The Loebl Conjecture for trees of small diameter

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Abstract

The Loebl Conjecture asks whether any graph on n vertices, at least half of which have degree $\frac{n}{2}$, contains any tree of order $\frac{n}{2}+1$ as a subgraph. We prove the conjecture for trees with diameter ≤ 5.

1 Introduction

The below conjecture, which is also called the $(\frac{n}{2}, \frac{n}{2}, \frac{n}{2})$-conjecture, was first formulated in 1994 in [3].

Conjecture 1 (Loebl Conjecture). *Any graph on n vertices of which at least $\frac{n}{2}$ have degree at least $\frac{n}{2}$, contains, as a subgraph, any tree with at most $\frac{n}{2}$ edges.*

This conjecture is trivially true for stars. It also holds for dumbbells, i.e. trees that consist of two stars with adjacent centers, as the set of vertices of large degree cannot be independent.

Bazgan, Li, and Woźniak [1] have proved the Loebl Conjecture for paths, and also for trees that are obtained from paths of which one vertex is identified with the centre of a star[4].

Zhao claims to have solved the conjecture completely, but a final version of his proof has not yet appeared.

If true, the Loebl Conjecture implies at once the following conjecture.

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[4]In fact, they prove these special cases for a generalisation of the Loebl Conjecture, the Loebl-Komlós-Sós Conjecture. This latter conjectures that for any $k \leq n$, any graph on n vertices of which $\geq \frac{n}{2}$ have degree $\geq k$, contains, as a subgraph, any tree with at most k edges.

Conjecture 2. *The Ramsey number of a tree with n edges is at most $2n$.*

Haxell, Łuczak, and Tingley [2] have solved this latter conjecture for trees of small maximum degree.

In this paper we prove the Loebl conjecture for trees $T = (V_T, E_T)$ with diameter $\max\{d(u,v),\ u,v \in V_T\} \leq 5$, where $d(u,v)$ is the usual distance between two vertices in a graph, i.e. the number of edges in the minimal path from u to v.

Theorem 3. *Let G be a graph of order n such that at least half of its vertices have degree at least $\frac{n}{2}$. Then any tree T of order at most $\frac{n}{2}+1$ with diameter at most 5 embeds into G.*

2 Proof of Theorem 3

In order to prove Theorem 3, assume that we are given a graph G on $n \in \mathbb{N}$ vertices, of which at least $\frac{n}{2}$ have degree at least $\frac{n}{2}$, and a tree T of order $\leq \frac{n}{2}+1$. Denote the set of the vertices in $V(G)$ that have degree $\geq \frac{n}{2}$ by V_1, and set $V_2 := V(G) \setminus V_1$. Observe that we may assume the set V_2 to be independent.

Furthermore, we may assume that exactly $\lceil \frac{n}{2} \rceil$ vertices of G have degree at least $\frac{n}{2}$. Indeed, otherwise we can delete any edge of G, decreasing $|V_1|$ by at most 2. Continue in this manner, until we arrive at a subgraph G' of G which has at most $\lceil \frac{n}{2} \rceil + 1$ vertices of degree $\geq \frac{n}{2}$. Now, if G' has no V_1–V_2 edges, then $G'[V_1]$ in the complete graph, and T embeds without a problem in G', and hence in G. So assume there is a V_1–V_2 edge e. Then $G' - e$ has exactly $\lceil \frac{n}{2} \rceil$ vertices of degree $\geq \frac{n}{2}$. If Theorem 3 holds for G', it certainly holds for G.

Let $\{r_1, r_2\}$ be the central edge of T (or some edge containing the center, for the case that the diameter of T is even). Set $P := N(r_1) \setminus \{r_2\}$, $Q := N(r_2) \setminus \{r_1\}$, $R := N(P) \setminus \{r_1\}$, $S := N(Q) \setminus \{r_2\}$. Then $V_T = \{r_1\} \cup \{r_2\} \cup P \cup Q \cup R \cup S$. Set $P' := N(R)$ and $Q' := N(S)$.

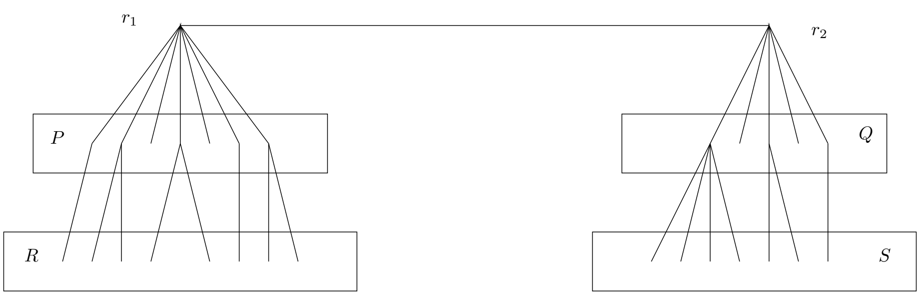

Figure 1: The tree T

Let us now refine our partition (V_1, V_2) of $V(G)$. First, let us divide V_1 into the two sets A and B, where A consists of all vertices in V_1 that send less than $\frac{n}{4}$ edges to the rest of V_1, and $B = V_1 \setminus A$. Then each vertex in A sends at least $\frac{n}{4}$ edges to V_2. Next, we divide V_2 into sets C and D. Set $C := \{v \in V_2 : deg(v) \geq \frac{n}{4}\}$ and $D := V_2 \setminus C$.

Lemma 4. *If there is an edge $xy \in E(G)$ such that $|N(x) \cap V_1| \geq \frac{n}{4}$, $|N(y) \setminus D| \geq \frac{n}{4}$, and $y \in V_1$, then T embeds in G.*

Proof. Since the union of $P \cup S$ and $R \cup Q$ has cardinality $< \frac{n}{2}$, one of the two sets has cardinality $< \frac{n}{4}$, say $|R \cup Q| < \frac{n}{4}$. Observe that $|R| < \frac{n}{4} - 1$, as otherwise Q would be empty, which contradicts $\{r_1, r_2\}$ being the central edge of T.

Now, embed r_2 in x and r_1 in y, and embed P' in $N(y) \setminus D$. Denote by $\tilde{P}$ the vertices of P' embedded in C. Then $l := |P' \setminus \tilde{P}|$ is the number of vertices of P' that are embedded in V_1.

Next, we want to embed $N(\tilde{P}) \cap R$. Since each vertex in $P' \setminus \tilde{P}$ has a neighbour in R, we have $|N(\tilde{P}) \cap R| \leq |R| - l < \frac{n}{4} - l - 1$. So, we can embed $N(\tilde{P}) \cap R$ appropriately into V_1. Indeed, each vertex of C has at least $\frac{n}{4}$ neighbours in V_1, of which at most $l + 2$ were used to embed r_1, r_2 and P'.

Up to now we have embedded at most $|R| + 2$ vertices in V_1. One of these is x, which by assumption has degree $\geq \frac{n}{4}$ in V_1. Hence, as $|R \cup Q| < \frac{n}{4}$, we can embed Q in $N(x) \cap V_1$. After this, all what is left to embed are leaves of T that are adjacent to vertices already embedded in V_1. This is easy, as the vertices in V_1 have large enough degree. □

Lemma 5. *If there is an edge $\{x, y\}$ such that $x \in B$, $y \in V_1$, and $|N(y) \cap V_1| \geq \frac{n}{8}$, then T embeds in G.*

Proof. Clearly, each vertex in $P' \cup Q'$ has degree at least 2. If $|P' \cup Q'| \geq \frac{n}{4}$, then $|E(T) \setminus \{r_1 r_2\}| \geq \frac{n}{2}$, contradicting the assumption that $|E(T)| \leq \frac{n}{2}$. Hence, $|P' \cup Q'| < \frac{n}{4}$. Without loss of generality, assume that $|P'| < \frac{n}{8}$. Embed r_2 in x and r_1 in y. Next, embed P' in $N(y) \cap V_1$, then embed the $< \frac{n}{4} - |P'|$ vertices of Q' in vertices of $N(x) \cap V_1$ not yet used. The rest of the tree are leaves of T, which are adjacent to vertices embedded in V_1, and as such easy to embed. □

Lemma 6. *G has an edge $\{x, y\}$, so that one of the following holds:*

(i) $x \in B$, $y \in V_1$ *and* $|N(y) \cap V_1| \geq \frac{n}{8}$, *or*

(ii) $y \in V_1$, $|N(y) \setminus D| \geq \frac{n}{4}$ *and* $|N(x) \cap V_1| \geq \frac{n}{4}$.

Proof. Suppose otherwise. Now, if $B = \emptyset$, then, since there is no edge satisfying (ii), we have $\frac{n}{2}\frac{n}{4} \leq |A|\frac{n}{4} \leq e(V_1, D) < |D|\frac{n}{4} \leq \frac{n}{2}\frac{n}{4}$, a contradiction. So we may assume that $B \neq \emptyset$. Moreover, B is independent, as otherwise it spans an edge that satifies (i). Hence,

$$|N(B) \cap A| \geq \frac{n}{4}. \tag{1}$$

As there is no edge for which (i) holds, we have that $|N(v) \cap D| > \frac{3n}{8} - |C|$ for all $v \in N(B) \cap A$. This together with (1) implies that

$$e(A, D) \geq |A|\frac{n}{4} + |N(B) \cap A|(\frac{n}{8} - |C|) \geq |A|\frac{n}{4} + \frac{n}{4}(\frac{n}{8} - |C|). \tag{2}$$

Now, if there is B–C edge, then it satisfies (i). Thus, there are no B–C edges, and so

$$e(B, D) \geq |B|^2, \tag{3}$$

because each $b \in B$ sends at least $\frac{n}{2} - |A| = |B|$ edges to V_2.
Observe that the function $f(a) = a\frac{n}{4} + (\frac{n}{2} - a)^2$ has its only extremum at $a = \frac{3n}{8}$. This is indeed a minimum, which implies that

$$|A|\frac{n}{4} + |B|^2 \geq \frac{3n}{8}\frac{n}{4} + (\frac{n}{8})^2. \tag{4}$$

Next, suppose that $|C| < \frac{n}{8}$. Then (2), (3), and (4) together imply that

$$\begin{aligned} e(V_1, D) &\geq |A|\frac{n}{4} + \frac{n}{4}(\frac{n}{8} - |C|) + |B|^2 \\ &\geq \frac{3n}{8}\frac{n}{4} + \frac{n}{4}(\frac{n}{8} - |C|) + (\frac{n}{8})^2 \\ &= 9(\frac{n}{8})^2 - \frac{n}{4}|C|. \end{aligned}$$

On the other hand, by definition of D,

$$e(V_1, D) < |D|\frac{n}{4} \leq (\frac{n}{2} - |C|)\frac{n}{4} = 8(\frac{n}{8})^2 - \frac{n}{4}|C|,$$

a contradiction.
Hence, we may asssume that $|C| \geq \frac{n}{8}$. Since there is no edge satisfying (ii), we have that $|N(v) \cap D| > \frac{n}{4}$ for all $v \in N(C)$. So, each vertex of A sends at least $\frac{n}{4}$ edges to D. Together with (3) and (4), this gives

$$\frac{3n}{8}\frac{n}{4} + (\frac{n}{8})^2 \leq |A|\frac{n}{4} + |B|^2 \leq e(V_1, D) < |D|\frac{n}{4} \leq \frac{3n}{8}\frac{n}{4},$$

a contradiction. □

Now, Theorem 3 follows directly from Lemmas 4, 5, and 6.

References

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