# The prism over the middle-levels graph is hamiltonian


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**Abstract**

Let $\mathbf{B}_k$ be the bipartite graph defined by the subsets of $\{1, \ldots, 2k+1\}$ of size $k$ and $k+1$. We prove that the prism over $\mathbf{B}_k$ is hamiltonian. We also show that $\mathbf{B}_k$ has a closed spanning 2-trail.


## 1 Introduction

Let $[2k+1]$ be the set $\{1, \ldots, 2k+1\}$. Consider the bipartite graph $\mathbf{B}_k$ whose vertices are all subsets of $[2k+1]$ of size $k$ or $k+1$, and whose edges represent the inclusion between two such subsets. The notorious Middle two levels problem is whether $\mathbf{B}_k$ is hamiltonian for all $k$. Most likely it was first asked by Havel [5] (see the account in [8]).

Many authors attempted to solve this problem. One approach was to prove the assertion for specific values of $k$. The best result in this direction was obtained by Shields and Savage [10] who proved that $\mathbf{B}_k$ is hamiltonian for $1 \leq k \leq 15$. Another approach aimed at identifying long cycles in $\mathbf{B}_k$. In [10], it was proved that $\mathbf{B}_k$ has a cycle of length $\geq 0.86\,|\mathbf{B}_k|$, where $|\mathbf{B}_k| = 2\binom{2k+1}{k}$ is the number of vertices of $\mathbf{B}_k$. The best lower bound is due to R. Johnson who proved [11] that there is a cycle of length $(1 - o(1))\,|\mathbf{B}_k|$. Yet another direction was to find other structures that hopefully would be useful for finding the elusive hamiltonian cycle in $\mathbf{B}_k$. For instance, since a hamiltonian cycle in $\mathbf{B}_k$ is a disjoint union of two 1-factors, one may hope to find a hamiltonian cycle by building a sufficiently large repertoire of 1-factors. Duffus et al. [3] proved that no two 1-factors of the lexicographic 1-factorization form a hamiltonian cycle. This motivated Kierstead and Trotter [8] to generalize the lexicographic factorizations to lexical factorizations. Still no hamiltonian cycle was discovered. Another paper, [2], introduced the modular matchings.

In this paper, we use modular matchings to prove that $\mathbf{B}_k$ is *close* to being hamiltonian. The word 'close' can be interpreted in several ways. For instance, one can view a hamiltonian cycle as a spanning closed walk that visits each vertex exactly once. One can also view a hamiltonian path as a spanning tree of maximum degree 2. It is then quite natural to explore the following modifications. Instead of searching for a hamiltonian cycle in a graph, search for a spanning, closed walk in which every vertex is visited at most twice (or, in general, $k$ times). Similarly, instead of searching for a hamiltonian path, one can look for a spanning tree of maximum degree 3 (or $k$). In accordance with the terminology of [6], we call these spanning structures *$k$-walks* and *$k$-trees*, respectively.

It was shown in [6] that any graph with a $k$-tree has a $k$-walk, and that the existence of a $k$-walk guarantees the existence of a $(k+1)$-tree, for any $k$. This results in the following hierarchy among families of graphs:

$$\begin{aligned}\text{1-walk (hamiltonian cycle)} &\implies \text{2-tree (hamiltonian path)}\\ &\implies \text{2-walk} \implies \text{3-tree} \implies \dots\end{aligned}$$

Clearly, for every connected graph $G$, there is a $k$ for which $G$ has a $k$-walk (just duplicate all edges to obtain an eulerian graph whose Euler trail visits every vertex at most $\Delta(G)$ times). Graphs with a $k$-walk for a smaller $k$ can be regarded as closer to being hamiltonian. For a nice survey of results on $k$-walks, $k$-trees and related topics, we refer the reader to Ellingham [4].

The *prism* over a graph $G$ is the Cartesian product $G \square K_2$ of $G$ with the complete graph $K_2$ [1, 7, 9]. Thus, it consists of two copies of $G$ and a 1-factor joining the corresponding vertices. It was observed in [7] that the property of having a hamiltonian prism is 'sandwiched' between the existence of a 2-tree and the existence of a 2-walk. That is:

$$\text{2-tree} \implies \text{hamiltonian prism} \implies \text{2-walk}$$

and both implications are sharp. This can be naturally interpreted as saying that graphs with a hamiltonian prism are closer to being hamiltonian than those which only have a 2-walk.

A hamiltonian cycle in a graph is a spanning 2-regular subgraph. In this note, we use the modular factorization to prove that $\mathbf{B}_k$ has a spanning 3-connected cubic subgraph. A direct consequence of this is that $\mathbf{B}_k$ has a hamiltonian prism and also a 2-*trail* (a 2-walk in which each edge is used at most once). As an aside, we note that in case $\mathbf{B}_k$ fails to be hamiltonian, these cubic subgraphs yield a family of cubic, 3-connected bipartite non-hamiltonian graphs.

## 2 Modular matchings

Our main tool is the concept of a modular matching in $\mathbf{B}_k$, as defined in [2]. We recall the related definitions, generally trying to keep in line with the notation of [2]. The *weight* $\sum B$ of a set $B \subset [2k+1]$ is defined to be the sum of all elements of $B$. The complement of $B$ is denoted by $\overline{B}$.

Let $A \subset [2k+1]$ be a *k-set* (set of size $k$). For an integer $i = 1, \dots, k+1$, let $\mathbf{m}_i(A)$ be the set obtained when one adds the $j$-th largest element of $\overline{A}$ to $A$, where

$$j \equiv i + \sum A \pmod{k+1}$$

and $1 \le j \le k+1$. Let $\mathbf{m}_i$ be the set of edges of $\mathbf{B}_k$ of the form $\{A, \mathbf{m}_i(A)\}$.

**Theorem 1 ([2])** *For $i = 1, \dots, k+1$, $\mathbf{m}_i$ is a matching in $\mathbf{B}_k$ and the set $\{\mathbf{m}_1, \dots, \mathbf{m}_{k+1}\}$ is a 1-factorization of $\mathbf{B}_k$.* □

An important observation, which is implicit in the proof of [2, Theorem 1], is the following:

**Lemma 2** *Define a mapping $\mathbf{b}_i : \mathbf{B}_{k+1} \to \mathbf{B}_k$ by setting $\mathbf{b}_i(B)$ to be the set obtained by removing the $j$-th smallest element from $B$, where*

$$j \equiv i + \sum B \pmod{k+1}$$

*and the index is based at 1. The composition $\mathbf{b}_i \circ \mathbf{m}_i$ is the identity.* □

It will be convenient to view the set $[2k+1]$ as ordered cyclically, with 1 being the successor of $2k+1$. A *segment* in a set $B \subset [2k+1]$ is a maximal contiguous sequence of elements of $B$. Since the elements 1 and $2k+1$ are considered to be adjacent, a segment may 'wrap around'.

# 3 A connected spanning subgraph of $\mathbf{B}_k$

In this section, we show that three suitably selected modular matchings in $\mathbf{B}_k$ form a connected spanning cubic subgraph of $\mathbf{B}_k$. To this end, we introduce the following notation. Throughout this section, let $A$ be a $k$-subset of $[2k+1]$. The elements of $\overline{A}$ can be labeled by numbers $+1, \dots, +(k+1)$ such that adding the element with label $+i$ to $A$, one obtains the set $B$ such that $\{A, B\} \in \mathbf{m}_i$ (thus, $B = \mathbf{m}_i(A)$). We shall use $A(+i)$ to denote the element of $[2k+1]$ labeled $+i$. By Lemma 2, the elements $A(+1), \dots, A(+(k+1))$ form a decreasing sequence (except for at most one increase caused by the wrap-around at 1).

Symmetrically, if $B$ is a $(k+1)$-subset of $[2k+1]$, then the elements of $B$ can be labeled by $-1, \dots, -(k+1)$ in such a way that removing the element labeled $-i$ from $B$ (we shall write $B(-i)$ for the element), one obtains the set $\mathbf{b}_i(B)$. Again by Lemma 2, the sequence $B(-1), \dots, B(-(k+1))$ is increasing (with a possible wrap-around at $2k+1$).

We need to be able to describe a sequence of additions and removals of elements of the above type. First, let $i, j \in [k+1]$. We write $A^{+i}$ for the set obtained by adding $A(+i)$ to $A$ (i.e., the set $\mathbf{m}_i(A)$). The symbol $A^{+i,-j}$ denotes the outcome of the removal of $A^{+i}(-j)$ from $A^{+i}$. The definition is extended to sequences like $A^{+i_1,-i_2,\dots,\pm i_r}$ (in which the signs must alternate) in a natural way. Expressions like $B^{-i_1,+i_2,\dots,\pm i_s}$, where $B$ is a $(k+1)$-set, are defined symmetrically.

To help the reader, we introduce a graphical notation for the above operations, used in Figure 1. A sequence of additions and deletions is represented using a rectangular grid, each of whose rows corresponds to a set involved in the sequence. For brevity, we identify the rows with such sets. Columns correspond to (and are identified with) elements of $[2k+1]$. A square in row $S$ and column $x$ is marked gray iff $x \in S$. A label like $+i$ in row $S$ denotes the element $S(+i)$. Labels on the left and on the top of a diagram mark special sets and elements. Finally, a square marked in bold represents the element whose addition/removal leads to the next set in the sequence. Observe that this is always a square with a label ($+i$ or $-i$).

**Lemma 3** *For any $k$-set $A \neq \{1,\dots,k\}$, the spanning subgraph of $\mathbf{B}_k$ formed by the edges in $\mathbf{m}_1 \cup \mathbf{m}_2 \cup \mathbf{m}_3$ contains a path $P$ that starts in $A$ and ends in a $k$-set of smaller weight.*

**Proof.** Assume first that some element of $A$ is larger than $a = A(+2)$ (see Figure 1a), and set $C = A^{+2,-3}$. Since

$$C(-3) = A^{+2}(-3) > A^{+2}(-2) = a,$$

the weight of $C$ is less than that of $A$. Thus, the path that starts at $A$ and follows first the edge of $\mathbf{m}_2$ and then the edge of $\mathbf{m}_3$ has the required property.

We may therefore assume that $a$ is the largest element of $A^{+2}$ (as in Figures 1b and c). Let $s$ and $z$ be the first and the last element of the last segment $\sigma$ of $A$ preceding $a$, respectively. Clearly, $s < a$ (although our definition allows a segment to wrap around). Furthermore, $s > 1$ since $A \neq \{1,\dots,k\}$.

Note that $A^{+2}(-1) = z$. Set $D = A^{+2,-1}$, observing that $D(+2) = s-1$. Furthermore, set $E = D^{+2,-3} = A^{+2,-1,+2,-3}$.

To interpret $E$, we distinguish two cases based on the length of $\sigma$. If $\sigma$ has length 1 (i.e., $s+1 \notin A$, see Figure 1b), then $D^{+2}(-3) = a$. Consequently, $E$ differs from $A$ in that it has $s-1$ in place of $s$. We infer that $\sum E < \sum A$. The desired path follows the matchings in the order $\mathbf{m}_2, \mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3$ starting from $A$.

It remains to consider the case that the length of $\sigma$ is more than 1 (Figure 1c). The element $D^{+2}(-3)$ is now $s$, so $E = A \cup \{a\} \setminus \{s\}$. Since $E^{+2} = z$, one has $E^{+2}(-3) = a$. Setting $F = E^{+2,-3}$, one has $F = A \cup \{s-1\} \setminus \{s\}$, and hence $\sum F < \sum A$. Recalling that $F = A^{+2,-1,+2,-3,+2,-3}$, one sees that a path from $A$ to $F$ uses edges of $\mathbf{m}_2, \mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3, \mathbf{m}_2$ and $\mathbf{m}_3$ in order. The proof is finished. $\square$

**Theorem 4** *The union $M$ of the matchings $\mathbf{m}_1$, $\mathbf{m}_2$ and $\mathbf{m}_3$ is a connected spanning cubic subgraph of $\mathbf{B}_k$.*

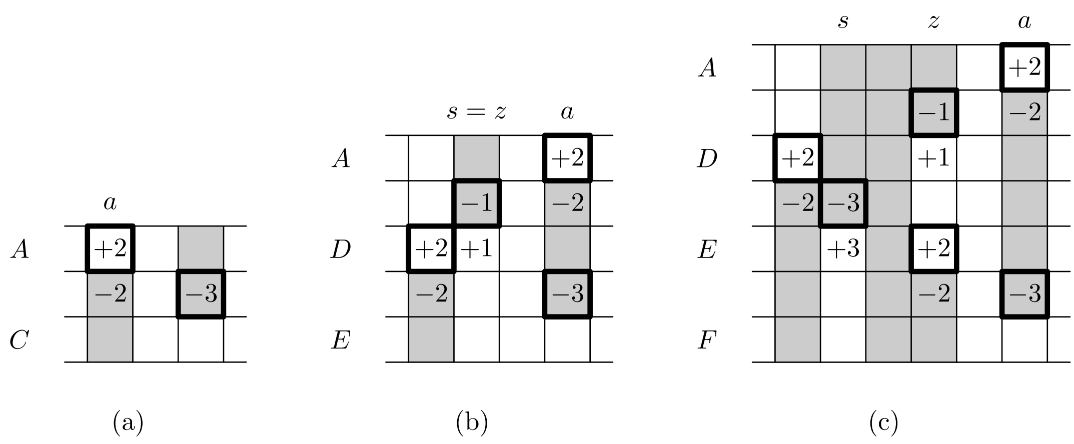


Figure 1: An illustration to the proof of Lemma 3.

**Proof.** Lemma 3 implies that every vertex different from $A_0 = \{1, \ldots, k\}$ is joined to $A_0$ by a path in $M$. Therefore, $M$ is connected. It is cubic since $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3$ are pairwise disjoint by Theorem 1. □

# 4 The subgraph $M$ is 3-connected

We now strengthen the result of Section 3 by showing that the spanning cubic subgraph $M$ of $\mathbf{B}_k$ is actually 3-connected. When working with elements of $[2k+1]$, we perform all our computations modulo $2k+1$, using $2k+1$ in place of 0. Thus, for instance, $(2k+1)+1$ is 1.

Let $A \subset [2k+1]$. The *shift* $\mathrm{sh}(A)$ of $A$ is the set

$$\mathrm{sh}(A) = \{x + 1 : x \in A\}.$$

Thus, as we consider the elements 1 and $2k+1$ to be adjacent, the shift of $A$ is obtained from $A$ by a translation by one to the right. Set $\mathrm{sh}^0(A) = A$ and, for $n > 0$, $\mathrm{sh}^n(A) = \mathrm{sh}(\mathrm{sh}^{n-1}(A))$. Clearly, $\mathrm{sh}^{2k+1}(A) = A$.

**Lemma 5** *Let $A$ be a subset of $[2k+1]$ with $|A| \in \{k, k+1\}$. Then $\mathrm{sh}^n(A) \neq A$ for all $n = 1, ..., 2k$.*

**Proof.** Let $n < 2k+1$ be smallest such that $\mathrm{sh}^n(A) = A$. Since $\mathrm{sh}^{2k+1}(A) = A$, the number $2k+1$ is clearly divisible by $n$. For $a \in A$, set

$$\tilde{a} = \{a + ni : 0 \leq i < (2k+1)/n\}.$$

The sets $\tilde{a}$, each of which is of size $(2k+1)/n$, partition $A$. It follows that $|A|$, which is $k$ or $k+1$, is divisible by $(2k+1)/n$. However, $(2k+1)/n$ divides $2k+1$, and so it can divide neither $k$ nor $k+1$, a contradiction. $\square$

**Lemma 6** *If* $\{A, B\} \in \mathbf{m}_i$, *then* $\{\mathrm{sh}(A), \mathrm{sh}(B)\} \in \mathbf{m}_i$ *as well.*

**Proof.** Suppose that $\{\mathrm{sh}(A), C\} \in \mathbf{m}_i$. Let $B - A = \{d\}$ and $C - \mathrm{sh}(A) = \{d'\}$. Clearly, to prove that $C = \mathrm{sh}(B)$ it is only needed to show that $d' = d + 1$. Set $n \equiv i + \sum A$, where $1 \leq n \leq 2k+1$. We will distinguish two cases. Suppose first that $2k+1 \in A$. Let $A = \{a_1, ..., a_{k-1}, a_k = 2k+1\}$. Then $\mathrm{sh}(A) = \{1, a_1 + 1, ..., a_{k-1} + 1\}$. We get

$$\begin{aligned} n &\equiv i + \sum_{j=1}^{k} a_j \pmod{k+1} \\ &\equiv i + (2k+1) + 2 + \sum_{j=1}^{k-1} (a_j + 1) \pmod{k+1} \\ &\equiv i + \sum \mathrm{sh}(A) \pmod{k+1}. \end{aligned}$$

Thus, $d$ and $d'$ is the $n$-th largest element of $\overline{A}$ and $\overline{\mathrm{sh}(A)}$, respectively. As $\mathrm{sh}(A)$ is the shift of $A$, $\overline{\mathrm{sh}(A)} = \mathrm{sh}(\overline{A})$. Since $2k+1 \notin \overline{A}$, for each $n$, the $n$-th largest element of $\overline{\mathrm{sh}(A)}$ is larger by one than the $n$-th largest element of $\overline{A}$. Hence, $d' = d + 1$.

Now suppose that $2k+1 \notin A$. Thus, for $A = \{a_j : 1 \leq j \leq k\}$, we have $\mathrm{sh}(A) = \{a_j + 1 : 1 \leq j \leq k\}$. We obtain

$$\begin{aligned} n &\equiv (i + \sum A) \pmod{k+1} \\ &\equiv i + (k+1) + \sum A \pmod{k+1} \\ &\equiv 1 + i + \sum_{j=1}^{k} (a_j + 1) \pmod{k+1} \\ &\equiv 1 + i + \sum \mathrm{sh}(A) \pmod{k+1}. \end{aligned}$$

That is, $d$ is the $n$-th largest element of $\overline{A}$, while $d'$ is the $(n-1)$-st largest element of $\overline{\mathrm{sh}(A)}$. Since $2k+1 \in \overline{A}$, for each $n$, the $(n-1)$-st largest element of $\overline{\mathrm{sh}(A)}$ is larger by one than the $n$-th largest element of $\overline{A}$. For $n = 1$, the statement means that if $d = 2k+1$, then $d' = 1$. Thus, also in this case, $d' = d + 1$. $\square$

The edges of $\mathbf{m}_i \cup \mathbf{m}_j$, $1 \leq i \neq j \leq k$, form a 2-factor of $\mathbf{B}_k$. The following lemmas describe some properties of the cycles of the 2-factors $\mathbf{m}_i \cup \mathbf{m}_{i+1}$.

**Lemma 7** *Let $C$ be a cycle in the 2-factor $\mathbf{m}_i \cup \mathbf{m}_{i+1}$, where $i \in \{1, \ldots, k-1\}$. If the set $A$ is on $C$ then, for all $t$, $\mathrm{sh}^t(A)$ is on $C$ as well.*

**Proof.** A segment of a set $A$ will be denoted by $[a, b]$, where $a$ and $b$ are the smallest and the largest numbers in the segment, respectively. Let $A$ be a $k$-set and $[a_j, b_j]$, $j = 1, \ldots, n$ be its segments. We label the segments in such a way that $[a_1, b_1]$ is the first segment to the right of $A(+i)$, and the segment $[a_j, b_j]$ is the first segment to the right of the segment $[a_{j-1}, b_{j-1}]$, with a possible wrap-around. It is easy to see that the path $P$ through the vertices given below is a part of the cycle of $C$ (see Figure 2 for an illustration).

$$\begin{aligned}
&A = \{[a_1, b_1], [a_2, b_2], \ldots, [a_{n-1}, b_{n-1}], [a_n, b_n]\},\\
&\{A(+i), [a_1, b_1], [a_2, b_2], \ldots, [a_{n-1}, b_{n-1}], [a_n, b_n]\},\\
&\{A(+i), [a_1+1, b_1], [a_2, b_2], \ldots, [a_{n-1}, b_{n-1}], [a_n, b_n]\},\\
&\{A(+i), [a_1+1, b_1+1], [a_2, b_2], \ldots, [a_{n-1}, b_{n-1}], [a_n, b_n]\},\\
&\{A(+i), [a_1+1, b_1+1], [a_2+1, b_2], \ldots, [a_{n-1}, b_{n-1}], [a_n, b_n]\},\\
&\{A(+i), [a_1+1, b_1+1], [a_2+1, b_2+1], \ldots, [a_{n-1}, b_{n-1}], [a_n, b_n]\},\\
&\qquad \vdots,\\
&\{A(+i), [a_1+1, b_1+1], [a_2+1, b_2+1], \ldots, [a_{n-1}+1, b_{n-1}+1], [a_n, b_n]\},\\
&\{A(+i), [a_1+1, b_1+1], [a_2+1, b_2+1], \ldots, [a_{n-1}+1, b_{n-1}+1], [a_n+1, b_n]\}\\
&\qquad = B.
\end{aligned}$$

If $A(+i) = b_n + 1$, then $B = \mathrm{sh}(A)$. Otherwise, the two vertices on $P$ that immediately follow $B$ are

$$\begin{aligned}
&\{A(+i), [a_1+1, b_1+1], [a_2+1, b_2+1], \ldots, [a_{n-1}+1, b_{n-1}+1],\\
&\quad [a_n+1, b_n+1]\},\\
&\{[a_1+1, b_1+1], [a_2+1, b_2+1], \ldots, [a_{n-1}+1, b_{n-1}+1], [a_n+1, b_n+1]\}\\
&\quad = \mathrm{sh}(A).
\end{aligned}$$

This shows that if a $k$-set $A$ is on $C$ then its shift is on $C$ as well. Applying the same argument repeatedly, we get that the vertex $\mathrm{sh}^t(A)$ is on $C$ for all $t > 1$. By Lemma 6, the same applies to each $(k+1)$-set on $C$. □

From the proof of the preceding lemma and Lemma 5, we immediately get:

**Corollary 8** *If $A$ is a $t$-set, then the cycle of $\mathbf{m}_i \cup \mathbf{m}_{i+1}$ containing $A$ is of length $2(2k+1)(t+\delta)$, where $\delta \in \{0, 1\}$. In particular, if two $m$-sets $A$ and $B$, $m \in \{k, k+1\}$, are on a cycle of $\mathbf{m}_i \cup \mathbf{m}_{i+1}$, then the number of segments of $A$ differs from the number of segments of $B$ by at most 1.*

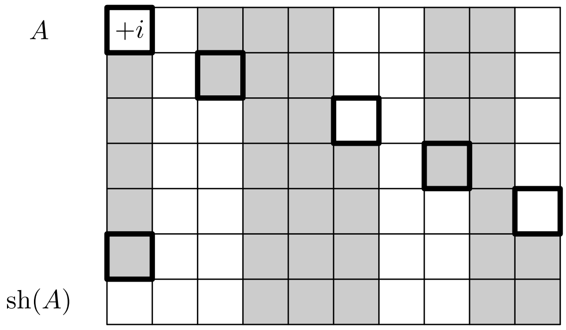


Figure 2: The path $P$ in the proof of Lemma 7.

If $A, B$ be vertices on a cycle, then $d_C(A, B)$ denotes the length of the shorter path from $A$ to $B$ on $C$. From Lemma 6 and from the proof of Lemma 7, we get:

**Corollary 9** *Let $A$ be on a cycle $C$ of $\mathbf{m}_i \cup \mathbf{m}_{i+1}$, where $i \in \{1, \ldots, k-1\}$. Then the vertices $A, \mathrm{sh}(A), \ldots, \mathrm{sh}^{2k}(A)$ are uniformly distributed on $C$, i.e.,*

$$d_C(A, \mathrm{sh}(A)) = d_C(\mathrm{sh}^i(A), \mathrm{sh}^{i+1}(A)),$$

*where $i \in \{1, \ldots, 2k\}$.*

**Corollary 10** *Let $A$ be a $k$-set, $C$ be a cycle of $\mathbf{m}_1 \cup \mathbf{m}_2$ containing $A$, and $\{A, B\} \in \mathbf{m}_3$. Then either $B$ is not on $C$, or $d_C(A, B) > d_C(A, \mathrm{sh}(A))$.*

**Proof.** From the proof of Lemma 6, we see that no $(k+1)$-set on the path $P$ from $A$ to $\mathrm{sh}(A)$, that is a part of $C$, contains the number $a_1$. Similarly, no $(k+1)$-set on the path from $\mathrm{sh}^{2k}(A)$ to $A$, that is a part of $C$, contains the element $b_n$. Thus, if $B$ is on $C$, then $d_C(A, B) > d_C(A, \mathrm{sh}(A))$. □

For $\{A, B\} \in \mathbf{m}_i$, we define $\mathbf{m}_i(A)$ to equal $B$.

**Theorem 11** *The union $M$ of the matchings $\mathbf{m}_1, \mathbf{m}_2$ and $\mathbf{m}_3$ is a 3-connected graph.*

**Proof.** By Theorem 4, $M$ is connected. Let $F$ be an edge-cut of $M$, and let $x \in F$, where $x \in \mathbf{m}_i$, $1 \le i \le 3$. As $x$ is on a cycle in the 2-factor $\mathbf{m}_i \cup \mathbf{m}_j$, where $i \ne j$ and $1 \le j \le 3$, there is an edge $y \ne x$ such that $y \in F$. Thus, $|F| \ge 2$. Assume, for the sake of a contradiction, that $|F| = 2$. If $y \in \mathbf{m}_j$, then $F$ contains at least two edges of the cycle of $\mathbf{m}_i \cup \mathbf{m}_k$ ($k \in \{1, 2, 3\} - \{i, j\}$) that passes through the edge $x$. This in turn implies that $|F| > 2$. Hence, both the edges $x$ and $y$ are from the same matching $\mathbf{m}_i$. Let the set of vertices of a component of $M - F$ be denoted by $R$, and set $S = V(M) - R$. Suppose first that $i = 1$.

Let $C$ be the cycle of $\mathbf{m}_1 \cup \mathbf{m}_2$ containing the edge $x$. Then, by Lemma 7 and Corollary 9, there is a vertex $A$ on $C$ such that $A \in R$ and $\mathrm{sh}^t(A) \in S$ for some $t \le 2k$. Let $C'$ be a cycle of $\mathbf{m}_2 \cup \mathbf{m}_3$ passing through $A$. Since the edge-cut $F$ does not contain any edge of $\mathbf{m}_2 \cup \mathbf{m}_3$, all the vertices of $C'$ are in $R$. Thus, by Lemma 6, $\mathrm{sh}^t(A) \in R$, a contradiction. An analogous argument applies if $i = 3$. Thus, we are left with the case $i = 2$.

Let $C$ be a cycle of $\mathbf{m}_1 \cup \mathbf{m}_2$ that contains the edge $x$. Then $C$ passes through $y$ as $|F| = 2$. Write $P_1$ and $P_2$ for the two paths of $C - \{x, y\}$, assuming $|P_1| \le |P_2|$. Let $A_x$ denote the vertex of $P_1$ incident with the edge $x$. Without loss of generality, we assume that $P_1 \subset R$. Suppose first that the vertex $B = \mathbf{m}_3(A_x)$ is on $C$ as well. Then, by Lemma 6 and Corollary 10, there is an $r$ with the property that $\mathrm{sh}^r(A_x) \in P_1$, but $B' \in P_2$, where $z = \{\mathrm{sh}^r(A), B'\} \in \mathbf{m}_3$. However, then $z \in F$, and $|F| > 2$. Thus, $B = \mathbf{m}_3(A_x) \notin C$. However, then $B$ is on the cycle $C'$ of $\mathbf{m}_2 \cup \mathbf{m}_3$ that passes through $x$. Clearly, $B \in R$, for otherwise $|F| > 2$. By the same token as above, $y$ is on $C'$ as well. Assume that, for some $t$, $\mathrm{sh}^t(B) \in S$. Consider a cycle $C''$ of $\mathbf{m}_1 \cup \mathbf{m}_2$ passing through $B$. As $B$ is not on $C$, all vertices of $C''$ are in $R$. From Lemma 7, all shifts of $B$ are in $C''$, which contradicts the fact that $\mathrm{sh}^t(B)$ is in $S$, and $|F| > 2$. We need to consider now the case that for all $t$, $\mathrm{sh}^t(B) \in R$. Let $T_1$ and $T_2$ denote the two paths of $C' - F$, and suppose that $B$ is on $T_1$. As $\mathrm{sh}^t(B)$ is on $T_1$ for all $t \ge 0$, then, by Corollary 9, for any $E$ on $C'$, there is at most one $t_e < 2k + 1$ so that $\mathrm{sh}^{t_e}(E)$ is on $T_2$. However, $A_x$ is on both $P_1$ and $C'$, and $|P_1| \le |P_2|$ leads to a contradiction with the previous statement as $|P_1| \le |P_2|$ implies (Corollary 9) that at least two distinct shifts of $A_x$ have to be on $P_2 \subset S$, hence on $T_2$. The proof is complete. □

**Corollary 12** *The prism over the graph* $\mathbf{B}_k$ *is hamiltonian.*

**Proof.** By [9] (see also [1]), any 3-connected cubic graph has a hamiltonian prism. Thus, the assertion follows from Theorem 11. □

We remark that Corollary 12 can also be directly derived from Theorem 4, by showing that a connected cubic bipartite graph has a hamiltonian prism. We conclude the paper with the following observation on 2-trails (defined in Section 1):

**Corollary 13** *The graph* $\mathbf{B}_k$ *has a* 2*-trail.*

**Proof.** Adding any matching $\mathbf{m}_i$ ($i \ge 4$) to $M$, we obtain a connected spanning 4-regular subgraph of $\mathbf{B}_k$. The Euler trail of this subgraph is a 2-trail in $\mathbf{B}_k$. □

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