The Grad of a Graph and Classes with Bounded Expansion

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Abstract

We introduce classes of graphs with *bounded expansion* as a generalization of both proper minor closed classes and degree bounded classes. Such classes are based on a new invariant, the *greatest reduced average density (grad) of* G *with rank* r, $\nabla_r(G)$. We generalize to these classes some results proved for proper minor closed classes and bounded degree graphs, such as the existence of low tree-width colorings and homomorphism dualities.

Keywords: Constrained orientation. Transitive fraternal augmentation. Homomorphism. Coloring. Tree-depth. Bounded expansion. Restricted duality.

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1 Tree-width, Tree-depth and Coloring

A *k-tree* is a graph which is either a single vertex, or is obtained from a smaller k-tree by adding a vertex joined to the vertices of a clique of size at most k. The *tree-width* $\mathrm{tw}(G)$ of a graph G is the smallest k such that G

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is a subgraph of a k-tree. Tree-width has been proved to be a fundamental parameter, especially in the study of minor closed classes of graphs.

In a paper motivated by a question of R. Thomas [6], DeVos, Ding, Oporowski, Sanders, Reed, Seymour and Vertigan [1] proved that for every graph K and integer $j \geq 1$ there is an integer $i_V = i_V(K, j)$, such that every graph with no K-minor has a vertex partition into i_V graphs, so that any $j' \leq j$ parts form a graph with tree-width at most $j' - 1$. This proof relies on an important result of Robertson and Seymour on the structure of graphs without a particular graph as a minor [5].

In a previous work [4], we introduced the *tree-depth* $\mathrm{td}(G)$ of graph G as the minimum height of a tree which closure contains G as a subgraph. This minor monotone invariant $\mathrm{td}(G)$ is obviously at least $\mathrm{tw}(G) + 1$ but $\mathrm{td}(G)$ fails to be bounded by a function of $\mathrm{tw}(G)$ (a path of length n has tree-width 1 and tree-depth $\lceil \log_2(n+1) \rceil$).

Using the above mentioned result [1], we proved in [4] that for any proper minor closed class of graphs $\mathcal{C}$ (that is: any minor closed class of graphs excluding at least one minor) and integer $j \geq 1$ there is an integer $F_j(\mathcal{C})$, such that every graph in $\mathcal{C}$ has a vertex partition into $F_j(\mathcal{C})$ graphs, so that any $j' \leq j$ parts form a graph with tree-depth at most j', and that the tree-depth is the maximum graph invariant for which such a statement holds.

For graph G and integer $j \geq 1$, define $\chi_j(G)$ has the smallest integer N, such that G may be N-colored in such a way that, for any $H \subseteq G$, H gets at least $\min(j, \mathrm{td}(H))$ colors. Then the previous statement may be restated as:

Theorem 1.1 *For any proper minor closed class of graphs $\mathcal{C}$ and any integer $j \geq 1$, $\chi_j(G)$ is bounded on $\mathcal{C}$.*

Notice that for any graph G of order n, $\chi_1(G) = 1$, $\chi_2(G)$ is the usual chromatic number $\chi(G)$. We have then

$$\chi_1(G) \leq \chi_2(G) \leq \cdots \leq \chi_n(G) = \mathrm{td}(G).$$

2 Homomorphism Duality

The previous result has then been applied to prove that any proper minor closed class $\mathcal{C}$ of graphs has a *restricted homomorphism duality* for any connected graph F, that is: For any connected graph F, there exists a graph $D_F^{\mathcal{C}}$ so that:

- F has no homomorphism to $D_F^{\mathcal{C}}$:

$$F \not\longrightarrow D_F^{\mathcal{C}}$$

- any graph $G \in \mathcal{C}$ has no homomorphism **from** F if and only it has a homomorphism **to** $D_F^{\mathcal{C}}$:

$$\forall G \in \mathcal{C}, \quad (F \not\longrightarrow G) \iff (G \longrightarrow D_F^{\mathcal{C}})$$

Such a restricted duality exists for proper minor closed classes of graphs [4] and for classes of bounded degree graphs [2,3].

3 Grad of a Graph and Classes with Bounded Expansion

The *greatest reduced average density (grad) of* G *with rank* r: $\nabla_r(G)$ is related to the maximum average degree of a minor obtained by contracting a family of disjoints connected subgraphs, each having radius bounded by r. Precisely, a *connected partition* of G is a partition $\mathcal{P} = (V_1, \dots, V_p)$ of G such that each subgraph $G[V_i]$ of G induced by V_i is connected. The *radius* $\rho(\mathcal{P})$ of a connected partition $\mathcal{P}$ is

$$\rho(\mathcal{P}) = \max_{X \in \mathcal{P}} \min_{x \in X} \max_{y \in X} \mathrm{dist}_{G[X]}(x, y)$$

and the *grad of rank* r of G is

$$\nabla_r(G) = \max_{\rho(\mathcal{P}) \le r} \frac{|E(G/\mathcal{P})|}{|\mathcal{P}|}$$

where the maximum is taken over all the connected partitions $\mathcal{P}$ of G having radius at most r and where $G/\mathcal{P}$ stands for the **simple** graph obtained by contracting each of the $G[V_i], V_i \in \mathcal{P}$ to a single vertex.

For any graph G of order n, $\nabla_0(G)$ is half of the maximum average degree (mad) of G and

$$\frac{\mathrm{mad}(G)}{2} = \nabla_0(G) \le \nabla_1(G) \le \dots \le \nabla_n(G) \le \mathrm{td}(G) - 1$$

The last inequality is straightforward: any minor of order n of a graph with tree-depth at most k has tree-depth at most k hence size at most $(k-1)n$.

Moreover, $\nabla_r(G)$ is obviously bounded by a constant for any proper minor closed class of graphs and by $O(\Delta^r)$ for any graph with maximum degree Δ.

A class $\mathcal{C}$ has a *bounded expansion* if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that

$$\forall r \in \mathbb{N}, \forall G \in \mathcal{C}, \quad \nabla_r(G) \leq f(r).$$

4 Transitive Fraternal Orientation and Augmentation

It is well known that a graph G is a k-tree (for some k) if and only if it admits an acyclic *fraternal orientation*, that is an orientation of its edges such that when (x, z) and (y, z) are both arcs of G then either (x, y) or (y, x) is an arc of G.

We will relax the properties of acyclic fraternal orientations to build some locally fraternal orientations. In the following, directed graphs may have, for distinct vertices x and y, one arc (at most) from x to y **and** one arc (at most) from y to x.

Let $\boldsymbol{G}$ be a directed graph. A 1-*transitive fraternal augmentation* of $\boldsymbol{G}$ is a directed graph $\boldsymbol{H}$ with the same vertex set, including all the arcs of $\boldsymbol{G}$ and such that, for any vertices x, y, z,

- if (x, z) and (z, y) are arcs of $\boldsymbol{G}$ then (x, y) is an arc of $\boldsymbol{H}$ (*transitivity*),
- if (x, z) and (y, z) are arcs of $\boldsymbol{G}$ then (x, y) or (y, x) or both are arcs of $\boldsymbol{H}$ (*fraternity*).

A *transitive fraternal augmentation* of a directed graph $\boldsymbol{G}$ is a sequence $\boldsymbol{G} = \boldsymbol{G}_1 \subseteq \boldsymbol{G}_2 \subseteq \cdots \subseteq \boldsymbol{G}_i \subseteq \boldsymbol{G}_{i+1} \subseteq \cdots$, such that $\boldsymbol{G}_{i+1}$ is a 1-transitive fraternal augmentation of $\boldsymbol{G}_i$ for any $i \geq 1$.

Theorem 4.1 *Let $\mathcal{C}$ be a class with bounded expansion. There exists a function $F : \mathbb{N}^2 \to \mathbb{N}$ such that for any integer Δ^-, there exists a sequence $\mathcal{C} = \mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \cdots \subseteq \mathcal{C}_i \subseteq \cdots$ of classes with bounded expansion, so that any orientation $\boldsymbol{G}$ of a graph $G \in \mathcal{C}$ with $\Delta^-(\boldsymbol{G}) \leq \Delta^-$ has a transitive fraternal augmentation $\boldsymbol{G} = \boldsymbol{G}_1 \subseteq \boldsymbol{G}_2 \subseteq \cdots \subseteq \boldsymbol{G}_i \subseteq \cdots$ where $\boldsymbol{G}_i$ is an orientation with maximum indegree $\Delta^-(\boldsymbol{G}_i) \leq F(\Delta^-, i)$ of some $G_i \in \mathcal{C}_i$.*

5 Main Results

Our first main result stands in a characterization of classes with bounded expansion, while the second one expresses the existence of restricted dualities for such classes.

Theorem 5.1 *Let $\mathcal{C}$ be a class of graphs. The following conditions are equivalent:*

- *$\mathcal{C}$ has bounded expansion,*
- *for any integer c, the class $\mathcal{C}[K_c] = \{G[K_c] : G \in \mathcal{C}\}$ has bounded expansion, where $G[K_c]$ denotes the lexicographic product of G and K_c;*
- *there exists a function $F : \mathbb{N}^2 \to \mathbb{N}$ such that any orientation $\boldsymbol{G}$ of a graph $G \in \mathcal{C}$ has a transitive fraternal augmentation $\boldsymbol{G} = \boldsymbol{G}_1 \subseteq \boldsymbol{G}_2 \subseteq \cdots \subseteq \boldsymbol{G}_i \subseteq \cdots$ where $\Delta^-(\boldsymbol{G}_i) \leq F(\Delta^-(\boldsymbol{G}), i)$;*
- *for any integer i, $\chi_i(G)$ is bounded on $\mathcal{C}$.*

From a computational point of view, it has to be noticed that, for any fixed class $\mathcal{C}$ with bounded expansion and any fixed integer j, the proof of the theorems yields a linear time algorithm to compute, for any graph $G \in \mathcal{C}$, a coloring using at most $N(\mathcal{C}, j)$ colors so that any $j' \leq j$ parts not only induce a subgraph of tree-depth at most j', but actually induce a subgraph which connected components have the property that some color appears exactly once in them, which is a stronger statement.

From the previous theorem, we further prove that classes with bounded expansion also admits restricted dualities:

Theorem 5.2 *Any class of graphs with bounded expansion has a restricted duality for any connected graph: for every connected graph F, there exists a graph $D_F^{\mathcal{C}}$ so that $F \not\longrightarrow D_F^{\mathcal{C}}$ and*

$$\forall G \in \mathcal{C}, \quad (F \not\longrightarrow G) \iff (G \longrightarrow D_F^{\mathcal{C}})$$

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