The Edmonds-Gallai Decomposition for the k-Piece Packing Problem

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Abstract

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Generalizing Kaneko's long path packing problem, Hartvigsen, Hell and Szabó consider a new type of undirected graph packing problem, called the *k-piece packing problem.* A k-piece is a simple, connected graph with highest degree exactly k so in the case $k = 1$ we get the classical matching problem. They give a polynomial algorithm, a Tutte-type characterization and a Berge-type minimax formula for the k-piece packing problem. However, they leave open the question of an Edmonds-Gallai type decomposition. This paper fills this gap by describing such a decomposition. We also prove that the vertex sets coverable by k-piece packings have a certain matroidal structure.

1 Introduction

In this paper all graphs are simple and undirected. Given a set $\mathcal{F}$ of graphs, an *$\mathcal{F}$-packing* of a graph G is a subgraph P of G such that each connected component of P is isomorphic to a member of $\mathcal{F}$. An $\mathcal{F}$-packing P is called *maximal* if there is no $\mathcal{F}$-packing P' with $V(P) \subsetneq V(P')$. An $\mathcal{F}$-packing is *maximum* if it covers a maximum number of vertices of G and it is *perfect* if it covers every vertex of G. The $\mathcal{F}$-packing problem is to describe the properties of the $\mathcal{F}$-packings of G. Finally, the $\mathcal{F}$-packing problem is *polynomial* if for all input graphs G the size of the maximum $\mathcal{F}$-packings of G can be determined in time polynomial in the size of G. (The size of a graph is the number of its vertices.)

Several polynomial $\mathcal{F}$-packing problems are known in the case $K_2 \in \mathcal{F}$. For instance, we get a polynomial packing problem if $\mathcal{F}$ consists of K_2 and a finite set of hypomatchable graphs [2, 3, 4, 6]. A complete classification of the $\{K_2, F\}$-packing problems for graphs F is given in [10]. In all known polynomial $\mathcal{F}$-packing problems with $K_2 \in \mathcal{F}$ it holds that each maximal $\mathcal{F}$-packing is maximum too; those vertex sets which can be covered by an $\mathcal{F}$-packing form a matroid (this is the *matroidal property*); and the analogue of the Edmonds-Gallai structure theorem holds.

The first polynomial $\mathcal{F}$-packing problem with $K_2 \notin \mathcal{F}$ was considered by Kaneko [7], who presented a Tutte-type characterization of graphs having a perfect packing by *long paths*, ie. by paths of length at least 2. A shorter proof for Kaneko's theorem and a min-max formula was subsequently found by Kano, Katona and Király [8] but polynomiality remained open. The long path packing problem was generalized by Hartvigsen, Hell and Szabó [5] by introducing the *k-piece packing problem*, ie. the $\mathcal{F}$-packing problem where $\mathcal{F}$

consists of all *connected graphs with highest degree exactly* k. Such a graph is called a *k-piece.* Note that a 1-piece is just K_2, thus the 1-piece packing problem is the classical matching problem. The 2-piece packing problem is equivalent to the long path packing problem because a 2-piece is either a long path or a circuit C of length at least 3 so deleting an edge from C results in a long path. The main result of [5] is a polynomial algorithm for finding a maximum k-piece packing. From this algorithm a characterization for graphs having a perfect k-piece packing and a min-max result for the size of a maximum k-piece packing are derived.

Neither the Edmonds-Gallai decomposition nor the matroidal property of packings is considered in [5]. This paper fills this gap by giving a canonical Edmonds-Gallai type decomposition for the k-piece packing problem. We also show that the vertex sets coverable by *maximal* k-piece packings have a certain matroidal structure, see Section 2. It turns out that in the k-piece packing problem maximal and maximum packings do not coincide and the maximal packings are of more interest than the maximum ones.

In Section 5 we present some results on *barriers* related to k-piece packings, for instance we prove that the intersection of two barriers is a barrier.

The number of connected components of a graph G is denoted by $c(G)$ and the highest degree of G by $\Delta(G)$. For $X \subseteq V(G)$ the subgraph induced by X is denoted by $G[X]$, and the set of vertices in $V(G) - X$ which are adjacent to a vertex in X is denoted by $\Gamma(X)$. We say that an edge e *enters* X if exactly one end-vertex of e is contained in X. For a subgraph P of G let $G - P = G[V(G) - V(P)]$. Finally, we say that an $\mathcal{F}$-packing P of G *misses* a vertex set $X \subseteq V(G)$ if $X \cap V(P) = \emptyset$ and that P *covers* X if $X \subseteq V(P)$.

2 The theorems

In this section we state the main theorems of the paper. The proofs are contained in Sections 4 and 7. Till Section 8, k is a fixed positive integer.

Definition 2.1. A *k-piece* is a connected graph G with $\Delta(G) = k$.

Definition 2.2. For a graph G we denote $I_G = G[\{v \in V(G) \colon \deg_G(v) \geq k\}]$.

Definition 2.3. A graph G is *hypomatchable* if $G - v$ has a perfect matching for all $v \in V(G)$.

In [5] it was revealed that *galaxies* play a central role in the k-piece packing problem.

Definition 2.4. [5] For an integer $k \geq 1$ the connected graph H is a *k-galaxy* if it satisfies the following properties:

- each component of I_H is a hypomatchable graph,
- for each $v \in V(I_H)$ there exist exactly $k-1$ edges between v and $V(H) - V(I_H)$, each being a cut edge in H.

A hypomatchable graph has no vertex of degree 1 so a k-galaxy has no vertex of degree k. Furthermore, each component of I_H is a hypomatchable graph on at least 3 vertices. Since k is fixed, we shall call a k-galaxy simply a *galaxy*. Galaxies generalize hypomatchable graphs because the 1-galaxies are exactly the hypomatchable graphs. The 2-galaxies were introduced by Kaneko under the name 'sun' [7]. See **Fig. 1** for some galaxies. The vertices of I_H are drawn as big dots and the edges of I_H as thick lines.

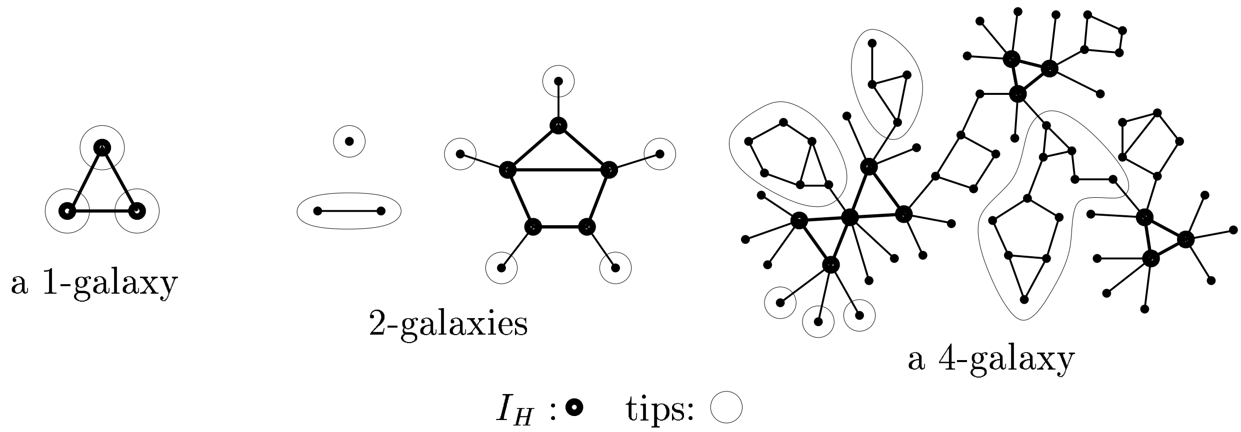


Fig. 1. Galaxies

The following important property of galaxies was proved in [5].

Lemma 2.5. [5] *A k-galaxy has no perfect k-piece packing.* □

Now we introduce special subgraphs of galaxies, called *tips*. Each tip is circled by a thin line in **Fig. 1** (except in the 4-galaxy of **Fig. 1** where not all tips are circled).

Definition 2.6. [5] If $k \geq 2$ then for a k-galaxy H the connected components of $H - V(I_H)$ are called *tips*. In the case $k = 1$ we call each vertex of H a *tip*. The union of vertex sets of the tips is denoted by $W_H \subseteq V(H)$.

So $W_H = V(H)$ if $k = 1$ and $W_H = V(H) - V(I_H)$ if $k \geq 2$. In the case $k \geq 2$ a k-galaxy may consist of only a single tip (a graph with highest degree at most $k-1$), but must always contain at least one tip.

The Edmonds-Gallai structure theorem can be formulated for the k-piece packing problem as follows. The classical Edmonds-Gallai theorem first defines the vertex set D to consist of those vertices which can be missed by a maximal matching. In the k-piece packing problem we have to use a different formulation. This causes the fact that Theorem 2.8 is not a direct generalization of the classical Edmonds-Gallai theorem.

Definition 2.7. For a graph G let

$$U_G = \{v \in V(G) \colon \text{there exists a } \textit{maximal } k\text{-piece packing } P \text{ of } G \text{ with } v \notin V(P)\}.$$

Theorem 2.8. *For a graph G let $D = \{v \colon |U_{G-v}| < |U_G|\}$, $A = \Gamma(D)$ and $C = V(G) - (D \cup A)$. Now*

1. *the connected components of $G[D]$ are k-galaxies,*
2. *for all $\emptyset \neq A' \subseteq A$ the number of those k-galaxy components of $G[D]$ which are adjacent to A' is at least $k|A'| + 1$,*
3. *$G[C]$ has a perfect k-piece packing,*
4. *a k-piece packing P of G is maximal if and only if*
   - (a) *exactly $k|A|$ connected components of $G[D]$ are entered by an edge of P and these components are completely covered by P,*
   - (b) *if H is a component of $G[D]$ not entered by P then $P[H]$ is a maximal k-piece packing of H,*
   - (c) *$P[C]$ is a perfect k-piece packing of $G[C]$,*
5. *for each maximal k-piece packing P of G, the graph $G - P$ has exactly $c(G[D]) - k|A|$ connected components.*

For proof, see Section 4. We could also choose $D = \{v \colon U_{G-v} \subsetneq U_G\}$ by Theorem 4.19.

It is a well known fact in matching theory that those vertex sets which can be covered by a matching form a matroid. In the k-piece packing problem this property holds only in the following weaker form. The proof is contained in Section 7.

Theorem 2.9. *There exists a partition π on $V(G)$ and a matroid $\mathcal{M}$ on π such that the vertex sets of the maximal k-piece packings are exactly the vertex sets of the form $\bigcup\{X : X \in \pi'\}$ where π' is a base of $\mathcal{M}$.*

3 Preliminaries

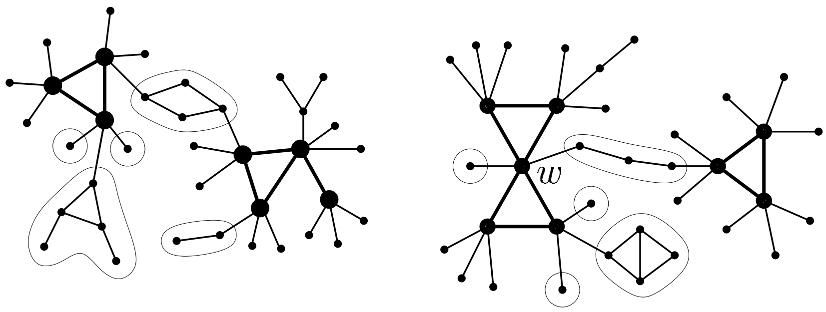
In this section we summarize the results and notions of [5] which are needed to prove the main theorems of the paper. First we introduce two other classes of graphs which are near to galaxies.

Definition 3.1. For an integer $k \geq 2$ the connected graph H is an *almost k-galaxy of type 1* if it satisfies the following properties:

- one of the components of I_H has a perfect matching and the others are hypomatchable,
- for each $v \in V(I_H)$ there exist exactly $k-1$ edges between v and $V(H) - V(I_H)$, each being a cut edge in H.

Definition 3.2. For an integer $k \geq 2$ the connected graph H is an *almost k-galaxy of type 2* if it satisfies the following properties:

- each component of I_H is a hypomatchable graph,
- there is a distinguished vertex $w \in V(I_H)$ such that for each $v \in V(I_H)$ each edge between v and $V(H) - V(I_H)$ is a cut edge in H, and the number of these edges is $k-1$ for $v \neq w$ and $k-2$ for w.



almost k-galaxy of type 1   almost k-galaxy of type 2

Fig. 2. Almost galaxies, $k = 4$

Fig. 2 shows some almost 4-galaxies. Just like in the case of galaxies, we define tips for almost galaxies. Some tips are circled by a thin line in **Fig. 2**.

Definition 3.3. For an almost galaxy H the connected components of $H - I_H$ are called *tips*.

Many properties of the galaxies are explained by the following lemma, which is implicit in [5].

Lemma 3.4. *Each almost k-galaxy has a perfect k-piece packing.*

Proof. First we prove the statement for almost galaxies of type 2. Let H be an almost k-galaxy of type 2. We proceed by induction on $|V(H)|$. Let K be the component of I_H containing the specified vertex w. K is a hypomatchable graph on at least 3 vertices so it is easy to see that w has two neighbors $w', w'' \in V(K)$ such that $K - \{w', w, w''\}$ has a perfect matching M. For each edge $uv \in M$ let P_{uv} be the subgraph of H induced by the vertex set

$$\{u, v\} \cup \bigcup \{V(T) : T \text{ is a tip of } H \text{ adjacent to } \{u, v\}\}.$$

Furthermore, let P_w be the subgraph of H induced by the vertex set

$$\{w', w, w''\} \cup \bigcup \{V(T) : T \text{ is a tip of } H \text{ adjacent to } \{w', w, w''\}\},$$

with the deletion of the edge $w'w''$ (if any). Clearly P_{uv} $(uv \in M)$ and P_w are disjoint k-piece subgraphs of H. Deleting these k-pieces from H, each connected component of the remaining graph is an almost k-galaxy of type 2 so we are done by induction.

Now let H be an almost k-galaxy of type 1. Denote by K the perfectly matchable component of I_H. For each edge uv of a perfect matching of K let P_{uv} be the k-piece subgraph of H induced by the vertex set

$$\{u, v\} \cup \bigcup \{V(T) : T \text{ is a tip of } H \text{ adjacent to } \{u, v\}\}.$$

Deleting these k-pieces from H, each connected component of the remaining graph is an almost k-galaxy of type 2 so we are done by the first part of the proof. □

Lemma 3.5. [5] *If T is a tip of a k-galaxy H then $H - T$ has a perfect k-piece packing.*

Proof. The statement holds for $k = 1$ by definition. Let $k \geq 2$. It is easy to see that each component of $H - T$ is an almost k-galaxy of type 2, which has a perfect k-piece packing by Lemma 3.4. □

For the proof of the following lemma see [5].

Lemma 3.6. [5] *If P is a k-piece packing of the k-galaxy H then there exists a tip T of H such that $V(P) \cap V(T) = \emptyset$.* □

The maximal matchings of a hypomatchable graph H are exactly the perfect matchings of $H - v$ for the vertices $v \in V(H)$. The characterization of the maximal k-piece packings of a k-galaxy can be stated by means of the tips.

Lemma 3.7. [5] *The maximal k-piece packings of a k-galaxy H are exactly the perfect k-piece packings of $H - T$ where T is a tip of H.*

Proof. By Lemmas 3.5 and 3.6. □

The next lemma is another generalization of the defining property 2.3 of hypomatchable graphs. This lemma is only implicit in [5].

Lemma 3.8. *If H is a k-galaxy and $v \in V(H)$ then there exists a vertex set $v \in X \subseteq V(H)$ such that $H[X]$ is connected, $\Delta(H[X]) \leq k - 1$ and $H - X$ has a perfect k-piece packing.*

Proof. The statement is trivial for $k = 1$ so assume that $k \geq 2$. If v is contained in a tip T then let $X = V(T)$. Now $H - X$ has a perfect k-piece packing by Lemma 3.5 so we are done. If $v \in V(I_H)$ then let

$$X = \{v\} \cup \bigcup \{V(T) : T \text{ is a tip of } H \text{ adjacent to } v\}.$$

Clearly $\Delta(H[X]) = k - 1$. It is easy to check that each component of $H - X$ is an almost k-galaxy of type 1 or 2. Hence $H - X$ has a perfect k-piece packing by Lemma 3.4. □

Definition 3.9. A connected graph G is a *k-solar-system* (see **Fig. 3**) if it has a vertex y, called *center*, such that $\deg_G(y) = k$ and $G - y$ has k connected components, each being a k-galaxy.

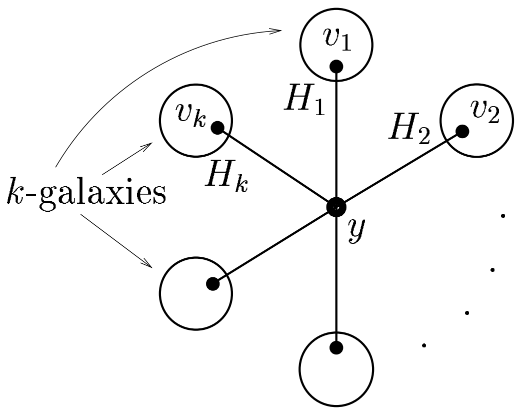


Fig. 3. A k-solar system

Lemma 3.10. *Each k-solar-system has a perfect k-piece packing.*

Proof. Let G be a k-solar-system with center y. Denote the neighbors of y by v_i $(1 \leq i \leq k)$ and denote the k-galaxy component of $G - y$ containing v_i by H_i. Lemma 3.8 implies that for all $1 \leq i \leq k$ there exists a vertex set $v_i \in X_i \subseteq V(H_i)$ such that $H_i - X_i$ has a perfect k-piece packing and $H_i[X_i]$ is a connected graph with highest degree at most $k-1$. The latter condition on $H_i[X_i]$ implies that $G[\{y\} \cup \bigcup_{1 \leq i \leq k} X_i]$ is a k-piece. □

[5] describes a polynomial algorithm finding a maximum k-piece packing in the input graph G. The algorithm consists of two phases and already the first phase obtains a maximal k-piece packing of G which is further refined in the second phase (called 'Re-Rooting procedure') to become a maximum k-piece packing. Now we are interested only in the first phase of the algorithm of [5] to which we simply refer as *the algorithm.* This algorithm is a direct generalization of the alternating forest matching algorithm of Edmonds. It builds certain alternating forests and it outputs a decomposition $V(G) = D \cup A \cup C$ where the sets D, A, C are pairwise disjoint. It also outputs a maximal k-piece packing P of G but we are not interested in it now. The algorithm may have different runs on the same graph G depending on the actual implementation. We refer to the outputs of these runs as *decomposition outputs.* In the next section we prove that the decomposition output is unique for all runs of the algorithm and it is *canonical* for the k-piece packing problem in a certain way. The following proposition is implicit in the description of the algorithm in [5], see **Fig. 4**.

Proposition 3.11. [5] *Each run of the algorithm outputs a decomposition $V(G) = D \cup A \cup C$ where D, A, C are pairwise disjoint and*

1. *the connected components of $G[D]$ are k-galaxies,*
2. *G contains no edge joining D to C,*
3. *for all $\emptyset \neq A' \subseteq A$ the number of those k-galaxy components of $G[D]$ which are adjacent to A' is at least $k|A'| + 1$,*
4. *$G[C]$ has a perfect k-piece packing.*

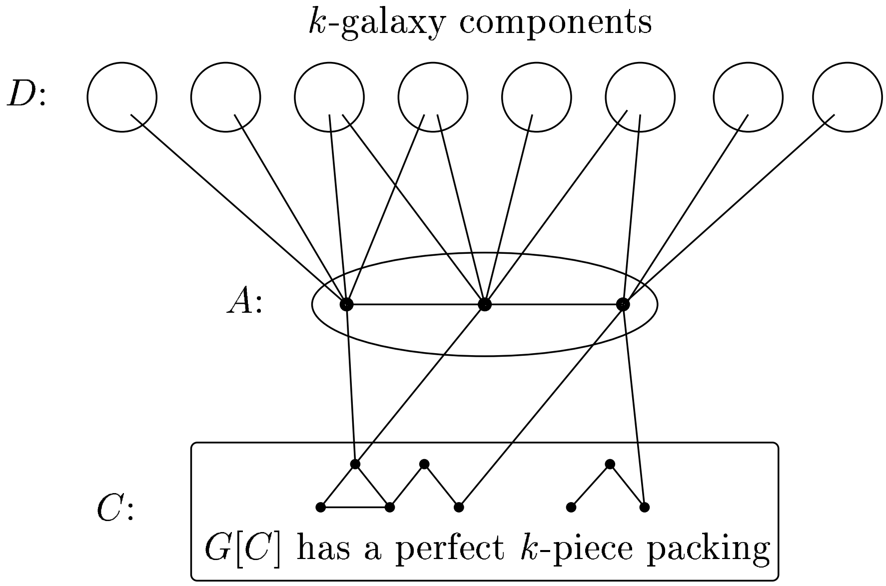


Fig. 4. A decomposition output of the algorithm, $k = 2$

Any decomposition output of the algorithm implies the Tutte-type existence theorem 3.13 for the k-piece packing problem, proved in [5].

Definition 3.12. Let k-gal(G) denote the number of those connected components of the graph G that are k-galaxies.

Theorem 3.13. [5] *A graph G has a perfect k-piece packing if and only if*

$$k\text{-gal}(G - A) \leq k|A|$$

for all set of vertices $A \subseteq V(G)$.

Proof. The "only if" part is straightforward using that a k-galaxy has no k-piece packing by Lemma 2.5. On the other hand, if G has no perfect k-piece packing then A in any decomposition output of the algorithm will do. □

4 The Edmonds-Gallai decomposition

In this section we prove that the decomposition output is unique for all runs of the algorithm and that this decomposition has the properties described in Theorem 2.8.

Definition 4.1. For $A \subseteq V(G)$ let

$$D^A = \bigcup \{V(H) \colon H \text{ is a } k\text{-galaxy component of } G - A\}.$$

We use the notation D^A_G if confusion may arise. Moreover, let $C^A = V(G) - (D^A \cup A)$ (or C^A_G).

Definition 4.2. The vertex set $A \subseteq V(G)$ has *k-surplus* if for all $\emptyset \neq A' \subseteq A$ the number of k-galaxy components of $G[D^A]$ adjacent to A' is at least $k|A'| + 1$. The vertex set A is *perfect* if C^A has a perfect k-piece packing.

Definition 4.3. We say that a vertex set $A \subseteq V(G)$ can be *k-matched into $X \subseteq V(G) - A$ by M* if M is a subgraph of G with $k|A|$ edges such that $\deg_M(v) = k$ for all $v \in A$ and exactly $k|A|$ connected components of $G[X]$ are entered by an edge of M (each by one edge). The vertex set A can be *k-matched into $X \subseteq V(G) - A$* if there exists a subgraph M of G such that A can be k-matched into X by M.

The following property (in fact, characterization) of the vertex sets with k-surplus is implied by Hall's theorem.

Lemma 4.4. *If $A \subseteq V(G)$ has k-surplus then A can be k-matched into $D^A - V(H)$ for each connected component H of $G[D^A]$.*

Using these definitions we can reformulate Proposition 3.11.

Proposition 4.5. *For any decomposition output $V(G) = D \cup A \cup C$ of the algorithm the set A is perfect with k-surplus.*

Proof. A k-galaxy has no perfect k-piece packing so $D^A = D$ and $C^A = C$. So Proposition 3.11, *3.* is tantamount to that A has k-surplus and *4.* to that A is perfect. ☐

The next lemma describes an important property of the galaxies.

Lemma 4.6. *If H is a k-galaxy and $\emptyset \neq X \subseteq V(H)$ then $k\text{-gal}(H - X) \leq k|X| - 1$.*

Proof. The statement is well-known for $k = 1$. Indeed, otherwise for $x \in X$ the number of hypomatchable components of $(H - x) - (X - x)$ is more than $|X - x|$ implying that $H - x$ has no perfect matching, a contradiction.

For $k \geq 2$ it is easier to prove the lemma for a broader set of graphs, called *pseudo galaxies.*

Definition. For an integer $k \geq 2$ the connected graph G is a *pseudo k-galaxy* if for each $v \in V(I_G)$ there exist exactly $k - 1$ edges between v and $V(G) - V(I_G)$, each being a cut edge in G.

Note, that this is just the definition of the k-galaxies with the relaxation that the connected components of I_G need not be hypomatchable. What we actually prove is Lemma 4.7 which immediately implies Lemma 4.6.

Lemma 4.7. *If G is a pseudo k-galaxy and $\emptyset \neq X \subseteq V(G)$ is a vertex set with the property that each vertex of $X \cap V(I_G)$ is contained in a hypomatchable component of I_G then $k\text{-gal}(G - X) \leq k|X| - 1$ holds.*

Proof. Suppose that G is a pseudo galaxy of minimum size for which a vertex set $\emptyset \neq X \subseteq V(G)$ fails Lemma 4.7, ie. $k\text{-gal}(G - X) \geq k|X|$ holds. $\deg_G(v) \leq k - 1$ for vertices $v \notin V(I_G)$ so clearly $X \cap V(I_G) \neq \emptyset$.

Let F be a hypomatchable component of I_G with $X_F = X \cap V(F) \neq \emptyset$. Assume that the number of k-galaxy components of $G - X_F$ is s and denote these components by $H_1, \ldots, H_s$. It is easy to see that the other components of $G - X_F$ are pseudo k-galaxies. Let their number be t and denote them by $G_1, \ldots, G_t$. Note that each component K of $G - X_F$ satisfies the condition of Lemma 4.7, ie. each vertex of $(X \cap V(K)) \cap V(I_K)$ is contained in a hypomatchable component of I_K. Let h (resp. g) denote the number of vertices $x \in X$ contained in a k-galaxy (resp. pseudo k-galaxy) component of $G - X_F$. Clearly $|X| = |X_F| + h + g$.

Let $X_i = X \cap G_i$ for $1 \leq i \leq t$. By induction, $k\text{-gal}(G_i - X_i) \leq k|X_i|$ for $1 \leq i \leq t$ independently of the emptiness of X_i. So the number of k-galaxy components of $G - X$ contained in a component G_i for $1 \leq i \leq t$ is at most kg.

Now we bound s. Let H_i be a k-galaxy component of $G - X_F$ such that $Y = V(H_i) \cap V(F) \neq \emptyset$. It is easy to see that $F[Y]$ is connected. This implies that $F[Y]$ is a component of I_{H_i} so it is hypomatchable. The number of such hypomatchable components $F[Y]$ is at most $k|X_F| - 1$ by the already proved case $k = 1$ of Lemma 4.6. Thus the number of k-galaxy components of $G - X_F$ which intersect $V(F)$ is at most $k|X_F| - 1$. On the other hand, the number of components of $G - X_F$ which do not intersect $V(F)$ is exactly $(k - 1)|X_F|$ because each vertex $v \in X_F \subseteq V(F)$ is incident with exactly $k - 1$ cut edges in G. So $s \leq |X_F| - 1 + (k - 1)|X_F| = k|X_F| - 1$.

Let s' be the number of those k-galaxy components H_i of $G - X_F$ for which $X^i = X \cap V(H_i) \neq \emptyset$. For such a component $k\text{-gal}(H_i - X^i) \leq k|X^i| - 1$ holds by the minimality of G. So these components contain altogether at most $kh - s'$ of the k-galaxy components of $G - X$. Finally, it is trivial that the number of k-galaxy components H_i of $G - X_F$ for which $X \cap V(H_i) = \emptyset$

is $s - s'$. Summarizing,

$$\begin{aligned} k\text{-gal}(G - X) &\leq kg + (kh - s') + (s - s') \leq k(h + g) + s \\ &\leq k(|X_F| + h + g) - 1 = k|X| - 1. \end{aligned}$$

□

Theorem 4.8. *If A_1, $A_2 \subseteq V(G)$ are perfect vertex sets with k-surplus then $A_1 = A_2$.*

Proof. Let $D_i = D^{A_i}$ and $C_i = C^{A_i}$ for $i = 1, 2$. Denote by g_i the number of components of $G[D_i]$ intersecting A_{3-i} for $i = 1, 2$. We prove that $g_1 = g_2 = 0$. Suppose that $g_1 \geq g_2$ and that $A_2' = A_2 \cap D_1 \neq \emptyset$. By the k-surplus of A_2, the vertex set A_2' is adjacent to at least $k|A_2'| + 1$ k-galaxy components of $G[D_2]$. Let K be a k-galaxy component of $G[D_2]$ which is adjacent to A_2'. If $V(K) \cap A_1 = \emptyset$ then $V(K) \subseteq D_1$ because $A_2' \subseteq D_1$ so K is contained in a k-galaxy component of $G[D_1]$. Thus the number of such components K with $V(K) \cap A_1 = \emptyset$ is at most $k|A_2'| - g_1$ by Lemma 4.6. So the number of components of $G[D_2]$ which are adjacent to A_2' and intersect A_1 is at least $g_1 + 1$. Thus $g_2 \geq g_1 + 1$, a contradiction. This implies $g_1 = g_2 = 0$.

Suppose that $A_1 \setminus A_2 \neq \emptyset$. By the k-surplus of A_1 the number of components of $G[D_1]$ which are adjacent to $A_1 \setminus A_2$ is at least $k|A_1 \setminus A_2| + 1$. These components do not intersect A_2 because $g_1 = 0$. Hence k-gal$(G[C_2] - (A_1 \setminus A_2)) \geq k|A_1 \setminus A_2| + 1$ implying that $G[C_2]$ has no perfect k-piece packing by Theorem 3.13, a contradiction.

So $A_1 \subseteq A_2$ and by symmetry, $A_1 = A_2$. □

Theorem 4.9. *The decomposition output is unique for all runs of the algorithm.*

Proof. Let $V(G) = D \cup A \cup C$ be any decomposition output of the algorithm. Proposition 4.5 implies that A is perfect with k-surplus hence it is unique by Theorem 4.8. Finally, a k-galaxy has no perfect k-piece packing so $D = D^A$ and $C = C^A$. □

Hence the following definition is sound:

Definition 4.10. The unique decomposition output of the algorithm is denoted by $V(G) = D_G \cup A_G \cup C_G$ and called the *canonical decomposition of* G with respect to the k-piece packing problem.

Proposition 4.5 and Theorem 4.8 imply

Corollary 4.11. *If $A \subseteq V(G)$ is perfect and has k-surplus then $A = A_G$.*

Now we investigate the structure of *maximal* k-piece packings of G.

Lemma 4.12. *Each maximal k-piece packing P of G has the following structure:*

1. *exactly $k|A_G|$ connected components of $G[D_G]$ are entered by an edge of P and these components are completely covered by P,*
2. *if H is a component of $G[D]$ not entered by P then $P[H]$ is a maximal k-piece packing of H, ie. there exists a tip T of H such that $P[H]$ is a perfect k-piece packing of $H - T$, and*
3. *$P[C_G]$ is a perfect k-piece packing of $G[C_G]$.*

Proof. Let P be a maximal k-piece packing of G. We construct a k-piece packing P' with $V(P') \supseteq V(P)$ such that if P fails any of properties *1.-3.* then $V(P') \supsetneq V(P)$ would hold. We need the theorem of Mendelsohn and Dulmage (see 1.4.3 in [11]).

Theorem 4.13. (Mendelsohn, Dulmage) *Let B be a bipartite graph with color classes U and V. If B has a matching covering $U' \subseteq U$ and another matching covering $V' \subseteq V$ then it has a matching covering $U' \cup V'$.*

We apply Theorem 4.13 to the bipartite graph B_A defined as follows.

Definition 4.14. We denote $kA_G = \{v^i : v \in A_G,\ 1 \le i \le k\}$. Let $V(B_A) = kA_G \cup \{H : H$ is a component of $G[D_G]\}$ and $E(B_A) = \{v^i H : 1 \le i \le k,\ v$ is adjacent to H in $G\}$.

B_A has a matching covering kA_G by the k-surplus of A_G. Moreover, P shows that B_A has a matching covering $\mathcal{H}_P = \{H : H$ is a component of $G[D_G]$ entered by an edge of $P\}$. So Theorem 4.13 implies that B_A has a matching M with vertex set $kA_G \cup \mathcal{H}_M$ where $\mathcal{H}_P \subseteq \mathcal{H}_M$. Using Lemma 3.10, this matching gives rise to a perfect k-piece packing P_1 in the subgraph induced by

$$A_G \cup \bigcup \{V(H) : H \in \mathcal{H}_M\}.$$

Let H be a component of $G[D_G]$ such that $H \notin \mathcal{H}_M$. By Lemma 3.6 there exists a tip T of H such that $V(P) \cap V(T) = \emptyset$. Take a perfect k-piece packing of $H - T$ guaranteed by Lemma 3.5 and denote the union of these k-pieces by P_2. Finally, let P_3 be a perfect k-piece packing of $G[C_G]$. With $P' = P_1 \cup P_2 \cup P_3$ we get that $V(P') \supseteq V(P)$.

Trivially $|\mathcal{H}_P| \leq k|A_G|$. In fact, $|\mathcal{H}_P| = k|A_G|$ holds here because otherwise the matching M of B_A would enter strictly more components of $G[D_G]$ than P, resulting in $V(P') \supsetneq V(P)$, a contradiction. Properties *1.* and *2.* are straightforward by the maximality of P and by Lemmas 3.7 and 3.10. For *3.* observe that P has no edge joining A_G to C_G because otherwise $|\mathcal{H}_P| < k|A_G|$ would hold. □

Observe that Lemma 4.12 holds also by replacing A_G by A, D_G by D^A and C_G by C^A where $A \subseteq V(G)$ is a perfect vertex set which can be k-matched into D^A. This observation will be needed in the proof of Theorem 4.19.

Lemma 4.15. *If P is a k-piece packing satisfying properties 1., 2. and 3. of* Lemma 4.12 *then P is maximal.*

Proof. Properties *1., 2.* and *3.* imply that $c(G-P) = c(G[D_G]) - k|A_G|$ and that each component of $G - P$ is a tip of some galaxy component of $G[D_G]$. Let $\mathcal{H}_P = \{H : H$ is a component of $G[D_G]$ entered by an edge of $P\}$. Suppose that P' is a k-piece packing covering $V(P)$ and one more vertex $v \notin V(P)$. Now v is contained in a tip of a galaxy $H \notin \mathcal{H}_P$. So Property *2.* implies that P' intersects each tip of H thus P' enters H by Lemma 3.6. Moreover, P' enters each component in $\mathcal{H}_P$ by Lemma 3.6. So P' enters at least $k|A_G| + 1$ components of $G[D_G]$ which is impossible because $\deg_{P'}(v) \leq k$ for $v \in A_G$. □

For characterizing D_G in the canonical decomposition first we need to characterize the union of the vertex sets of tips in $G[D_G]$. Recall that U_G was introduced in Definition 2.7.

Definition 4.16. Let $W_G = \bigcup\{W_H : H$ is a k-galaxy component of $G[D_G]\}$.

Lemma 4.17. $W_G = U_G$.

Proof. Lemma 4.12 implies that $U_G \subseteq W_G$. On the other hand, let $v \in W_G$ be a vertex contained in a tip T of a k-galaxy component H_0 of $G[D_G]$. A_G has k-surplus so A_G can be k-matched into $D_G - V(H_0)$ by a subgraph M of G. Let $\mathcal{H}_M = \{H : H$ is a component of $G[D_G]$ entered by an edge of $M\}$. Using Lemma 3.10, M gives rise to a perfect k-piece packing P_1 in the subgraph induced by $A_G \cup \bigcup\{V(H) : H \in \mathcal{H}_M\}$. By Lemma 3.7, for each component $H \notin \mathcal{H}_M$ of $G[D_G]$ we can take a perfect k-piece packing of $H - T_H$ where T_H is any tip of H. Take care to choose $T_{H_0} = T$. The union of these k-pieces is denoted by P_2. Finally, let P_3 be a perfect k-piece packing of $G[C_G]$. By Lemma 4.15, the k-piece packing $P_1 \cup P_2 \cup P_3$ is maximal and it misses $v \in W_G$. □

In the matching case (ie. in the case $k = 1$) it holds that $W_G = D_G$ thus Lemma 4.17 itself characterizes the canonical D_G. In the general case only $W_G \subseteq D_G$ holds so we have to go one step further in order to characterize D_G in Theorem 4.19. First we need the following lemma.

Lemma 4.18. *If H is a k-galaxy and $v \in V(H)$ then each component of $H - v$ is either a k-galaxy or has a perfect k-piece packing. Moreover,*

$$\bigcup\{W_K : K \text{ is a } k\text{-galaxy component of } H - v\} \subsetneq W_H.$$

Proof. The statement is well-known for $k = 1$ so assume $k \geq 2$. If v is contained in a tip then clearly each component of $H - v$ is either a k-galaxy or an almost k-galaxy of type 2. Each almost k-galaxy component has a perfect k-piece packing by Lemma 3.4. Furthermore,

$$\bigcup\{V(T) : T \text{ is a tip in a component of } H - v\} = W_H - v$$

so we are done. If $v \in I_H$ then $H - v$ consists of k-galaxy components (the number of which is exactly $k - 1$), and almost galaxy components of type 1, the number of which is at least 1. Each almost k-galaxy component has a perfect k-piece packing by Lemma 3.4. Moreover,

$$\bigcup\{V(T) : T \text{ is a tip in a component of } H - v\} = W_H,$$

but each almost galaxy component contains at least one tip of H, yielding that

$$\bigcup\{V(T) : T \text{ is a tip in an almost } k\text{-galaxy component of } H - v\} \neq \emptyset.$$

□

Theorem 4.19. *$D_G = \{v : U_{G-v} \subsetneq U_G\} = \{v : |U_{G-v}| < |U_G|\}$ holds for all graphs G.*

Proof. We investigate the canonical decomposition of the graph $G - v$.

1. Let $v \in C_G$. Denote the graph $G[C_G - v]$ by G'. Observe that in the graph $G - v$ the set $A_G \cup A_{G'}$ is perfect with k-surplus. So $A_{G-v} = A_G \cup A_{G'}$ by Corollary 4.11, yielding that $W_{G-v} \supseteq W_G$, ie. $U_{G-v} \supseteq U_G$ by Lemma 4.17.

2. Let $v \in A_G$. In the graph $G - v$ the set $A_G - v$ is perfect with k-surplus so $A_{G-v} = A_G - v$ by Corollary 4.11. Hence $W_{G-v} = W_G$ or equivalently, $U_{G-v} = U_G$ by Lemma 4.17.

3. Finally, suppose that $v \in V(H)$ for a k-galaxy component H of $G[D_G]$. $\emptyset$ is perfect and has k-surplus in the graph $H - v$ by Lemma 4.18 so $A_{H-v} = \emptyset$ by Corollary 4.11, yielding that

$$D_{H-v} = \{V(K) \colon K \text{ is a } k\text{-galaxy component of } H - v\} \text{ and}$$

$$C_{H-v} = \{V(K) \colon K \text{ comp. of } H - v \text{ with a perfect } k\text{-piece packing}\}.$$

Let $D' = D_{A_G}^{G-v} = (D_G \setminus V(H)) \cup D_{H-v}$, $C' = C_{A_G}^{G-v} = C_G \cup C_{H-v}$ and $W' = \{V(T) \colon T \text{ is a tip in a component of } G[D']\}$. Lemma 4.18 implies that $W' \subsetneq W_G$. In the graph $G - v$ the set A_G is perfect because $G[C'']$ has a perfect k-piece packing. Moreover, A_G can be k-matched into D' in $G - v$ because A_G has k-surplus in G. So the statement of Lemma 4.12 holds for A_G in the graph $G - v$, as we mentioned after the proof of 4.12. This especially implies that each maximal k-piece packing of $G - v$ misses only vertices in W'. So $U_{G-v} \subseteq W' \subsetneq W_G = U_G$ and we are done.

□

At this point the proof of Theorem 2.8 is straightforward using the results of this section.

Proof of Theorem 2.8. $D = D_G$, $A = A_G$ and $C = C_G$ by Theorem 4.19. Now Property *1.* holds by definition. A_G is perfect with k-surplus which is just tantamount to Properties *2.* and *3.* Property *4.* is equivalent to Lemmas 4.12 and 4.15. Finally, *5.* follows from Property *4.* □

By Theorem 2.8 the graph G has a canonical decomposition $V(G) = D_k \cup A_k \cup C_k$ for each $k \geq 1$. Here $D_1 \cup A_1 \cup C_1$ is the classical Edmonds-Gallai decomposition. Observe that $A_k = C_k = \emptyset$ if $k \geq \Delta(G) + 1$ and $D_k = A_k = \emptyset$ if $k = \Delta(G)$. Nevertheless, there does not seem to be any nice relation between the decompositions for different k's.

5 The calculus of barriers

In this section we prove some properties of *barriers* which we define to be those vertex sets A which maximize $k\text{-gal}(G - A) - k|A|$. Not all of the following results generalize the theory of barriers described by Lovász and Plummer [11] because they count the odd size components instead of the hypomatchable components as we do.

Definition 5.1. For $A \subseteq V(G)$ the *deficiency* of A is $\text{def}(A) = k\text{-gal}(G - A) - k|A|$. The *deficiency* of G is

$$\text{def}(G) = \max\{\text{def}(A) \colon A \subseteq V(G)\}.$$

Finally, $A \subseteq V(G)$ is a *barrier* if $\text{def}(A) = \text{def}(G)$.

Theorem 3.13 is tantamount to saying that G has a perfect k-piece packing if and only if $\text{def}(G) = 0$. In this case $\emptyset$ is a barrier with deficiency 0.

Proposition 5.2. *A_G is a barrier of G.*

Proof. Let P be a maximal k-piece packing of G. Lemma 4.12 implies that $c(G - P) = k\text{-gal}(G - A_G) - k|A_G| = \text{def}(A_G)$. On the other hand, let A be a barrier of G. The number of components of $G[D^A]$ which are not entered by P is clearly at least $k\text{-gal}(G - A) - k|A| = \text{def}(A)$. Thus $c(G - P) \geq \text{def}(A)$ by Lemma 2.5. This implies that $\text{def}(A_G) \geq \text{def}(A)$ and so that A_G is a barrier. □

In the matching case (ie. when $k = 1$) each maximum (and so each maximal) matching misses $\text{def}(G)$ vertices of G. This property fails for general k because a maximal k-piece packing of a galaxy may miss an arbitrary number of vertices instead of only one (namely, the vertices of a tip). What is salvaged, is that $c(G - P) = \text{def}(G)$ for each maximal k-piece packing P by Lemma 4.12 and Proposition 5.2.

Lemma 5.3. *Each barrier is perfect.*

Proof. Let A be a barrier of G. Assume that $G[C^A]$ has no perfect k-piece packing. Then by Theorem 3.13 there exists a set $X \subseteq C^A$ such that $k\text{-gal}(G[C^A] - X) - k|X| > 0$. But then $\text{def}(A \cup X) > \text{def}(G)$ would hold, a contradiction. □

Theorem 5.4. *If A is a barrier then $A_G \subseteq A$ and $D_G \subseteq D^A$.*

Proof. Let A be a barrier of G and let $\mathcal{H} = \{H : H \text{ is a component of } G[D^A]\}$. For $\mathcal{J} \subseteq \mathcal{H}$ let

$$\Gamma(\mathcal{J}) = \left\{ v \in A : v \text{ is adjacent to } \bigcup\{V(H) : H \in \mathcal{J}\} \right\}.$$

Consider the following function f on $\mathcal{H}$: for $\mathcal{J} \subseteq \mathcal{H}$ let $f(\mathcal{J}) = |\mathcal{J}| - k|\Gamma(\mathcal{J})|$. Clearly $f(\mathcal{J}) \leq \text{def}(G)$ for $\mathcal{J} \subseteq \mathcal{H}$ and f is a supermodular function. Suppose that $f(\mathcal{J}_1) = f(\mathcal{J}_2) = \text{def}(G)$ for $\mathcal{J}_1$, $\mathcal{J}_2 \subseteq \mathcal{H}$. Now $2 \cdot \text{def}(G) = f(\mathcal{J}_1) + f(\mathcal{J}_2) \leq f(\mathcal{J}_1 \cap \mathcal{J}_2) + f(\mathcal{J}_1 \cup \mathcal{J}_2) \leq 2 \cdot \text{def}(G)$ implying that $f(\mathcal{J}_1 \cap \mathcal{J}_2) = \text{def}(G)$. $f(\mathcal{H}) = \text{def}(G)$ thus there exists an inclusion-wise minimum set $\mathcal{H}_0 \subseteq \mathcal{H}$ with $f(\mathcal{H}_0) = \text{def}(G)$. Let $A_0 = \Gamma(\mathcal{H}_0)$. The set A_0 has k-surplus because $\mathcal{H}_0$ is minimum.

Let $D' = \bigcup\{V(H) : H \in \mathcal{H} - \mathcal{H}_0\}$. We state that $A - A_0$ can be k-matched into D' by a subgraph M of G. This is due to Hall's theorem: if $Y \subseteq A - A_0$ was adjacent to less than $k|Y|$ components of $G[D']$ then $\text{def}(A - Y) > \text{def}(A) = \text{def}(G)$ would hold because Y is not adjacent to any component $H \in \mathcal{H}_0$. Moreover, $k|A - A_0| = |\mathcal{H} - \mathcal{H}_0|$ so M gives rise to a perfect k-piece packing in $D' \cup (A - A_0)$ using Lemma 3.10. Moreover, by Lemma 5.3, $G[C_A]$ has a perfect k-piece packing so A_0 is perfect.

Summarizing, A_0 is perfect with k-surplus so $A_G = A_0 \subseteq A$ by Corollary 4.11. Moreover, clearly $D_G = D^{A_0} = D^A - D'$. □

Note that in this proof, $A - A_0$ is adjacent to at most $k|A - A_0|$ components in $G[D^A]$ hence if A has k-surplus then $A - A_0 = \emptyset$. This implies that A_G is the only barrier with k-surplus.

Theorem 5.5. *The intersection of two barriers is a barrier.*

Proof. Let A_1, A_2 be barriers of G. We let $D_i = D^{A_i}$ and $C_i = C^{A_i}$ for $i = 1, 2$. Denote by g_i the number of components of $G[D_i]$ intersecting A_{3-i}. Wlog. we may assume that $g_1 \leq g_2$. Furthermore,

- g_C is the number of components of $G[D_1]$ contained in C_2,
- g_D is the number of components of $G[D_1]$ contained in D_2 and not adjacent to $A_1 \cap D_2$,
- g'_D is the number of components of $G[D_1]$ contained in D_2 and adjacent to $A_1 \cap D_2$.

Now

$$k|A_1| + \mathrm{def}(G) = k\text{-gal}(G - A_1) = g_C + g_1 + g_D + g'_D.$$

The graph $G[C_2]$ has a perfect k-piece packing by Lemma 5.3 so

$$g_C \leq k|A_1 \cap C_2|.$$

The components of $G[D_1]$ which are contained in D_2 but which are not adjacent to $A_1 \cap D_2$ are connected components of $G - (A_1 \cap A_2)$ as well so

$$g_D \leq k\text{-gal}(G - (A_1 \cap A_2)).$$

Each component of $G[D_1]$ which is contained in D_2 and which is adjacent to $A_1 \cap D_2$ is contained in some component H of $G[D_2]$. The number of such components H was denoted by g_2. Hence Lemma 4.6 implies that

$$g'_D \leq k|A_1 \cap D_2| - g_2.$$

Summarizing,

$$k|A_1| + \mathrm{def}(G) \leq k|A_1 \cap C_2| + k|A_1 \cap D_2| + g_1 - g_2 + k\text{-gal}(G - (A_1 \cap A_2)) \leq$$

$$\leq k|A_1| + k\text{-gal}(G - (A_1 \cap A_2)) - k|A_1 \cap A_2|.$$

So $\mathrm{def}(G) \leq \mathrm{def}(A_1 \cap A_2)$, ie. $A_1 \cap A_2$ is a barrier. □

Theorem 5.6. *If A_1 and A_2 are barriers such that there is no edge between $A_1 \cap D^{A_2}$ and $A_2 \cap D^{A_1}$ then $A_1 \cup A_2$ is a barrier.*

Proof. Let $D_i = D^{A_i}$ and $C_i = C^{A_i}$ for $i = 1, 2$. We prove that $A_1 \cap D_2$ and $A_2 \cap D_1$ are empty. Assume that $A_1 \cap D_2 \neq \emptyset$ and let K be a component of $G[D_2]$ such that $X = A_1 \cap V(K) \neq \emptyset$. $X \subseteq A_1$ is adjacent to at least $k|X|$ components of $G[D_1]$ since otherwise $\mathrm{def}(A_1 - X) > \mathrm{def}(G)$ would hold. Let $v \in D_1$ be a vertex adjacent to $x \in X$. $v \notin C_2$ since G contains no edge between D_2 and C_2. $v \notin A_2$ either by the condition of the theorem. Hence

v is contained in the same component of $G[D_2]$ than x, ie. $v \in V(K)$. But then Lemma 4.6 implies that X can have at most $k|X|-1$ neighbors among the components of $G[D_1]$, a contradiction.

So $A_1 \cap D_2 = \emptyset$ and by symmetry $A_2 \cap D_1 = \emptyset$. Let

- g_C^1 be the number of components of $G[D_1]$ contained in C_2,
- g_C^2 be the number of components of $G[D_2]$ contained in C_1 and
- $g_D = c(G[D_1 \cap D_2])$.

Clearly

$$k \cdot |A_1| + \mathrm{def}(G) = k\text{-gal}(G - A_1) = g_C^1 + g_D,$$

$$k \cdot |A_2| + \mathrm{def}(G) = k\text{-gal}(G - A_2) = g_C^2 + g_D \quad \text{and}$$

$$k \cdot |A_1 \cap A_2| + \mathrm{def}(G) = k\text{-gal}(G - (A_1 \cap A_2)) \geq g_D.$$

These inequalities sum up to $g_D + g_C^1 + g_C^2 \geq k \cdot |A_1 \cup A_2| + \mathrm{def}(G)$. It is easy to see that $k\text{-gal}(G - (A_1 \cup A_2)) \geq g_D + g_C^1 + g_C^2$ and so $A_1 \cup A_2$ is a barrier. □

Theorem 5.6 fails for arbitrary barriers. For example, let $k = 2$ and P_3 be the path of length 3 with vertices v_1, v_2, v_3, v_4 in this order. P_3 has a perfect 2-piece packing so $C_{P_3} = V(P_3)$. The barriers of P_3 are $A_{P_3} = \emptyset$, $\{v_2\}$ and $\{v_3\}$ but $\{v_2, v_3\}$ is not a barrier.

In the matching theory, the deficiency is usually defined as $q(G-A)-|A|$ where $q(G-A)$ is the number of odd size components of $G-A$. For this 'odd-deficiency' it holds that $A_G \cup C_G$ is the union of inclusion-wise maximal barriers. This property fails for our deficiency, see P_3 defined in the previous paragraph.

For the odd-deficiency it also holds that A_G is the intersection of the inclusion-wise maximal barriers. This property fails in our case as well. For example, let P_2 be the path of length 2 with vertices v_1, v_2, v_3 in this order. P_2 has a perfect 2-piece packing and its barriers are $A_{P_2} = \emptyset$ and $\{v_2\}$.

Nevertheless, Theorem 5.5 fails for the classical odd deficiency.

6 Two more properties of galaxies

First we show a characterization of k-galaxies which is a direct generalization of the defining property 2.3 of the hypomatchable graphs.

Theorem 6.1. *A graph G satisfies properties 1. and 2. if and only if G is a k-galaxy.*

1. *G has no perfect k-piece packing.*

2. *For each $v \in V(G)$ there exists a vertex set $v \in X \subseteq V(G)$ such that $G[X]$ is connected, $\Delta(G[X]) \leq k-1$ and $G - X$ has a perfect k-piece packing.*

Proof. If G is a k-galaxy then *1.* follows from Lemma 2.5 and *2.* from Lemma 3.8.

For the reverse direction, suppose that G satisfies the above two properties. First, if $A_G = \emptyset$ then either $C_G = V(G)$ which contradicts to *1.* by Theorem 2.8 property *3.*, or $D_G = V(G)$. In this latter case each component of G is a k-galaxy. However, G cannot have more than one component since then *2.* would yield a perfect k-piece packing of G contradicting to *1.* Second, assume that $A_G \neq \emptyset$. Choose a vertex $v \in A_G$ and let X be the vertex set guaranteed by *2.* Now $\deg_{G[X]}(v) \leq k-1$ since $\Delta(G[X]) \leq k-1$. Adjoin $k - \deg_{G[X]}(v)$ new isolated vertices to G and join each new vertex to v by an edge. The new graph is denoted by G'. Now X and the set of new vertices induce a k-piece in G'. This k-piece together with the perfect k-piece packing of $G - X$ gives a perfect k-piece packing of G'. However, $k\text{-gal}(G' - A_G) \geq k|A_G| + 1$ by Theorem 2.8, property *2.*, which is a contradiction by Theorem 3.13. ☐

In the case $k = 1$ Theorem 6.1 *2.* is equivalent to the defining property 2.3 of hypomatchable graphs. This implies property *1.* as well by parity arguments when $k = 1$. However, parity has no consequence in the case $k \geq 2$. Another easy characterization of galaxies is the following corollary of Theorem 4.19.

Proposition 6.2. *The following statements are equivalent for a connected graph G.*

1. *G is a k-galaxy.*

2. *$|U_{G-v}| < |U_G|$ for all $v \in V(G)$.*

3. *$U_{G-v} \subsetneq U_G$ for all $v \in V(G)$.*

Proof. *1.* $\Rightarrow$*2.* and *1.* $\Rightarrow$*3.*: $\emptyset$ is a perfect set with k-surplus so $A_G = \emptyset$ by Corollary 4.11. So $D_G = V(G)$ and both *2.* and *3.* are implied by Theorem 4.19.

2. $\Rightarrow$*1.* and *3.* $\Rightarrow$*1.*: Theorem 2.8 yields that $D_G = V(G)$ hence G is a k-galaxy by Theorem 2.8 property *1.* and by the connectivity of G. □

7 The matroidal property and maximum packings

Definition 7.1. We say that the $\mathcal{F}$-packing problem is *matroidal* if for all graphs G those vertex sets $X \subseteq V(G)$ which can be covered by an $\mathcal{F}$-packing of G form a matroid.

Loebl and Poljak conjecture [9] that for graph sets $\mathcal{F}$ with $K_2 \in \mathcal{F}$ the $\mathcal{F}$-packing problem is polynomial if and only if it is matroidal. This conjecture is still open. In [5] it was shown that the k-piece packing problem is not matroidal in the case $k \geq 2$. For an example, let $k = 2$ and G be a claw (ie. a 3-star) with one of its edges subdivided by a new vertex. Still, the k-piece packing problem has the matroidal property in a somewhat weaker form. So Theorem 2.9 gives another support for the validity of the conjecture of Loebl and Poljak.

Theorem. 2.9. *There exists a partition π on $V(G)$ and a matroid $\mathcal{M}$ on π such that the vertex sets of the maximal k-piece packings are exactly the vertex sets of the form $\bigcup\{X : X \in \pi'\}$ where π' is a base of $\mathcal{M}$.*

Proof. Lemmas 4.12, 4.15 and the k-surplus of A_G imply that the following considerations hold.

$\pi = \{\{v\} \colon v \notin W_G\} \cup \{V(T) \colon T$ is a tip of a k-galaxy component of $G[D_G]\}$.

Denote by $\mathcal{N}$ the matroid with ground set $\mathcal{H} = \{H : H$ is a component of $G[D_G]\}$ such that a set $\mathcal{H}' \subseteq \mathcal{H}$ of size $k|A_G|$ is a base in $\mathcal{N}$ if and only if A_G can be k-matched into $\bigcup\{V(H) \colon H \in \mathcal{H}'\}$. Observe that $\mathcal{N}$ is indeed a matroid, it is the transversal matroid of the bipartite graph B_A, see Definition 4.14. Now for each component H of $G[D_G]$ replace H in $\mathcal{N}$ by $\mathcal{T}_H = \{V(T) : T$ is a tip of $H\} \subseteq \pi$ such that the elements of $\mathcal{T}_H$ are in series with each other. The resulting matroid is $\mathcal{N}'$ with ground set $\{V(T) : T$ is a tip of a k-galaxy component of $G[D_G]\} \subseteq \pi$. Add as a direct sum to $\mathcal{N}'$ the elements $\{v\}$ as a bridge for $v \notin W_G$. The resulting matroid is $\mathcal{M}$. □

The co-rank of $\mathcal{N}$ and $\mathcal{N}'$ are $\mathrm{def}(G)$ thus the co-rank of $\mathcal{M}$ is $\mathrm{def}(G)$ too. Note that for each maximal k-piece packing P of G, every vertex set of π is either fully covered or fully missed by P and the number of the fully missed sets is $\mathrm{def}(G)$. In the case $k = 1$ a tip has exactly one element so π is the partition into singletons. In the case $k = 2$ a tip has one or two elements so the vertex sets of π are of size one or two. Finally, for $k \geq 3$ a tip may be of arbitrary size thus a vertex set of π can be of arbitrary size as well.

Because the ground set of the matroid $\mathcal{M}$ is a partition into different size sets, in the k-piece packing problem a *maximal* packing is not necessarily *maximum*, as it is the case in the polynomial packing problems with $K_2 \in \mathcal{F}$. Still, the vertex sets which can be covered by maximum k-piece packings admit a similar matroid: take the maximum weight bases of $\mathcal{M}$ with the weight function $X \mapsto |X|$ for $X \in \pi$. This weighted matroidal approach yields a proof for the Berge-type formula of [5] on the size of a maximum k-piece packing. Indeed, the maximum weight bases of $\mathcal{M}$ correspond to the minimum weight bases of $\mathcal{N}$ (defined in the proof of Theorem 2.9) with the weight function $H \mapsto$ (the minimum size of a tip of H). So one can apply the greedy method to the k-galaxy components of $G[D_G]$. In fact, a little additional work is needed for proving Theorem 7.2 since it is stated in a more compact form in [5]. Let $k\text{-gal}_i(G)$ denote the number of k-galaxy components H of the graph G with the property that each tip of H has size at least i.

Theorem 7.2. [5] *If G is a graph of size n then the size of the maximum k-piece packings of G is*

$$n - \max \sum_{i=1}^{n} \left(k\text{-gal}_i(G - A_i) - k|A_i| \right),$$

taken over all sequences of vertex sets $V(G) \supseteq A_1 \supseteq A_2 \supseteq \ldots \supseteq A_n$.

A_1 can be chosen to be the canonical barrier A_G. The sequence of vertex sets is related to the structure of the minimum weight bases of the transversal matroid $\mathcal{N}$. We do not go into details. In the case $k = 1$ we get the Berge-Tutte theorem on maximum matchings [1]. The case $k = 2$ was proved by Kano, Katona and Király [8].

8 The (l,u)-piece packing problem

As a generalization of the k-piece packing problem, the (l,u)*-piece packing problem* is introduced in [5]. It turns out that all the above results hold with the straightforward modifications. We do not go into details, only illustrate this relation using the reduction to the k-piece packing problem shown in [5].

Let two integer bounds $u(v) \geq l(v) \geq 0$ be given for each vertex $v \in V(G)$. A connected subgraph P of G is an (l,u)*-piece* if $\deg_P(v) \leq u(v)$ holds for each $v \in V(P)$ and there exists at least one vertex $w \in V(P)$ with $\deg_P(w) \geq l(w)$. Note that $l \equiv u \equiv k$ gives the k-piece packing problem. Galaxies and tips change in the following way.

Definition 8.1. Given the bounds $l,u\colon V(H) \to \mathbb{N}$, the graph H is an (l,u)-*galaxy* if it satisfies the following properties:

- denoting by I_H the graph induced by the vertices v with $\deg_G(v) \geq l(v)$, each component of I_H is a hypomatchable graph,
- $l(v) = u(v) \geq 1$ for $v \in V(I_H)$,
- for each $v \in V(I_H)$ there exist exactly $l(v) - 1$ edges between v and $V(H) - V(I_H)$, each being a cut edge in H.

The *tips* are the connected components of $H - V(I_H)$ together with the vertices $v \in V(I_H)$ with $l(v) = u(v) = 1$ as single vertex subgraphs.

The difference in the definition of the galaxies and tips can be explained by the following reduction to the k-piece packing problem, described in [5]. Let $k = 1 + \max\{u(v)\colon v \in V(G)\}$. For each vertex $v \in V(G)$ let M_v and N_v be disjoint sets of new vertices with $|M_v| = u(v) - l(v) + 1$ and $|N_v| = k - u(v) - 1$. Now for each $v \in V(G)$ take a complete graph on M_v and join the vertices of $M_v \cup N_v$ to v. Denote the new graph by G_k. It is easy to see that G_k has a perfect k-piece packing if and only if G has a perfect (l,u)-piece packing, and that G is an (l,u)-galaxy if and only if G_k is a k-galaxy. With the help of this reduction one can see that all the above considerations for the k-piece packings hold for the (l,u)-piece packings as well, with the necessary modifications. For illustrating this, we briefly describe how to get the canonical decomposition of G related to the (l,u)-piece packing problem.

Let $V(G_k) = D_k \,\dot\cup\, A_k \,\dot\cup\, C_k$ be the canonical decomposition of G_k related to the k-piece packing problem. Due to the k-surplus of A_k, each vertex of

A_k has degree at least $k+1$ in G_k. Because the new vertices of G_k (ie. the vertices in $V(G_k) - V(G)$) have degree at most $u(v) - l(v) + 1 \leq k$, we get that $A_k \subseteq V(G)$. So the deletion of the new vertices yields a partition $V(G) = D \dot\cup A \dot\cup C$ where $D = D_k \cap V(G)$, $A = A_k$ and $C = C_k \cap V(G)$. This canonical partition has all the properties listed in Theorem 2.8, for example the connected components of $G[D]$ are (l,u)-galaxies, for all $\emptyset \neq A' \subseteq A$ the number of those (l,u)-galaxy components of $G[D]$ which are adjacent to A' is at least $u(A') + 1$, and C has a perfect (l,u)-piece packing. This partition is unique, because if $V(G) = D' \dot\cup A' \dot\cup C'$ is another partition with these properties then in G_k the set A' is a perfect barrier with k-surplus, hence by Corollary 4.11 it equals to A_k. The analogue of Theorem 4.19 also holds.

This Edmonds-Gallai type theorem for the (l,u)-piece packing problem becomes quite compact in the case $l(v) = l < u = u(v)$ for all $v \in V(G)$, so we include this. Here an (l,u)-piece packing is a packing with connected graphs F with $l \leq \Delta(F) \leq u$. Call such a packing an *$(l<u)$-packing.* The simplicity of this structure theorem comparing to the general case is due to the fact that here an (l,u)-galaxy is just a graph with highest degree at most $l-1$. So it always consists of only one tip.

Theorem 8.2. *For a graph G let $D = \{v \in V(G)\colon v$ can be missed by a maximal $(l<u)$-packing of $G\}$. Let $A = \Gamma(D)$ and $C = V(G) - (D \cup A)$. Now*

1. *$\Delta(G[D]) \leq l-1$,*
2. *for all $\emptyset \neq A' \subseteq A$ the number of those components of $G[D]$ which are adjacent to A' is at least $u|A'|+1$,*
3. *$G[C]$ has a perfect $(l<u)$-packing, and*
4. *for each maximal $(l<u)$-packing P of G, the graph $G-P$ has exactly $c(G[D]) - u|A|$ connected components.*

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