On the computational complexity of partial covers of Theta graphs ⋆

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Abstract

By use of elementary geometric arguments we prove the existence of a special integral solution of a certain system of linear equations. The existence of such a solution then yields the NP-hardness of the decision problem on the existence of locally injective homomorphisms to Theta graphs with three distinct odd path lengths.

Key words: computational complexity, graph homomorphisms, Theta graphs

1 Introduction and background

Graphs considered in this paper are finite, undirected and simple, i.e., without loops or multiple edges. The vertex set of a graph G is denoted by V_G, the edge set by E_G, and edges are considered as two-element sets of vertices, with the notation (u, v) for the edge connecting vertices u and v. The neighborhood of a vertex is the set of vertices adjacent to it, formally $N_G(u) = \{v : (u, v) \in E_G\}$ (the subscript is omitted if it is clear from the context in which graph the

⋆ Research supported in part by project KONTAKT 525 — DIMACS-DIMATIA-Rényi

[1] Supported by the Ministry of Education of the Czech Republic as project 1M0021620808.

neighborhood if considered). The size of its neighborhood is called the degree of the vertex.

A graph homomorphism $f : G \to H$ is an edge-preserving vertex-mapping, i.e., a mapping $f : V_G \to V_H$ such that $(f(u), f(v)) \in E_H$ whenever $(u, v) \in E_G$. A homomorphism from a graph G into the complete graph on k vertices corresponds to a k-coloring of G (coloring of vertices by k colors such that adjacent vertices receive distinct colors), a notion that finds applications in many optimization problems (scheduling, broadcasting etc.). In this sense a homomorphism from a graph G to a graph H is often referred to as an H-coloring of G. For a thorough introduction to the theory of graph homomorphisms as an important part of algebraic graph theory, the reader is referred to the recent monograph [12].

Observe that every homomorphism $f : G \to H$ maps the neighborhood $N_G(u)$ of each vertex $u \in V_G$ into the neighborhood $N_H(f(u))$ of its image in H. Based on the properties of the restricted mappings we say that f is *locally injective* (*locally surjective, locally bijective*, respectively) if for every vertex $u \in V_G$, the neighborhood $N_G(u)$ is mapped injectively (surjectively, bijectively) into $N_H(f(u))$ via the mapping f. Locally injective homomorphisms find application in generalized distance constrained graph labelings and the channel assignment problem and this connection will be discussed in more detail in the concluding section. Locally bijective homomorphisms have been studied in topological graph theory as *graph covers* [2] and locally surjective ones relate to so called role assignment graphs studied in sociological applications [4]. Since a graph G allows a locally injective homomorphism to a graph H if and only if G is an induced subgraph of a graph G' which covers H (i.e. G' allows a locally bijective homomorphisms to H), locally injective homomorphisms are also referred to as *partial covers*.

We are interested in the computational complexity of the following decision problem

> H-LIHOM
> **Input:** A graph G.
> **Question:** Does there exist a locally injective homomorphism $f : G \to H$?

where the final goal would be a complete characterization of the complexity this problem, depending on the parameter graph H. This study has been initiated and partial results presented in [7] and [6]. In this paper we extend the classification for a variety of so called Theta graphs.

In a similar manner we refer to H-LBHOM, H-LSHOM and H-HOM as the problems whether an input graph G allows a locally bijective, locally surjective or locally unconstrained (respectively) homomorphism to the parameter graph H.

The characterization of the complexity of the H-HOM problem was given by Hell and Nešetřil [11] who proved that the problem is polynomially solvable for bipartite graphs H and NP-complete otherwise. The computational complexity of locally surjective homomorphisms was studied by Kristiansen and Telle [18] and the full characterization was completed by Fiala and Paulusma [9]: The H-LSHOM problem is NP-complete for every connected graph H with at least three vertices.

The study of the computational complexity of locally bijective homomorphisms was initiated by Bodlaender et al. [3] (who proved NP-hardness if H is regarded as part of the input) and Abello et al. [1] (who identified the first polynomial and NP-complete instances of the H-LIHOM problem and asked for its characterization). The study was carried on in a series of papers of Kratochvíl, Proskurowski and Telle [14–16] who proved, among other results, that H-LBHOM is NP-complete for simple regular graphs H of valency at least three.

It is proven in [7] that for every graph H, the H-LBHOM is polynomially reducible to the H-LIHOM. Hence the NP-hardness results on locally bijective homomorphisms translate directly to hardness results on locally injective ones. And therefore from the point of view of locally injective homomorphisms, it makes sense to concentrate on those graphs H for which locally bijective homomorphisms are polynomially time solvable. The simplest such graphs are those with at most two vertices of degree greater than 2 [16], and the simplest of these are the Theta graphs. Surprisingly, even for such simple graphs both NP-complete and nontrivial polynomial time solvable instances have been identified (in [6] and in this paper).

For a collection of at least three positive integers $a_1 \leq a_2 \leq \ldots \leq a_n$, with $a_2 \geq 2$, denote by $\Theta(a_1, a_2, \ldots, a_n)$ the graph consisting of two vertices of degree n, say u and v, and for each i, a path of length a_i connecting u and v (apart from the end-vertices u and v, these paths are disjoint and since $a_2 \geq 2$, the resulting graph has no multiple edges).

If the collection consists of only two distinct integers, say $a_1 = \cdots = a_k \neq a_{k+1} = \cdots = a_n$, the complexity of $\Theta(a_1, \ldots, a_n)$-LIHOM problem was fully described in [6] (and a polynomial/NP-completeness dichotomy holds true in this case). The next case, which remained open for a while, was $H = \Theta(1, 3, 5)$. Extending the methods developed in [5], we prove the following more general theorem.

Theorem 1 *For every three distinct odd positive integers* a, b, c*, the* $\Theta(a, b, c)$*-LIHOM problem is* NP*-complete.*

The reduction is based on a number theoretic result (Theorem 3) which we find interesting in its own and which has previously been asked as an open problem at several occasions. The paper is organized as follows. In Section 2 we briefly describe the reduction from [5] in a more general form. Section 3 contains the proof of Theorem 3 and Section 4 the proof of our main result Theorem 1. Concluding remarks are gathered in Section 5.

2 The reduction

Lemma 2 *If $a < b < c$ and m are odd positive integers such that*

(i) There is no integer solution $x, y, z \geq 0$ of the equation $xa + yb + zc = m$ such that x, y and z satisfy the triangle inequalities $x + y \leq z$, $x + z \leq y$ and $y + z \leq x$;
(ii) There is an integer solution $x, y, z \geq 0$ of the equation $xa + yb + zc = m$ such that $x + y = z - 1$;
(iii) There is an integer solution $x, y, z \geq 0$ of the equation $xa + yb + zc = m$ such that $y + z = x - 1$;
(iv) There is an integer solution $x, y, z \geq 0$ of the equation $xa + yb + zc = m$ such that $z + x = y - 1$,

then the $\Theta(a, b, c)$-LIHOM problem is NP*-complete.*

Proof: It is well known that the edges of every cubic bipartite graph can be properly colored by three colors (i.e., so that each color induces a matching), while for general cubic graphs the existence of such a 3-coloring is NP-complete to decide [13]. However, deciding if a precoloring of edges of a cubic bipartite graph can be extended to a proper 3-edge-coloring of the entire graph is also NP-complete, as recently shown in [5]. This problem remains NP-complete if only two colors are used in the precoloring (as a matter of fact, it would become polynomially solvable if only one color were used, cf. [17]). We reduce this problem to $\Theta(a, b, c)$-LIHOM.

Given a cubic bipartite graph G with some edges precolored by two colors, say amber and black (the third color will be cyan), we construct G' from G by replacing every amber edge by a path of length a, every black edge by a path of length b, and every edge which is not precolored by a path of length m. As the problem parameters a, b, c, and m are constant, the size of the graph G' is linear in the size of G. If m satisfies the above stated properties, then G' allows a locally injective homomorphism into $\Theta(a, b, c)$ if and only if the edge precoloring of G can be extended to a proper 3-edge-coloring of the whole graph. This follows from the fact that the vertices of degree 3 in G' must map onto u or v and the paths joining them must each map onto a sequence of

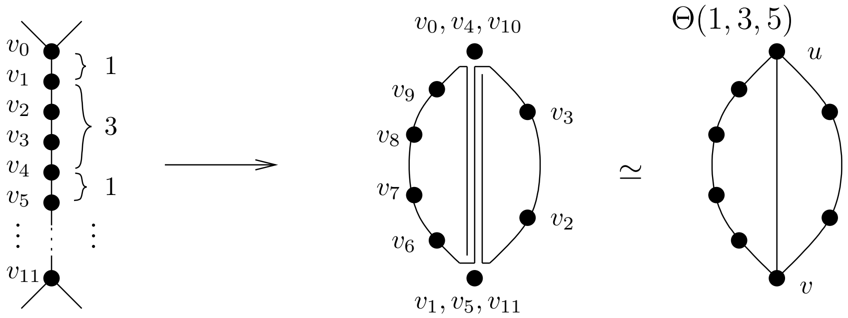

Fig. 1. Example of a mapping of a path of length $m = 11$ into $\Theta(1, 3, 5)$ according to the pattern $1 + 3 + 1 + 5 + 1 = 11$.

paths of length a, b, c with no two consecutive paths having the same length. (See Figure 1 for an example.)

If x, y, z are the numbers of occurrences of the lengths a, b, c (respectively) in such a sequence for a path of length m, the condition (i) implies that the lengths of the initial and last segments in each such sequence are the same, and conditions (ii-iv) guarantee that the path of length m can have both the initial and the last segment mapped onto the path of length a (and both onto the path of length b, and as well c). Hence these three options encode the colors (a = amber, b = black and c = cyan).

A locally injective homomorphism from G' into $\Theta(a, b, c)$ thus corresponds to a proper 3-edge-coloring of G, since both degree-3 vertices of $\Theta(a, b, c)$ are incident with exactly one path of length a, one path of length b and one path of length c. And this coloring must extend the precoloring of G, since a path of length a in G' can only map onto the path of length a in $\Theta(a, b, c)$ (and similarly for the paths of length b). □

The geometric meaning of the condition of Lemma 2 is illustrated in Figure 2. The triangle determined in the plane $xa + yb + zc = m$ by the triangle-inequalities cone must contain no integer points, but each segment parallel with one of its sides and shifted by 1 away must contain at least one integer point. It turns out that after performing a rotation of the coordinate axes such that this triangle is transformed into the whole triangle determined on $xa + yb + zc = m$ by the coordinate planes, the statement can be proved by an essentially elementary geometric argument.

3 The geometric reformulation

Theorem 3 *Let A, B, C be distinct positive integers. Then a positive integer M exists such that*

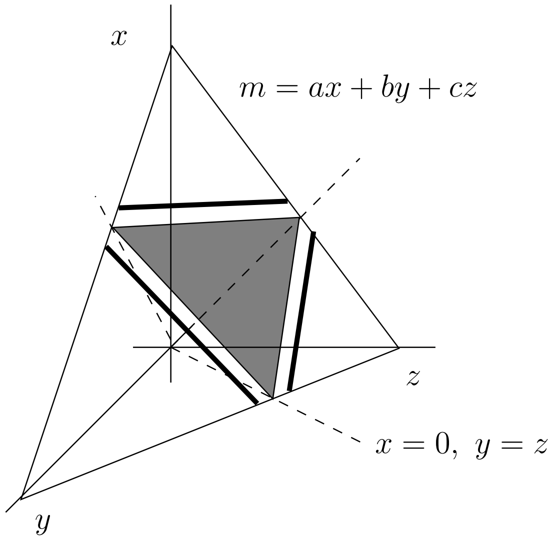

Fig. 2. The geometric meaning of Lemma 2. The thick segments shall contain an integer point.

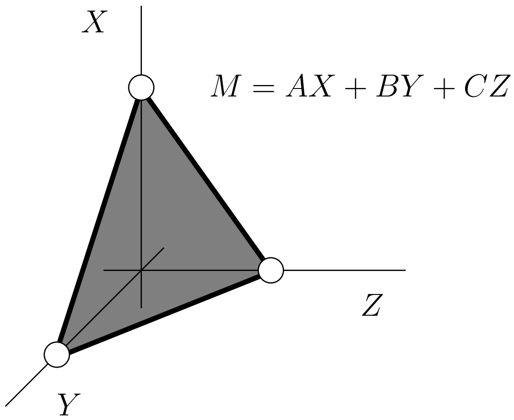

Fig. 3. The triangle Δ_M.

(I) There is no integer solution $X, Y, Z \geq 1$ *of the equation* $XA + YB + ZC = M$*;*
(II) There is an integer solution $X, Y \geq 1$ *of the equation* $XA + YB = M$*;*
(III) There is an integer solution $Y, Z \geq 1$ *of the equation* $YB + ZC = M$*;*
(IV) There is an integer solution $Z, X \geq 1$ *of the equation* $XA + ZC = M$*.*

The geometric meaning of this theorem is that there always exists a shift of the plane $XA + YB + ZC = 0$ such that the triangle (further referred to as Δ_M) determined in the translated plane by the halfspaces $X \geq 0, Y \geq 0, Z \geq 0$ contains at least one integer point inside each of its sides, but none inside the triangle. (See Figure 3.)

Proof: Let π be the plane of points (X, Y, Z) for which $XA + YB + ZC = 0$. Denote by L the 2-dimensional lattice that is the intersection of π and the 3-dimensional lattice of integer points. Every translate of π intersects the 3-

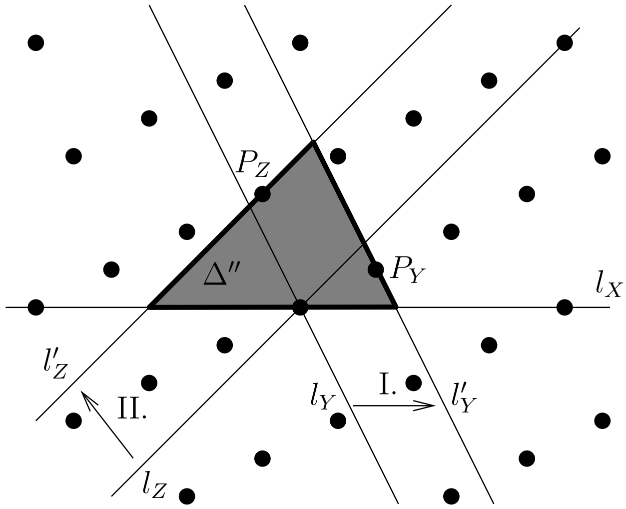

Fig. 4. Finding points P_Y and P_Z in the plane π.

dimensional integer lattice either in a translate of L, or in the empty set. Let l_X be the intersection line of π and the coordinate plane $X = 0$. Define similarly l_Y and l_Z. Note that l_X, l_Y and l_Z are parallel to the sides of the triangle Δ_M for every $M \neq 0$. The lines l_X, l_Y and l_Z intersect in the origin.

Let P_Y be (one of) the lattice point(s) of L lying in the angle determined by l_X and l_Z and being on the closest line parallel to the line l_Y (for every line parallel to l_Y, this triangle contains only finite number of integer points). Shift the line l_Y into l'_Y that passes through P_Y, thus obtaining a triangle Δ' with lattice points inside the sides lying on l_X and l'_Y, but with no lattice points in its interior. Similarly, let P_Z be a point of L lying in the angle determined by l_X and l'_Y and closest to the line l_Z. Shift l_Z to l'_Z passing through P_Z, obtaining a triangle Δ'' with lattice points inside each of its sides, but with no lattice points in its interior. (See Figure 4.)

Let $P_Y = (y_1, y_2, y_3)$ and $P_Z = (z_1, z_2, z_3)$ be the coordinates of these points. For $M = -By_2 - Cz_3$ (an integer), the triangle Δ_M is the translate of Δ'' by the integer vector $(0, -y_2, -z_3)$, and hence it contains the integer point $(0, -y_2, -z_3)$ on its side parallel to l_X, the integer point $(y_1, 0, y_3 - z_3)$ on its side parallel to l_y and the integer point $(z_1, z_2 - y_2, 0)$ on its side parallel to l_Z, but no integer point in the interior. Thus $M = -By_2 - Cz_3$ satisfies (I-IV). $\square$

4 Proof of the main theorem

We show that for every three distinct odd positive integers a, b, c, there exists a positive odd integer m such that the conditions (i-iv) of Lemma 2 are satisfied.

Given $a < b < c$, set

$$A = b + c, B = a + c, C = a + b.$$

Let M be the number guaranteed by Theorem 3.

Lemma 4 *This M satisfies (I-IV) of Theorem 3 if and only if $m = M - a - b - c$ satisfies (i-iv) of Lemma 2.*

Proof: Note first that since a, b, c are odd, A, B, C are all even and so is M. It follows that $m = M - a - b - c$ is odd. Consider the dual transformations given by

$$(X, Y, Z) \to (x = Y + Z - 1,\ y = Z + X - 1,\ z = X + Y - 1)$$

and

$$(x, y, z) \to (X = \frac{y + z - x + 1}{2},\ Y = \frac{z + x - y + 1}{2},\ Z = \frac{x + y - z + 1}{2}).$$

A simple calculation shows that

$$AX + BY + CZ =$$

$$\frac{(b + c)(y + z - x + 1)}{2} + \frac{(a + c)(z + x - y + 1)}{2} + \frac{(a + b)(x + y - z + 1)}{2} =$$

$$a + b + c + xa + yb + zc$$

and hence

$$AX + BY + CZ = M \quad \text{if and only if} \quad ax + by + cz = m.$$

Obviously, x, y, z are integers if X, Y, Z are. On the other hand, if x, y, z are integers solving $xa + yb + zc = m$, then $x + y + z \equiv 1 \mod 2$ and X, Y, Z are also integers. Thus the transformations provide a bijection among integer solutions of $xa + yb + zc = m$ and $XA + YB + ZC = M$.

It is straightforward that $X = 0$ if and only if $y + z = x - 1$, and that under this assumption $Y \geq 1, Z \geq 1$ imply $x \geq 0, y \geq 0$ and $z \geq 0$, as well as $x, y, z \geq 0$ imply $Y = \frac{z+x-y+1}{2} = \frac{2z+1}{2} > 0$ and $Z = \frac{x+y-z+1}{2} = \frac{2y+1}{2} > 0$. Hence the conditions (ii-iv) of Lemma 2 of and (II-IV) of Theorem 3 are equivalent.

Similarly, $X > 0$ if and only $y + z > x - 1$, and since all the involved variables are integers, this means that $X \geq 1$ if and only if $y + z \geq x$. Since the inequalities are symmetric, the equivalence of the conditions (i) of Lemma 2 and (I) of Theorem 3 follows. $\square$

5 Concluding remarks

The recently intensively studied notion of $L(2,1)$-labelings of graphs stems from the applications in the frequency assignment and radio coloring problems. It asks for labeling the vertices of a given graph by nonnegative integers so that adjacent vertices are assigned labels that differ by at least two, while vertices at distance two must be assigned distinct labels. This notion was generalized in several ways ($L(p_1,\dots,p_k)$-labelings, the so called channel assignment problem, etc.) to capture more complex channel environments.

A natural generalization is the notion of $H(2,1)$-labelings [7], where it is assumed that the frequency space is equipped with a metric described by a graph H. An $H(2,1)$-*labeling* of a graph G is a vertex mapping $f: V_G \to V_H$ such that $\mathrm{dist}_G(u,v) + \mathrm{dist}_H(f(u),f(v)) \geq 3$ for all pairs of vertices $u,v \in V_G$. To explain the parameters $(2,1)$ observe that the condition requires that neighbors in G are mapped onto vertices non-adjacent in H, i.e., onto vertices which are at least 2 apart. At the same time vertices with a common neighbor are mapped onto distinct vertices of H, i.e., on vertices that are at least 1 apart. Hence, as noted in [7], f is an $H(2,1)$-labeling if and only if it is a locally injective homomorphism from G to $\overline{H}$, the complement of H.

An $L(2,1)$-labeling of span k is thus a $P_k(2,1)$-labeling where P_k denotes the path of length k. The original optimization problem of minimizing the span of an $L(2,1)$-labeling of G was shown NP-complete in [10]. The fixed parameter version, i.e., assuming that the span k is bounded, leads to the class of decision problems on the existence of $P_k(2,1)$-labelings. These were shown NP-complete in [8] for every $k \geq 4$. The case of the smallest NP-hard span, i.e., and path of length $k = 4$, is nothing else but the $\Theta(1,2,3)$-LIHOM problem, since $\overline{P_4} = \Theta(1,2,3)$. This provides another connection to partial covers of Theta graphs.

Distance constrained labelings in the circular metric considered by van den Heuvel et al. [20] and Liu and Yhu [19] correspond to $C_k(2,1)$-labelings. There NP-completeness for every fixed $k \geq 6$ (as shown in [7]) follows from the fact that $\overline{C_k}$ is a regular graph of valency $k-3 \geq 3$ for $k \geq 3$, and the already mentioned result $\overline{C_k}$-LBHOM $\propto$ $\overline{C_k}$-LIHOM.

The natural question to characterize the computational complexity of deciding the existence of an $H(2,1)$-labeling, depending on the parameter graph H, is thus equivalent to characterizing the computational complexity of the H-LIHOM problems. However, the full characterization is not yet in sight. Still we believe that the core of the hardness of this characterization problem will prove to be the Theta graphs.

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