Towards Mathematical Aesthetics[*]

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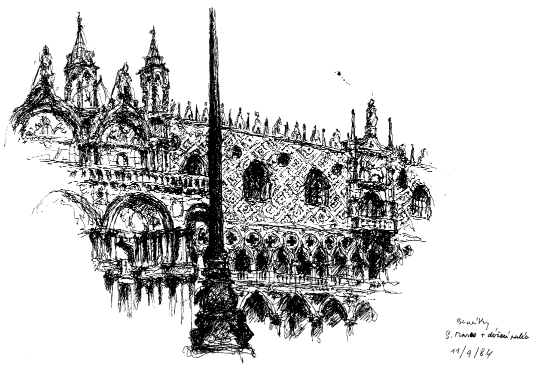
Abstract

We discuss a possibility of visual data global analysis with respect to the aesthetic (harmonious) criteria. We explain our approach that is mathematical based and expressed by the Hereditary Entropy Thesis. We give some computational evidence which outlines aims and indicates meaning of this project.



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If there is not possible to change this world
which does not deserve love, then what is left to us?
Not to allow to be cheated.
To see and to know. To know how to see.
M. Kundera

1 The Main Problem

The contemporary art and technology produce a vast amount of visual data. We believe that the management, identification and sorting of these data is one of the main problems today. Several areas of contemporary science address these issues on both theoretical and practical level. These areas include e.g. fields as diverse as chemistry, artificial intelligence, image processing, graph drawing, robotics, and complexity theory.

This research is performed in all possible directions and tested broadly on (mostly) technical data. In fact, many algorithms are designed to suit data and situations of a very special kind (such as graphs schemata or surface materials modeling). Yet exactly this variety calls for unification and coding of basic underlying principles of visual analysis. We believe that the presence of an aesthetic aspect of this analysis is one of the unifying principles.

It seems that an aesthetic aspect is omni-present in the judgment of visual information. Even in most technical and "anti-art" areas an aesthetic judgment occurs. Sometimes it is present quite modestly in a sense of harmony, well-being, pleasing, and attraction. Visual information judgment always includes aesthetic criteria.

How to analyze the aesthetic aspect, how to express it, is it possible to measure it and to analyze it in the hope that we can then synthetize it by exact means? This is the main problem which we address here. We are continuing here our earlier attempts [1, 2, 14, 16, 17] which in turn were motivated by earlier attempts of both artists and scientists [3] one should even quote the pioneering works [10, 11, 12] which present a remarkably concise and coherent views. Even this supports our view that aesthetic aspects of visual information should be tested on art (and that artists have to get involved). Of course, neither semantics nor the ontology of aesthetic feeling is known. Remarkably several works (see e.g. [23]) trace this problems in the neurobiology context and present a convincing (although sometimes speculative) evidence of physiological explanation of the evolution and description of aesthetic feeling.

We want to pursue here a different line—and in a sense dual—approach (see Section 3). We would like to express some global aesthetic qualities in exact terms, to analyze visual data with respect to their aesthetic qualities (or perhaps more modestly and to the point: harmonious qualities). Of course, there is an extensive literature concerning it; convincingly demonstrated by the long development of the history of art as well as architecture, design theory, and "axioms of beauty" in very diverse fields (such as animal breeding).

Very schematically, we want to proceed as follows: starting from the visual information itself (which we call briefly a *picture* from now on), we want to derive its essential characteristics, to develop its "abstract core" which in turn could be used in analysis, identification, and sorting of data. To be more concrete, we speak about pictures in the sense of any individually observed (isolated) 2-

dimensional visual information. So the picture is an art-work as well as a scheme, a fixed scene as well as a sketch or a photo of a screen.

We do not aim to compare pictures with some canonical templates (related to, say, golden section and the geometric rendering) but we want to derive essential parameters from the picture itself. Our goal is that these parameters (which we call invariants, see Section 3) will reflect some essential aesthetic qualities of the picture. Our analysis is context free. We analyze individual pictures (and compare obtained parameters). This fits to the technical situations (such as graph analysis) where the history and the context have usually a little meaning. The individual picture is our primary source which we analyze as an isolated item. But the analysis we propose reflects our art and technology experience which is projected back to the picture.

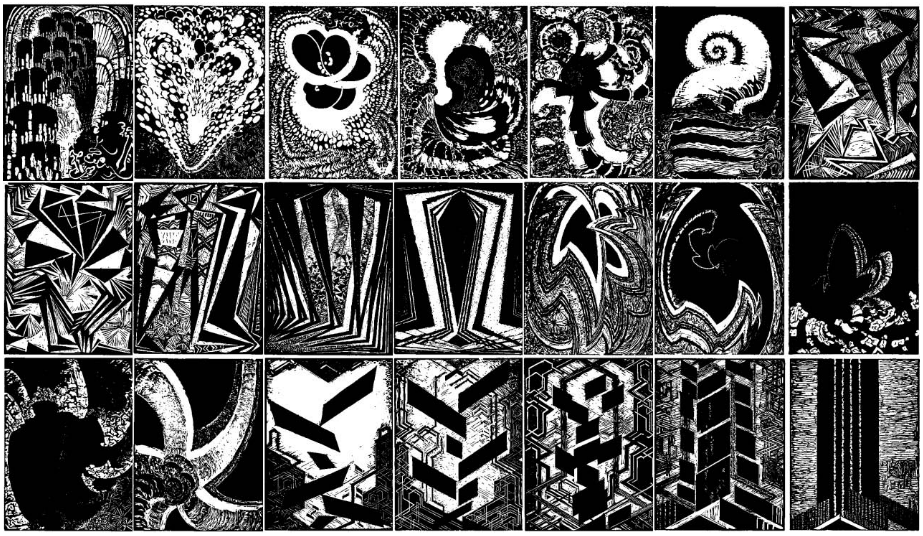


Figure 1: Kupka's archetypes, based on [12].

2 Why Art?

Mathematical and computer-science approach should justify itself by analyzing most of technical data where the history and the context have a little meaning. Why then we want to illustrate our methods and results on art and artistically charged objects?

We believe that analysis of visual information in its extreme variety of inputs calls for test examples which are complex and reflect this variety. Visual artists did and continue to do exactly this by exploring and exploiting every possibility to the limit (and sometimes seems to us beyond it). Visual art produces examples which are not only of great variety but they are also aesthetically charged (of course positively or negatively depending on individual feelings) as opposed to (usual) aesthetically neutral scientific data. Thus this aesthetic dimension presents an additional dimension which we may use to sharpen our tools and test the quality of our approach (and algorithms).

Abstract algorithms when tested on such complex examples more likely reveal their bottlenecks and because objects of art are aesthetically charged the aesthetic judgment can be used in this algorithm analysis, too. We have to try and we have to use art to widen our understanding. One would like to say: we would like to put the aesthetics in the service of complexity.

Therefore the analysis of art is both necessary (for the generality of out approach) and convenient even in the cases when our input data are of technical (machine made) inputs.

And our approach can perhaps be useful even for analysis of art itself; for example as a contribution to the ongoing discussion on Braque–Picasso cooperation in the 1905 early stages of cubism (see e.g. [19]). The objective data (collected to illustrate the Hereditary Entropy Thesis, see the explanation bellow in Section 6) may provide some additional information.

0,87	0,69	0,74	0,85	0,70	1,12	1,02	1,03
1,59		1,40		1,90		1,95	
0,95	0,66	0,58	0,65	0,81	1,14	0,93	0,99
	2,96				3,53		
0,58	0,50	0,62	0,92	0,82	0,73	0,86	0,95
1,22		1,67		1,62		1,60	
0,65	0,69	1,16	0,68	0,80	0,80	0,75	0,69
			6,65				
0,63	0,85	1,03	0,99	0,48	0,94	0,77	0,86
1,40		1,85		1,26		1,74	
0,60	0,83	0,99	0,46	0,48	0,71	1,01	0,97
	3,36				3,02		
0,91	0,77	0,80	0,59	0,90	1,01	1,12	0,78
1,74		1,42		1,70		1,80	
0,91	0,73	0,61	0,96	0,74	0,79	0,86	0,99

0,92	0,57	0,63	0,93	0,75	0,53	0,56	0,60
1,41		1,43		1,09		0,98	
0,84	0,55	0,53	0,82	0,55	0,37	0,39	0,45
	2,89				2,16		
0,72	0,64	0,66	0,71	0,65	0,48	0,68	0,45
1,31		1,53		1,27		1,06	
0,74	0,53	0,90	0,75	0,66	0,69	0,52	0,55
			4,55				
0,83	0,24	0,41	0,52	0,64	0,79	0,44	0,53
1,07		0,96		1,26		0,89	
0,77	0,39	0,45	0,42	0,48	0,65	0,46	0,44
	2,08				1,84		
0,64	0,56	0,42	0,35	0,37	0,45	0,40	0,36
1,24		0,72		0,82		0,81	
0,68	0,51	0,33	0,39	0,36	0,51	0,40	0,49

Figure 2: G. Braque: Viaducto en L'Estaque, early 1908 and P. Picasso: Paysage avec un pont, spring 1909.

3 Macro vs. Micro – a Duality

Which features—invariants—of pictures one should investigate? Where to find these invariants?

This of course encodes the present time of history, physiology and technology state of art. The choice of parameters and their interpretation is the place where we manifest our approach and it should be the place where it is manifested (supposed and presently unknown) ontology of the aesthetic perception. On the very basic level one asks how the aesthetic feeling is acquired, how it develops, and how it is maintained.

One possibility how to approach these questions is to consider aesthetics in a gnoseological context and view aesthetic feeling and judgments as an extension of general function of the brain. This is very nicely expressed by S. Zeki in [24]:

"The chief characteristic of an efficient knowledge-acquiring system is its capacity to abstract, which frees it from enslavement to the particular. But abstraction also leads to the formation of concepts and ideals. Similarly, all art is, in a sense, abstraction. And art is a translation onto canvas of concepts formed by the brain through abstraction. In this way, art, too, rises above the particular and gives general knowledge about categories."

This approach—viewing aesthetics as a typical function of brain which consists from unintelligibly complicated interactions and the resulting abstraction—could be called as a *micro analysis* of aesthetics, the aesthetics of the visual perception. This analysis seems to be expressed by the notion of a group: the elementary operations are mutually combined and are also invertible. The *algebra* governing such abstraction is particularly reflected by the group operations as perhaps first advocated by Piaget [18].

Our approach is different: when trying to define guidelines for picture analysis, we have to adopt a more *global approach.* We have to see aesthetic problems in total of the picture and we aim for a *macro analysis.* Perhaps, in a similar sense to topology of colors by H. Damish (see [5]) we aim for *topology of aesthetics.*

How should we understand this global approach?

Based on our understanding of pictures and based on the technological availability we isolate a certain set of measurable and verifiable concepts which we call *invariants.* The choice of these invariants is pivotal and of prime importance as it not only influences the quality and meaning of the output but also should reflect the underlying framework of the picture analysis and its topology expressed in algebraic terms.

What then is the *algebra* of this macro analysis? Rather than a combination of local perceptions it is a perception of the whole picture, of whole parts of the picture. This perception is then refined to perception of parts which in their turn may be further refined. In mathematical terms we work with dual objects like partition lattice and cogroup. We have tried to symbolize it by the scheme of Figure 3.

Of course macro- and micro- analysis combine each other but it seems that only macro-analysis (i.e. top–down analysis) yields concepts sufficiently general to our purposes.

The global (topological) approach studies and concentrates on features which are similar to individual art-work. Surely one characteristic of art is its richness and diversity. But considering individual work ("context free") we aim for similarities and hope that a global picture will emerge.

We developed basics of such analysis by a combination of techniques from integral geometry, model theory, and theory of fractals. Using that we can rigorously define *algebra*, *scale*, *space*, and *entropy* of a picture.

The interplay of these notions leads to the *Hereditary thesis* (see Section 6) which we view as a general criterion for a harmonious picture.

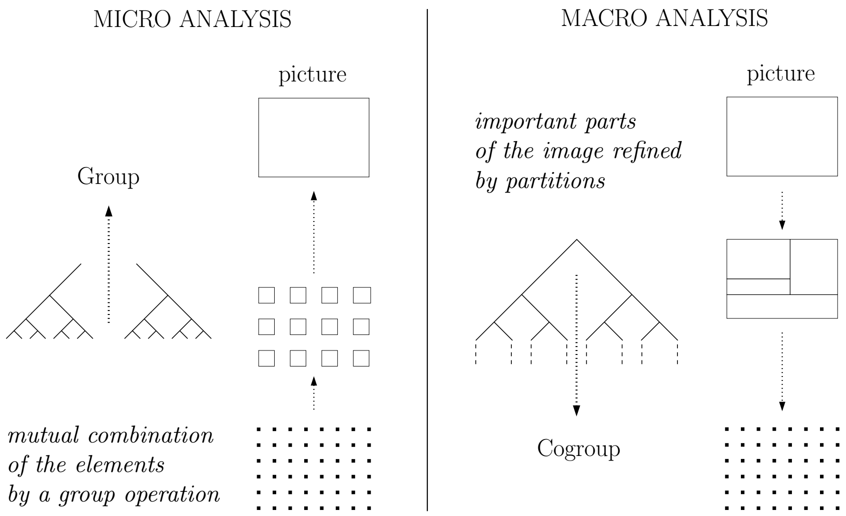
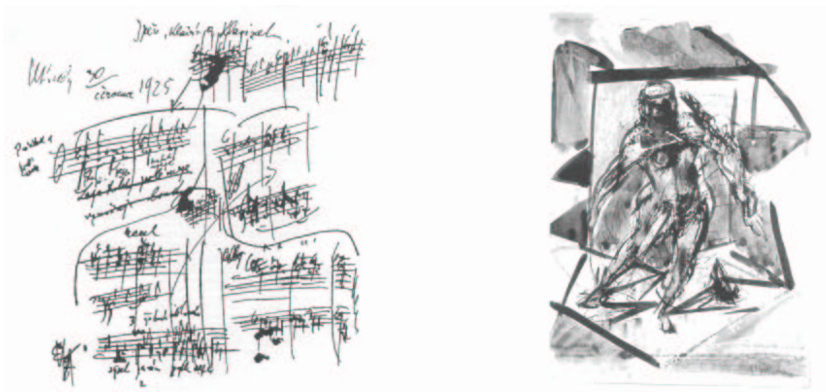


Figure 3: Scheme of a micro- and macro- analysis of an image.

4 A Case Analysis – Invariants for Pictures

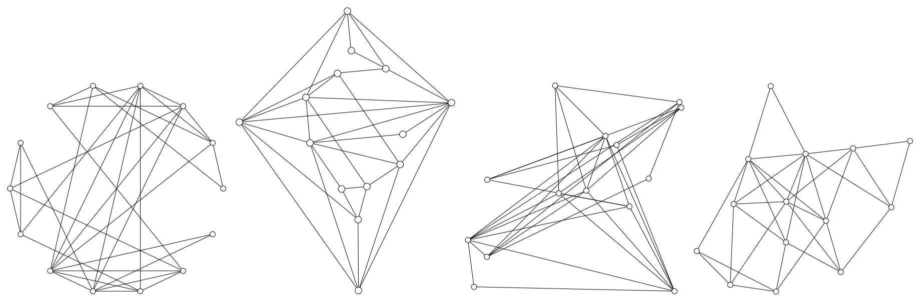
What makes the following pictures similar and what makes them different?



18.55 17.87

Figure 4: A sketch of a musical score by Janáček [21] and one of the *Moduli* – a sketch by Načeradský and the second author [15].

And, very simply, can we distinguish or order in some systematic way the following four drawings of the same graph?



12.48 9.47 11.81 7.87

Figure 5: Different drawings of a graph (from left): vertices placed regularly on a circle; human produced version emphasizing symmetries and spatial visualization; random drawing ; computer generated drawing using string model. The drawings are distinguished by numbers (entropies) given below.

The modern version of these questions is *not* how to teach a gifted and collaborating child what is nice and beautiful. Instead we need to teach an individual which is not collaborating at all and who takes every our information deadly seriously and exploits it to the last bit—a computer. People usually do not react this way (and if so, then only in comedies like [7] or [9]; the fact that these great novels have a military setting is then not an accident). In order to "teach" a computer (and even without ambition for teaching, just dealing with it) we need a precision. And a precision in the other words calls for some concrete measures of our phenomena, in the other words for invariants.

Consequently, the traditional principal problem of aesthetics (and art history)—to explain and to predict artistic and aesthetically pleasing—took recently an unexpected twist. We do not explain and deal with individual instances, we have to *classify* a vast amount of data and we have to *design* procedures with likely harmonious output. This problem in its manifold variety is interesting already when our objects are well defined compositions composed from simple building blocks such as lines, squares, sticks, etc. This in fact is a familiar exercise and training ground of schools of design and architecture and (traditional) art academies. This illustrates difficulty and variety of solutions even of simple situations. This should be not surprising if we realize how many simple lines needed e.g. Rembrandt or Picasso to produce full images (drawings usually use 50 lines or even less!).

For our "simple composition from simple building blocks" we would like to create an invariant which would help us to categorize and order these compositions. It is difficult to say even on this simple level what it is an invariant, but we can certainly state which properties such an invariant should have:

i. invariant should be an (easy) *computable* aspect of the structure;

ii. invariant should be *consistent* (or invariant, meaning it should not change) under chosen modification of structure;

iii. invariant should be *useful* in that it can be used to catalogue, to order (which structure is "better"), to classify, to distinguish.

We propose here an invariant—called *hereditary combinatorial entropy*—to measure an aesthetic quality of a visual data (drawing, scheme, painting, note score, molecular data output, and others). This invariant is presented in the next section.

5 Measuring

Papers [1], [17] suggested a different "aesthetics" which are based on the techniques of integral geometry and the geometric probability (for mathematical details see e.g. [20], [8]). The defined parameter *Fractional Length* [13] or *Combinatorial Entropy* [1] (two terms with the same meaning) defined only for line drawings and has several advantages:

- the drawing (picture, visual data) need not be given analytically, the input may be a scanned picture. This allows us to analyze, sort and compare real pictures, scenes, photos and visual art in general;
- Combinatorial Entropy is scale and rotation invariant;
- Combinatorial Entropy is easy to determine and it is "robust".

Generalization of Combinatorial Entropy for gray-scale pictures can be found in [2]. All the results of present paper were computed by means of these definitions. Without giving any further details and derivations we present the following formulae:

For a picture given as a (connected) line drawing we have defined Combinatorial Entropy E by

$$E = \frac{2L}{C},$$

where L is the total length of the drawing and C is the perimeter of convex hull of the drawing.

For a gray-scale picture we have similar definition of Combinatorial Entropy:

$$E = \frac{\pi F}{C},$$

where C is again the perimeter of convex hull of the picture (in this case it is usually the perimeter of framing rectangle) and F is total area occupied by the image. This formula is not precisely scale invariant (due to variable shade scales) but it can still be used for most of operations we need.

Both the formulas have common nice property that it can be easily approximated from the given image just by looking at the number of intersections of pictures with random lines. And this is exactly how we calculate numbers for any picture.

But we do not calculate only Combinatorial Entropy of the whole picture but we compute it also for the parts of the picture. Values in all the parts cannot be derived from the Combinatorial Entropy of the picture and are essentially dependent on the picture itself. It is one of the reasons why we think that the distribution of Combinatorial Entropies gives some valuable information about the structure of the picture. This is leading to our main criterion for a balanced, harmonious or aesthetically pleasing picture.

6 Main Thesis

So we can measure (rather easily) Combinatorial Entropy of a picture. But what is its meaning? It is an average density, an approximation to the density. Note that the distribution of the density in turn determines the picture itself (this is the key part of the technique of computer tomography [6]).

But the semantic and aesthetic contents of every picture induce covering and refinement of the global information. We are led by experience, by the "logic" of the picture to inspect its parts, to compare them. Not all parts have the same importance and the resulting hierarchical structure is the integral part of the image processing.

What is the abstract core of this hierarchical structure?

With the help of model theory we view it as countable homogeneous weighted atom-less lattice which is being truncated at the level of the fractal dimension of the picture. This in turn is approximated by our senses as fuzzy truncated blots which are inseparable. In the technical and machine drawings these may be pixels and the digital topology is here relevant and produces important results (and insights). The comparison and mutual evaluation of parts of a picture is expressed by our main thesis [1], [16]:

Hereditary Thesis: A picture is harmonious if any two of its similarly meaningful parts have similar combinatorial entropies.

The similarity of meaning is not defined but it is not a primitive notion. The similarity is governed by topological and algebraical considerations. This leads to algebra of a picture. This in turn is defined as a fuzzy factorization of the homogenean atom-less space of the picture. However in concrete instances this complex description is often not a handicap. Besides if no preferences in a picture are given then the uniform partitions serve as a good model.

Figure 6: Antropogeometry, [15]

7 Examples

We include several examples together with the results of measuring. We leave to a reader to interpret relevance of our experiments. The combinatorial entropies of whole picture together with entropies of its regular subdivisions are depicted by a table where the entropy of a rectangle is stated in its middle. In this way are stated entropies of early paintings of both fathers of cubism (Figure 2), see [19].

In the case where we indicate only the entropy of the whole picture then the number is listed simply under the picture (such as in Figures 4,5,8). It seems that the entropy measurements fit nicely with our intuitive feeling about the "density" of a picture. This is true even when comparing photographs with drawings (such as in Figure 7) where we compared a title photo of [22] by a renown Prague photographers S. Tůma with the second author drawing (which in fact motivated it).

Figure 8 contains two version of a drawing of our institute in Prague by the second author. One is the copy of original drawing while the other has been processed by thinning algorithm to get only thin lines. Its density is substantially higher.

We have selected pictures which are balanced so that we can analyze the picture by means of simple algebra of regular subdivisions.

Let us finish with a hereditary analysis of Venice drawing by the second author which is part of the abstract.

1,33	1,30	1,73	1,98	2,06	1,34	1,47	1,85
1,04	1,15	1,57	1,26	2,01	1,66	1,18	1,27
0,94	0,54	0,92	0,96	1,02	0,85	0,73	0,76
0,80	1,00	0,92	0,64	0,43	0,67	0,46	0,48
0,82	0,73	0,52	0,64	0,36	0,43	0,73	0,66
0,64	0,77	1,02	0,84	0,53	0,66	1,11	1,04
0,64	0,88	0,88	1,50	1,18	0,91	0,84	1,05
0,73	0,76	1,07	1,26	1,14	0,90	0,33	0,75

2,38	3,24	3,47	2,82
1,65	1,71	1,48	1,20
1,44	1,56	0,95	1,66
1,56	2,25	2,05	1,54

4,55	4,35
3,38	2,85

7,82

0,11	0,11	0,42	0,65	0,71	0,27	0,55	0,12
1,36	0,84	1,81	2,01	1,49	1,85	0,77	0,96
1,90	1,81	1,84	2,18	1,78	1,81	1,69	1,89
1,75	1,40	1,80	1,93	2,06	2,02	1,63	1,87
2,12	1,35	1,38	2,02	1,93	1,73	1,50	1,44
1,89	1,67	1,81	1,82	1,87	1,94	1,14	1,80
1,77	2,08	1,52	2,05	1,82	1,70	1,70	1,74
1,78	1,90	2,02	2,08	1,93	1,99	1,75	

1,26	2,78	2,39	1,19
3,41	3,87	3,96	3,43
3,36	3,74	3,63	2,63
3,91	3,62	3,64	2,51

5,94	5,70
7,53	5,92

12,95

Figure 7: Malá Strana, view from the Department of Applied Mathematics.

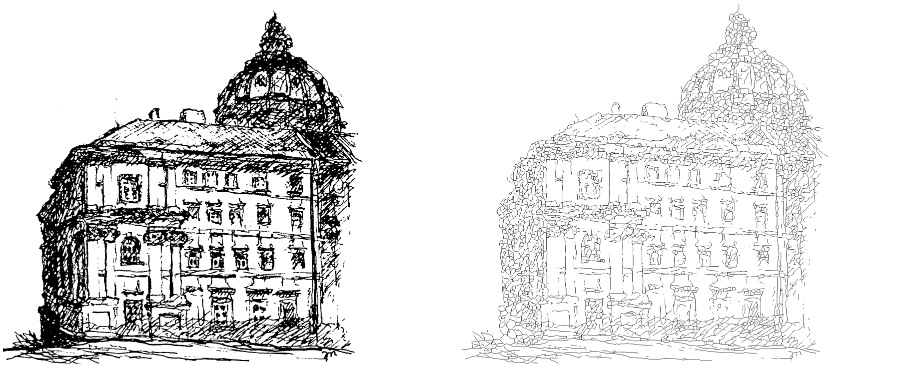


Figure 8: Building of Department of Applied Mathematics. A drawing (28.46) and its thinned version (35.28).

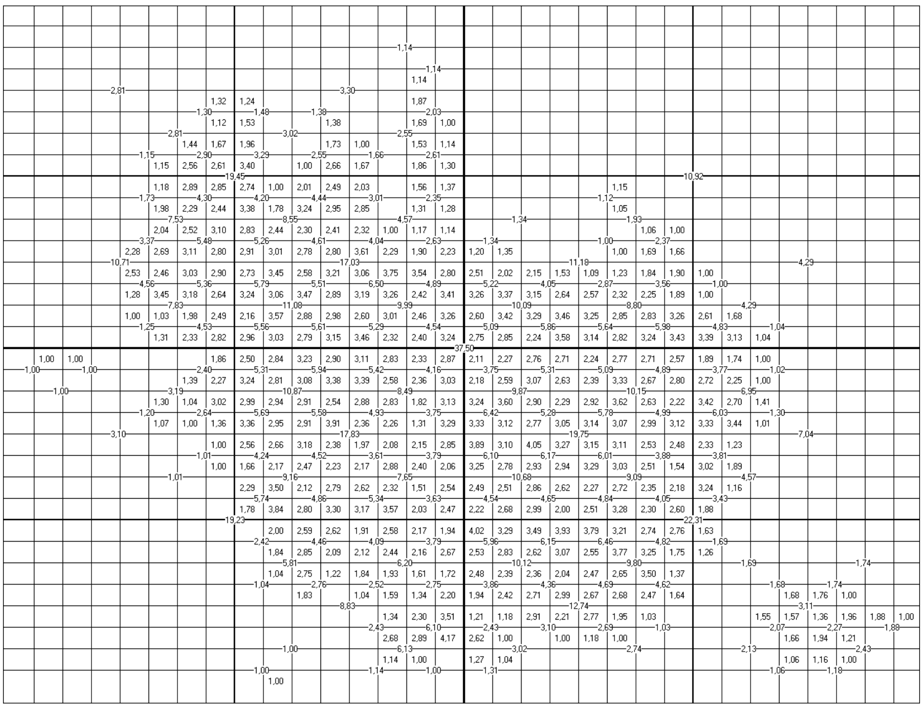


Figure 9: Hereditary analysis of Venice drawing.

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