is not contained in one line, and the angle between the line l and any line containing at least two points of A is smaller than δ.*

Proof. Let S be any strip of width $1/(3\delta)$ bounded by two lines parallel to l. Then its intersection with the infinite grid $\mathbb{Z} \times \mathbb{Z}$ contains an affine image A of $L(m \times m)$. If $M := \lfloor 1 + \operatorname{diam}(A) \rfloor$ then A can be shifted so that $A \subseteq L(M \times M)$. Claim 1 follows. $\square$

Below we shall use complete geometric graphs, i.e. graphs in which every pair of vertices is connected by an edge (segment). However, we should keep in mind that for any fixed $k \geq 4$ there are different non-isomorphic complete geometric graphs on k vertices.

Claim 2 *For any natural number h there is a $m = m(h)$ with the following property. If G_h is a complete geometric graph on h vertices, then there is a ${G_h}' \sim G_h$ such that $V({G_h}')$ is a subset of the $m \times m$ grid.*

The proof of Claim 2 is based on the fact that for any h there are finitely many non-isomorphic complete geometric graphs on h vertices. Details are left to the reader.

Claim 3 *For any planar geometric graph G, there is an edge $xy \in E(G)$ such that if $z \in V(G)$ and $xz, yz \in E(G)$ then the triangle xyz contains no other vertex of G.*

The proof of Claim 3 can be found in [6] or in [13], exercise 5.38.

Advancing the proof of Theorem 2 we first prove the statement in the special case $k = 1$:

Lemma 1 *Let G be a planar geometric graph and let H be a geometric graph. Then $T(G, H)$ holds.*

Proof. We proceed by induction on $n = |V(G)|$. If $n \leq 2$, then the assertion is trivial.

Now suppose that the assertion is true if $|V(G)| \leq n - 1$. We will use *generalized geometric graphs* which may contain collinear triples of points; otherwise they are defined analogously as geometric graphs. Let H be an arbitrary but fixed geometric graph. We can suppose that G is a triangulation. We consider an edge xy satisfying Claim 3. Since G is a triangulation, the edge xy lies on the boundary of exactly two triangles, xyz_1 and xyz_2, in G. We contract the edge xy to a point p and remove one edge from each pair of parallel edges. We obtain a planar graph G_- with $n - 1$ vertices. By the inductive hypothesis and using the compactness argument one can find a complete generalized geometric graph Y in the plane, such that if we color the edges of Y by blue and red then either there is a blue planar geometric graph ${G_-}' \sim G_-$ or a red geometric graph $H' \sim H$ in Y. If needed, the vertex set of Y is slightly perturbed so that Y becomes a geometric graph. For three points a, b, c, let $d(a, b, c)$ be the distance of a to the line bc. We define $d(Y) = \min d(a, b, c)$, where the minimum is taken over all triples of distinct points $a, b, c \in Y$.

We replace each vertex of Y by a scaled copy of $L(N \times N)$ having diameter $d(Y)/10$. N will be specified later (now it is as big as necessary). We denote the copies of $L(N \times N)$ by $L_1, \ldots, L_{|Y|}$. The size of the L_i's ensures that if we choose one point from each L_i, then the obtained configuration ("transversal") is isomorphic to Y and no three points in it determine an angle in some range $(\pi - \delta, \pi + \delta)$ (here $\delta = \delta(Y) > 0$ is independent of N). Let us denote the obtained complete generalized geometric graph on $N^2|Y|$ vertices by Z (see Fig. 3).

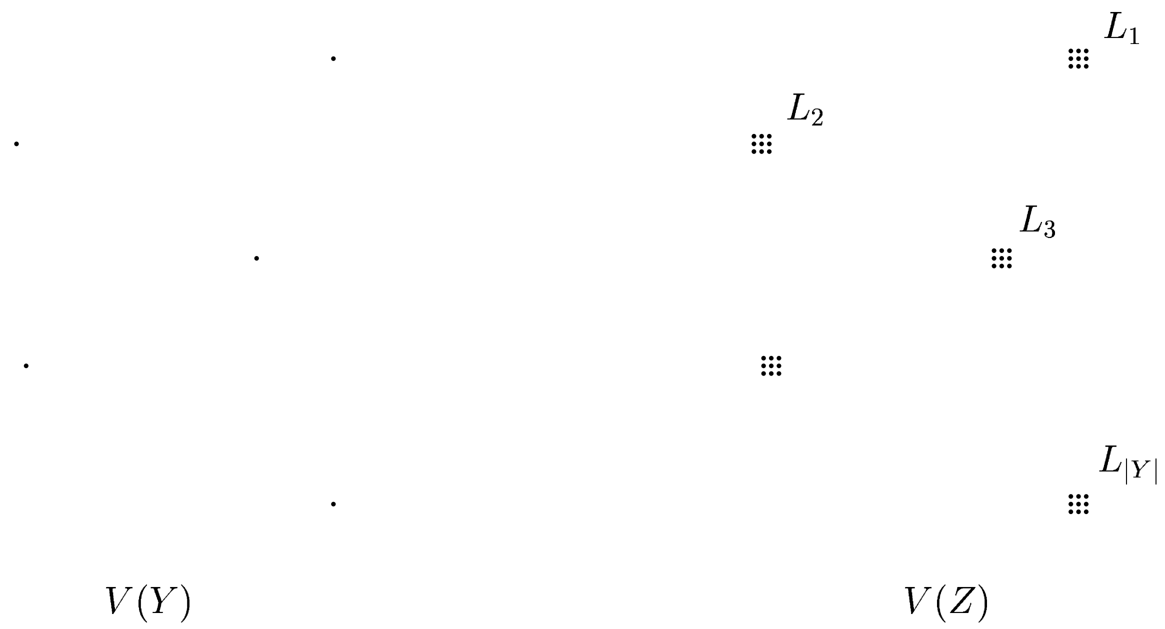

Figure 3: The vertex sets of the complete geometric graphs Y, Z.

Suppose that we have a coloring of the edges of Z by blue and red, such that there is no red isomorphic copy of H in Z. Then we have a blue isomorphic copy of G_- in each of the above mentioned transversals $Y' \subset Z$, $Y' \sim Y$. There are $(N^2)^{|Y|}$ such transversals. Each copy of G_- is contained in at most $(N^2)^{|Y|-(n-1)}$ transversals, so we have at least $N^{2(n-1)}$ different blue "transversal" copies of G_-. At least $N^{2(n-1)}/\binom{|Y|}{n-1}$ of them lie in the union of some fixed $n-1$ L_i's (say, in $L_1 \cup \ldots \cup L_{n-1}$). The $n-2$ vertices of G_- different from p may be distributed in $L_1 \cup \ldots \cup L_{n-1}$ in at most $(n-1)!(N^2)^{n-2}$ different ways. Thus, at least $\left(N^{2(n-1)}/\binom{|Y|}{n-1}\right) / \left((n-1)!(N^2)^{n-2}\right) = cN^2$ of the above blue "transversal" copies of G_- differ only in the image of p, where

$c := 1/(\binom{|Y|}{n-1}(n-1)!)$. Moreover, the image of p in these $\geq cN^2$ blue copies of G_- always lies in the same L_i. According to Theorem 4, if $N \geq N(c,t)$ then this L_i contains a (scaled and shifted) grid $L(t \times t)$ formed by images of p. Altogether, this gives us a blue graph which is an isomorphic copy of G_- in Z with the vertex p replaced by a small (scaled and shifted) copy of $L(t \times t)$, which will be denoted L_0.

We now look at the colors of the segments connecting points of L_0. We say that a segment connecting two points of L_0 is *separating* if the line containing this segment separates the images of z_1 and z_2. If there is a blue separating segment $x'y', x', y' \in L_0$, then substituting

$$x \to x', y \to y'$$

or

$$x \to y', y \to x',$$

we get a blue subgraph of Z isomorphic to G. So we can suppose that every separating edge is red. Now we use Claim 1. For any fixed m one can find a number M depending only on Y such that if $t \geq M$ then wherever the images of p, z_1, z_2 lie in Z, L_0 always contains a non-degenerate affine image A of the $m \times m$ grid in which every pair of points determines a separating (and therefore red) segment. Using Claim 2 it follows that if m is large enough then we have a red $H' \sim H$ in A. $\square$

We are ready to complete the proof of Theorem 2.

Proof of Theorem 2. For $k = 1$, $T(G_1, H)$ holds according to Lemma 1. We further proceed by induction on k.

Suppose that $T(G_2, G_3, \ldots, G_k, H)$ holds. Thus, by the compactness argument (eventually followed by a small perturbation), there is a geometric graph J with the following property. If we color the edges of J with k colors $2, \ldots, k, k+1$, then for some $i \in \{2, \ldots, k\}$ there is a $G_i' \subset J$ isomorphic to G_i with all edges in the i-th color or there is a $H' \subset J, H' \sim H$, with all edges in the $(k+1)$-th color.

We have $T(G_1, J)$ from above, and if the second color represents k colors $2, \ldots, k, k+1$, we get $T(G_1, \ldots, G_k, H)$. This finishes the proof of Theorem 2. $\square$

3 Proof of the colored Wagner–Fáry theorem

In this section we derive Corollary 1 from Theorem 2.

Proof of Corollary 1. Suppose that the line segments in the plane are partitioned into k classes. If Corollary 1 was satisfied for none of the classes, then $T(G_1, \ldots, G_k)$ would be violated for some planar geometric graphs $G_1, \ldots, G_k$ — a contradiction with Theorem 2.

Suppose that the class C is unique, and let H be a geometric graph having no isomorphic copy drawn by segments of C. Without loss of generality, let C be the last of the k classes. Then $T(G_1, \ldots, G_{k-1}, H)$ is violated for some planar geometric graphs $G_1, \ldots, G_{k-1}$ — a contradiction with Theorem 2. □

4 Pseudoplanar graphs

In this section we prove Theorem 3.

Lemma 2 *Let G, H be two geometric graphs such that $T(G, H)$ holds. Further, let G' be a geometric graph obtained from G by one of the operations (R), (D). Then $T(G', H)$ holds.*

Proof (sketch). The assertion is trivial if G' is obtained from G by operation (R). Thus, we may suppose that G' was obtained by operation (D). We use a similar method as in the proof of Lemma 1.

Since $T(G, H)$ holds, there is a complete geometric graph Y such that if we color the edges of Y by two colors (red and blue) then there is a monochromatic red isomorphic copy of G or a blue isomorphic copy of H. Let us replace all the vertices of Y by small copies of $L(N \times N)$ in the same way as in the proof of Lemma 1. We now prove that if N is large enough then the obtained complete geometric graph Z is Ramsey for $T(G', H)$.

If Z contains a blue copy of H then we are done. So we may suppose that there are many red "transversal" copies of G. There are still many of them for which $V(G) \setminus \{u\}$ is fixed and the images of u cover a positive fraction of one of the $L(N \times N)$'s. According to Theorem 4, some of the images of u form a (scaled and shifted copy of) $L(k \times k)$.

Similarly as in the proof of Theorem 2, we either find a red pair of points in this $L(k \times k)$ determining a line separating the images of $V(G) \setminus \{u\}$ in the "right" way, or there is a non-degenerate affine image of a large grid where all edges are blue. In the first case we find a red copy of G', in the second case we find a blue copy of H. □

Proof of Theorem 3. Lemma 2 gives a generalization of Lemma 1 when G is allowed to be pseudoplanar. The rest of the proof is analogous as the proof of Theorem 2 and of Corollary 1. □

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