GRAD AND CLASSES WITH BOUNDED EXPANSION I. DECOMPOSITIONS

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Abstract. We introduce classes of graphs with *bounded expansion* as a generalization of both proper minor closed classes and degree bounded classes. Such classes are based on a new invariant, the *greatest reduced average density (grad) of G with rank r*, $\nabla_r(G)$. For these classes we prove the existence of several partition results such as the existence of low tree-width and low tree-depth colorings. This generalizes and simplifies several earlier results (obtained for minor closed classes).

1. Introduction

Let us start with the following particular case which illustrates some of the motivation of this paper: It is well known that not only the chromatic number of planar graphs is bounded but so are various of its variants such as acyclic or star chromatic number (by 5 and 20, see for instance [2] and [1]). For which other classes of graphs does this hold? While these variants of chromatic number are unbounded even for bipartite graphs, we proved in [12] that any proper minor closed class of graphs has a bounded star chromatic number: For any minor closed class of graphs $\mathcal{C}$ excluding at least one graph — what we shall call a proper minor closed class — there exists an integer $N(\mathcal{C})$ such that any graph $G \in \mathcal{C}$ has a colorex by $N(\mathcal{C}$ colors such that any two colors induce a star forest. Thus also acyclic chromatic number of graphs from a proper minor closed class is bounded. This particular case also follows from a recent result of DeVos et al. [6] who proved, using the Structural Theorem of Robertson and Seymour [17], that for any fixed integer $p \geq 1$, any proper minor closed class of graphs has a bounded coloring such that any $i \leq p$ parts induce a graph of tree-width at most $(i-1)$. Such a coloring is called low tree-width coloring.

In [15], we presented a strengthened version of [6]: we introduced the tree-depth of a graph and proved that for any fixed p, any proper minor closed class of graphs has a bounded coloring such that any $i \leq p$ parts induce a graph of tree-depth at most i. We also proved that tree-depth is the best graph invariant with this property (see [15]

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and below for more details). Also this result uses [6] and thus also the Structural Theorem. Such a coloring is called low tree-depth coloring and this naturally leads to a sequence $\chi_1, \chi_2, \dots$ of chromatic numbers χ_p, where χ_1 is the usual chromatic number, χ_2 is the star chromatic number and, more generally, χ_p is the minimum number of colors such that any $i \leq p$ parts induce a graph with tree-depth at most i.

It is well known that χ_1 is bounded on a class of graphs if the maximum average degree of graphs in the class is bounded. In [12], we actually proved that χ_2 is bounded if the graphs obtained by contracting star forests have bounded maximum average degree. Also, if χ_2 is bounded then so is the maximum average degree (Assume $\chi_2(G) \leq N$. Then for any two colors $i \neq j, i, j \leq N$, orient the edges of G such that any vertex has indegree at most one in the star forest induced by colors i and j. Then the indegree of any vertex is at most $\binom{N}{2}$ and thus the graph has maximum average degree at most $2\binom{N}{2}$.)

This indicates that the minor closed classes are perhaps not the most natural restriction in the context of graph partitions. One is naturally led to the study of minors with bounded depth (of the contracted forest) and their edge densities. This in turn leads to the notion of bounded expansion which is the central notion of this paper.

Very schematically this relationship between the χ_p's and the bounded depth minors naturally leads to the following two questions:

Do there exist integral functions f_1 and f_2 such that, for any integer p:

- If the minors of depth at most $f_1(p)$ of the graphs of a class $\mathcal{C}$ have bounded maximum average degree then the graphs in $\mathcal{C}$ have bounded χ_p,
- If the graphs in $\mathcal{C}$ have bounded $\chi_{f_2(p)}$ then all the minors of depth at most p of the graphs of a class $\mathcal{C}$ have bounded maximum average degree.

In this paper, we prove that both questions have a positive answer. This is the main result of this paper formulated below as Theorem 8.1. It implies the above result of [15]. Perhaps more interestingly our proof does not rely on the Structural Theorem and yield an effective algorithm (in fact a linear algorithm, see our companion paper [13]).

Let us describe this development in a greater detail: The concept of tree-width [8],[16],[23] is central to the analysis of graphs with forbidden minors done by Robertson and Seymour and gained much algorithmic attention thanks to the general complexity result of Courcelle about monadic second-order logic graph properties decidability for graphs with bounded tree-width [3],[4]. This computational property (and similar algorithmic aspects), as well as a question of R. Thomas [20], motivated the study of graph partitions where k parts induce a subgraph of tree-width at most $(k-1)$. Such partitions have been proved to exists by DeVos et al. for proper minor closed classes of graphs [6], relying on Structural Theorem of Robertson and Seymour on the structure of graphs without a particular graph as a minor [17]. This result has been extended by the authors to tree-depth decompositions in [15]. Advancing the definition of tree depth let us recall the definition of the tree width by means of k-trees: A *k-tree* is a graph which is either a clique of size at most k or is obtained from a smaller k-tree by adding a vertex adjacent to at most k vertices which are pairwise adjacent. The *tree-width* $\mathrm{tw}(G)$ of a graph G is the smallest integer k such that G is a subgraph of a k-tree, that is: a *partial k-tree*. The *tree-depth* $\mathrm{td}(G)$ of a connected graph G is the minimum height of a rooted tree which closure contains G as a subgraph (height is defined here as the maximum number of vertices of a path from the root to a leaf of the tree; the closure of a rooted tree is the graph formed by the ancestor relation). (The tree depth of a disconnected graph G is the maximal tree depth of a component of G.)

The tree depth is a minor monotone invariant. It is related to the tree-width by $\mathrm{tw}(G) + 1 \leq \mathrm{td}(G) \leq \mathrm{tw}(G) \log_2 n$, where n is the order of G and is actually equal to the *vertex ranking number* [5][18] and to the minimum height of an *elimination tree* [5]. For our purposes it is important that $\mathrm{td}(G)$ has an alternative definition by means of *centered coloring*: a coloring of the vertices of a graph G is called centered if in any connected subgraph G' of G some color appears exactly once (thus a centered coloring is necessarily proper). It may be seen then that the tree-depth of a graph G is the minimum number of colors in a centered coloring of G. As well as graphs with large tree-width may be characterized by large grid minors, tree-depth may be characterized by excluded paths: a graph has large tree-depth if and only if it includes a long path.

Generalizing [6] we proved in [15] the following:

Theorem 1.1 (Corollary 5.3 of [15])**.** *For any proper minor closed class of graphs $\mathcal{K}$ and for any fixed integer $p \geq 1$, $\chi_p(G)$ is bounded on $\mathcal{K}$.*

An alternative way to look at this result is the following: for any integer k and any proper minor closed class of graphs $\mathcal{K}$, there exists an integer $N(\mathcal{K}, k)$ such that any subgraph $H \subseteq G$ gets at least $\min(k, \mathrm{td}(H))$ colors (hence $i < k$ parts induce graphs of tree-depth at most i).

In[15] we proved that this statement is optimal in the following sense: Let ϕ be an integral graph function (i.e. we assume that $\phi(G)$ is an integer for any graph G). Assume that for any integer k and for any proper minor closed class $\mathcal{K}$ there exists an integer $N(\mathcal{K}, k)$ such that any graph $G \in \mathcal{K}$ has a partition into $\leq N(\mathcal{K}, k)$ parts with the property that any subgraph $H \subseteq G$ gets at at least $\min(k, \phi(H))$ colors. Then $\phi(H) \leq \mathrm{td}(H)$.

Here we extend Theorem 1.1 to more general classes of graphs. In fact it appears that proper minor closed classes are unnecessary restrictive for the validity of Theorem 1.1.

Let f be a function assigning to every positive integer n a real value $f(n)$. Instead of dealing with proper minor closed classes we shall work with classes of graphs with f-bounded expansion. This definition is introduced in Section 4. Informally, a graph G is said to have f-bounded expansion if every minor G' of G which we obtain by contracting a disjoint union of connected subgraphs of radius $\leq r$ and then deleting some vertices have edge density bounded by $f(r)$. The main consequence of our approach here is a generalization of Theorem 1.1 to the classes of graphs with f-bounded expansion. This is indeed a generalization as each proper minor closed class has expansion bounded by a constant. Also bounded degree graphs are fitting into this scheme (they are bounded by an exponential function). (See Section 4 where the bounded expansion is defined and discussed in detail.) Actually, we not only extend Theorem 1.1 to classes with bounded expansion but prove that it cannot be extended further: classes with bounded expansion may be actually characterized by the validity of Theorem 1.1.

It is perhaps surprising that one can prove the full analogy of Theorem 1.1 on this level of generality. The main reason for this is that we approach the decomposition theorem via graph orientations and their local properties. Note that triangulated graphs, like k-trees, have orientations with strong local properties. A digraph $\vec{G}$ is *fraternally oriented* if $(x,z) \in E(\vec{G})$ and $(y,z) \in E(\vec{G})$ implies $(x,y) \in E(\vec{G})$ or $(y,x) \in E(\vec{G})$. This concept was introduced by Skrien [19] and a characterization of fraternally oriented digraphs having no symmetrical arcs has been obtained by Gavril and Urrutia [7], who also proved that triangulated graphs and circular arc graphs are all fraternally orientable graphs. An orientation is *transitive* if $(x,y) \in E(\vec{G})$ and $(y,z) \in E(\vec{G})$ implies $(x,z) \in E(\vec{G})$. It is obvious that a graph has an acyclic transitive fraternal orientation in which every vertex has indegree at most $(k-1)$ if and only if it is the closure of a rooted forest of height k. It follows that tree-depth and transitive fraternal orientation are closely related.

This paper is organized as follows: In Sections 2,3,4 we introduce the above notions in a greater detail. The key notion is the notion of the *greatest reduced average density (grad)* $\nabla_r(G)$ of rank r of a graph G. We then derive several results about local properties of orientations. This is the reason why we use or introduce relaxed versions, like *p-centered colorings* (in which in every subgraph, either some color appears exactly once or at least p colors appear), or *transitive fraternal* *augmentations* (each augmentation step consists in adding the missing arcs while applying the fraternity and transitivity rules on the initial arcs). The Section 5 is devoted to the proof of the stability of the notion of classes with bounded expansion with respect to the lexicographic product with an arbitrary fixed size complete graph (Lemma 5.2). This key lemma will allow to prove in Section 6 the existence of transitive fraternal augmentations with indegrees bounded as a function of the grad. These augmentations will be used in Section 7 to exhibit p-centered colorings, eventually leading us to Theorem 8.1 in Section 8.

Further corollaries and applications of our method will appear in the 3 companion papers, see [13, 14, 11].

2. Low tree-width coloring

A *k-tree* is recursively defined as a single vertex graph or a graph obtained from a smaller k-tree by adding a vertex adjacent to a clique of size at most k. The *tree-width* $\mathrm{tw}(G)$ of a graph G is the minimum integer k such that G is a subgraph of a k-tree.

A class $\mathcal{C}$ has a *low tree-width coloring* if, for any integer $p \geq 1$, there exists an integer $N(p)$ such that any graph $G \in \mathcal{C}$ may be vertex-colored using $N(p)$ colors so that each of the connected components of the subgraph induced by any $i \leq p$ parts has tree-width at most $(i-1)$. According to this definition, the result of DeVos et al. may be expressed as

Theorem 2.1 ([6])**.** *Any minor closed class of graphs excluding at least one graph has a low tree-width coloring.*

3. Low tree-depth coloring and p-centered colorings

In [15], we introduced the *tree-depth* $\mathrm{td}(G)$ of a graph G as follows:

A *rooted forest* is a disjoint union of rooted trees. The *height* of a vertex x in a rooted forest F is the number of vertices of a path from the root (of the tree to which x belongs to) to x and is noted $\mathrm{height}(x, F)$. The *height* of F is the maximum height of the vertices of F. Let x, y be vertices of F. The vertex x is an *ancestor* of y in F if x belongs to the path linking y and the root of the tree of F to which y belongs to. The *closure* $\mathrm{clos}(F)$ of a rooted forest F is the graph with vertex set $V(F)$ and edge set $\{\{x, y\} : x \text{ is an ancestor of } y \text{ in } F, x \neq y\}$. A rooted forest F defines a partial order on its set of vertices: $x \leq_F y$ if x is an ancestor of y in F. The comparability graph of this partial order is obviously $\mathrm{clos}(F)$. The *tree-depth* $\mathrm{td}(G)$ of a graph G is the minimum height of a rooted forest F such that $G \subseteq \mathrm{clos}(F)$. As a consequence, we have:

Lemma 3.1 ([15])**.** *Let G be a graph and let $G_1,\dots,G_p$ be its connected components. Then:*

$$\operatorname{td}(G)=\begin{cases}1, & \text{if } |V(G)|=1;\\ 1+\min_{v\in V(G)}\operatorname{td}(G-v), & \text{if } p=1 \text{ and } |V(G)|>1;\\ \max_{i=1}^{p}\operatorname{td}(G_i), & \text{otherwise.}\end{cases}$$

As we introduced low tree-width coloring, we say that a class $\mathcal{C}$ has a *low tree-depth coloring* if, for any integer $p\geq 1$, there exists an integer $N(p)$ such that any graph $G\in\mathcal{C}$ may be vertex-colored using $N(p)$ colors so that each of the connected components of the subgraph induced by any $i\leq p$ parts has tree-depth at most i. As $\operatorname{td}(G)\geq \operatorname{tw}(G)-1$, a class having a low-tree depth coloring has a low tree-width coloring. In [15] is proved a strengthening of Theorem 2.1:

Theorem 3.2 ([15])**.** *Any minor closed class of graphs excluding at least one graph has a low tree-depth coloring.*

Notation 3.1. Following [15], we will make use of the notation $\chi_p(G)$ for the minimum number of colors need for a vertex coloring of G such that $i<p$ parts induce a subgraph of tree-depth at most i.

Theorem 3.2 relies on p-centered colorings, which have also been introduced in [15]: A *p-centered coloring* of a graph G is a vertex coloring such that, for any (induced) connected subgraph H, either some color $c(H)$ appears exactly once in H, or H gets at least p colors.

For the sake of completeness we recall some lemmas of [15]:

Lemma 3.3 ([15])**.** *Let G,G_0 be graphs, let $p=\operatorname{td}(G_0)$, let c be a q-centered coloring of G where $q\geq p$. Then any subgraph H of G isomorphic to G_0 gets at least p colors in the coloring of G.* □

From this lemma follows that p-centered colorings induce low tree-depth colorings:

Corollary 3.4. *Let p be an integer, let G be a graph and let c be a p-centered coloring of G.*

Then $i<p$ parts induce a subgraph of tree-depth at most i

Proof. Let G' be any subgraph of G induced by $i<p$ parts. Assume $\operatorname{td}(G')>i$. According to Lemma 3.1, the deletion of one vertex decreases the tree-depth by at most one. Hence there exists an induced subgraph H of G' such that $\operatorname{td}(H)=i+1\leq p$. According to lemma 3.3 (choosing $G_0=H$), H gets at least p colors, a contradiction. □

Lemma 3.5 ([15])**.** *Let p,k be integers. Then there exists an integer $N(p,k)$ such that any graph G with tree width at most k has a p-centered coloring using $N(p,k)$ colors.* □

The following lemma is proved in [15] for the particular case of proper minor closed classes of graphs and tree-width. We shall state it here in its general form.

Lemma 3.6. *Let $\mathcal{C}$ be a class of graphs. Assume that for any integer $p \geq 1$ there exists a class of graphs $\mathcal{C}_p$ such that:*

- *there exists an integer $N(\mathcal{C}_p, p)$, such that any graph $G \in \mathcal{C}_p$ has a p-centered coloring using at most $N(\mathcal{C}_p, p)$ colors,*
- *there exists an integer $C(p)$ such that any $G \in \mathcal{C}$ has a $C(p)$ vertex coloring such that p classes induce a graph in $\mathcal{C}_p$.*

Then there exists an integer $X(p)$, such that every graph in $\mathcal{C}$ has a p-centered coloring using $X(p)$ colors.

Proof. Let $G \in \mathcal{C}$. According to the assumption, there exists a vertex partition into $C(p)$ parts, such that any p parts form a graph in $\mathcal{C}_p$. This partition will be defined as a coloring $\bar{c} : V(G) \longrightarrow \{1, 2, \dots, C(p)\}$. For any set P of p parts let G_P be the graph induced by all the parts in P. According to the assumption, each of the G_P has p-centered coloring c_P using $N(\mathcal{C}_p, p)$ colors. Consider the following ("product") coloring c defined as

$$c(v) = (\bar{c}(v), (c_P(v); |P| = p, P \subset \{1, 2, \dots, C(p)\})).$$

This is the product of the coloring of G by $C(p)$ colors and of the colorings of the G_P. This new coloring of G (with $X(p) = C(p)N(\mathcal{C}_p, p)^{\binom{C(p)}{p}}$ colors. Let H be a connected subgraph of G. Then, either H gets at least $p+1$ colors, or $V(H)$ is included in some subgraph G_P of G induced by p parts. In the later case, some color appears exactly once in H. □

Theorem 3.7. *Let $\mathcal{C}$ be a class of graphs having low tree-width colorings and let p be an integer. Then there exists integer $X(p)$, such that every graph in $\mathcal{C}$ has a p-centered coloring using $X(p)$ colors.*

Proof. Let $\mathcal{C}_p$ be the class of graphs with tree-width at most $(p-1)$. According to Theorem 2.1 and Lemma 3.5, the conditions of Lemma 3.6 are satisfied hence $X(p)$ exists. □

As a consequence we have the following equivalence of the various (seemingly unrelated) above notions:

Theorem 3.8. *Let $\mathcal{C}$ be a class of graphs. Then the following conditions are equivalent:*

- *$\mathcal{C}$ has a low tree-width coloring,*
- *$\mathcal{C}$ has a low tree-depth coloring,*
- *for any integer p, $\{\chi_p(G) : G \in \mathcal{C}\}$ is bounded,*
- *for any integer p, there exists an integer $X(p)$ such that any graph $G \in \mathcal{C}$ has a p-centered colorings using at most $X(p)$ colors.*

Our main result (Theorem 8.1) is a non-trivial extension of this equivalence.

4. The grad of a graph and classes with bounded expansion

Recall that the *maximum average degree* $\mathrm{mad}(G)$ of a graph G is the maximum over all subgraphs H of G of the average degree of H, that is $\mathrm{mad}(G) = \max_{H\subseteq G} \frac{2|E(H)|}{|V(H)|}$. The *distance* $d(x,y)$ between two vertices x and y of a graph is the minimum length of a path linking x and y, or ∞ if x and y do not belong to the same connected component.

We introduce several notations:

- The *radius* $\rho(G)$ of a connected graph G is:

$$\rho(G) = \min_{r\in V(G)} \max_{x\in V(G)} \mathrm{d}(r,x)$$

- A *center* of G is a vertex r such that $\max_{x\in V(G)} \mathrm{d}(r,x) = \rho(G)$.

Definition 4.1. Let G be a graph. A *ball* of G is a subset of vertices inducing a connected subgraph. The set of all the families of balls of G is noted $\mathfrak{B}(G)$.

Let $\mathcal{P} = \{V_1,\dots,V_p\}$ be a family of balls of G.

- The *radius* $\rho(\mathcal{P})$ of $\mathcal{P}$ is $\rho(\mathcal{P}) = \max_{X\in\mathcal{P}} \rho(G[X])$
- The *complexity* of $\mathcal{P}$ is $\zeta(\mathcal{P}) = \max_{v\in V(G)} |\{i : v \in V_i\}|$.
- The *quotient* $G/\mathcal{P}$ of G by $\mathcal{P}$ is a graph with vertex set $\{1,\dots,p\}$ and edge set $E(G/\mathcal{P}) = \{\{i,j\} : (V_i \times V_j) \cap E(G) \neq \emptyset \text{ or } V_i \cap V_j \neq \emptyset\}$.

We introduce several invariants that refine the notion of maximum average degree:

Definition 4.2. The *greatest reduced average density (grad) of G with rank r and complexity c* is

$$\overset{c}{\nabla}_r(G) = \max_{\substack{\mathcal{P}\in\mathfrak{B}(G)\\ \rho(\mathcal{P})\leq r, \zeta(\mathcal{P})\leq c}} \frac{|E(G/\mathcal{P})|}{|\mathcal{P}|}.$$

For the sake of simplicity, we also define:

- The *grad of G with rank r*:

$$\nabla_r(G) = \overset{1}{\nabla}_r(G) = \max_{\substack{\mathcal{P}\in\mathfrak{B}(G)\\ \rho(\mathcal{P})\leq r, \zeta(\mathcal{P})=1}} \frac{|E(G/\mathcal{P})|}{|\mathcal{P}|}$$

- The *grad of G*:

$$\nabla(G) = \max_r \nabla_r(G) = \max_{H\preceq G} \frac{|E(H)|}{|V(H)|}$$

Notice that we have:

$$\frac{\mathrm{mad}(G)}{2} = \nabla_0(G) \leq \nabla_1(G) \leq \dots \leq \nabla_{\rho(G)}(G) = \nabla(G) \tag{1}$$

and that $\nabla(G)$ is related to the Hadwiger number $h(G)$ of G (that is the maximum order of a complete graph which is a minor of G) by:

$$\frac{h(G)-1}{2} \leq \nabla(G) \leq O(h(G)\sqrt{\log h(G)}), \tag{2}$$

Proof. Let $h = h(G)$. As K_h is a $(h-1)$-regular minor of G , $\frac{h-1}{2} \leq \nabla(G)$. Moreover, there exists a constant C such that if $\nabla(G) > C(h+1)\sqrt{\log(h+1)}$ then G has a minor with minimum degree at least $\gamma(h+1)\sqrt{\log(h+1)}$ hence a minor K_{h+1} as proved by Kostochka [9] and Thomason [21] (extending earlier work of Mader [10]; see [22] for an tight value of constant γ). □

Also notice the following well known facts (usually expressed by means of the maximum average degree):

Fact 4.1. *Let G be a graph. Then G has an orientation such that the maximum indegree of G is at most k if and only if $k \geq \nabla_0(G)$.*

Fact 4.2. *Let G be a graph. Then G is $\lfloor 2\nabla_0(G)\rfloor$-degenerated, hence $\lfloor 2\nabla_0(G)+1\rfloor$-colorable.*

The grad actually appears to be related to low tree-depth colorings:

Lemma 4.1. *For any graph G and any integer r:*

$$\nabla_r(G) \leq (2r+1)\binom{\chi_{2r+2}(G)}{2r+2} \tag{3}$$

Proof. Consider a vertex coloring c of G with $N = \chi_{2r+2}(G)$ colors such that any $i \leq 2r+2$ colors induce a subgraph of tree-depth at most i. For any $J \in \binom{[N]}{2r+2}$, let $G_J = G[c^{-1}(J)]$ and let Y_J be a rooted forest of height $\mathrm{td}(G_J) \leq 2r+2$ such that $G_J \subseteq \mathrm{clos}(Y_J)$.

Let $\mathcal{P} = \{X_1, \dots, X_p\}$ be a family of balls of G with radius r and complexity 1 achieving the bound $\nabla_r(G)$ (that is: such that $\nabla_r(G) = \frac{|E(G/\mathcal{P})|}{|\mathcal{P}|}$). Let $x_1, \dots, x_k$ be centers of $X_1, \dots, X_k$. If X_i and X_j are adjacent in $G/\mathcal{P}$ then there exists a path $P_{i,j}$ of length at most $2r+1$ linking x_i and x_j. Let $I_{i,j} \in \binom{[N]}{2r+2}$ be such that $I_{i,j} \supseteq c(V(P_{i,j}))$. Then the path $P_{i,j}$ is included in some connected component of $G_{I_{i,j}}$. It follows that there exists in $P_{i,j}$ a vertex $v_{i,j}$ which is minimum with respect to the partial order defined by $Y_{I_{i,j}}$. As $\{x_i, x_j\} \subseteq V(P_{i,j}) \subseteq X_i \cup X_j$ and as $X_i \cap X_j = \emptyset$ (because $\zeta(\mathcal{P}) = 1$), $v_{i,j}$ either belongs to X_i or to X_j. Depending on the case, $v_{i,j}$ is a vertex of X_i which is an ancestor of x_j in $Y_{I_{i,j}} \cap X_i$ or a vertex of X_j which is an ancestor of x_i

in $Y_{I_{i,j}} \cap X_j$. Thus:

$$\begin{aligned} p\nabla_r(G) &\leq \sum_{I\in\binom{[N]}{2r+2}} \sum_{1\leq i\leq p} \sum_{\substack{1\leq j\leq p\\ j\neq i}} |\{v : v \text{ ancestor of } x_i \text{ in } Y_I \cap X_j\}| \\ &\leq \sum_{I\in\binom{[N]}{2r+2}} \sum_{i=1}^{p} |\{v : v \text{ ancestor of } x_i \text{ in } Y_I\}| \\ &\leq \binom{N}{2r+2} \times p \times (2r+1) \end{aligned}$$

Hence

$$\nabla_r(G) \leq (2r+1)\binom{N}{2r+1}$$

□

This lemma motivates the following definition:

Definition 4.3. A class of graphs $\mathcal{C}$ has *bounded expansion* if there exists a function $f : \mathbb{N} \to \mathbb{R}$ such that for every graph $G \in \mathcal{C}$ and every r holds

$$\nabla_r(G) \leq f(r) \tag{4}$$

Theorem 4.2. *If a class $\mathcal{C}$ has low tree-width colorings then $\mathcal{C}$ has bounded expansion.*

Proof. As low tree-width colorings and low tree-depth colorings are equivalent, the theorem is a direct consequence of Lemma 4.1. □

The main theorem of this paper may be seen as a converse of Theorem 4.2.

5. Grad stability over lexicographic product

Let G, H be graphs. The *lexicographic product* $G \bullet H$ is defined by $V(G \bullet H) = V(G) \times V(H)$ and $E(G \bullet H) = \{\{(x,y),(x',y') : \{x,y\} \in E(G)$ or $x = x'$ and $\{y,y'\} \in E(H)\}$.

Let us note at this place that the lexicographic product (or blowing up of vertices) is an operation which is incompatible with the minors. One can see easily that every graph is a minor of a graph of the form $G \bullet K_2$ for a planar graph G. But the lexicographic product is naturally related to the notion of complexity we have introduced for grad:

Lemma 5.1. *For any graph G and any integers c, r, we have:*

$$\overset{c}{\nabla}_r(G) = \nabla_r(G \bullet K_c)$$

Proof. Let $\mathcal{P} = \{V_1, \dots, V_p\}$ be a ball family of G with complexity $c = \zeta(\mathcal{P})$ and radius $r = \rho(\mathcal{P})$. As $\zeta(\mathcal{P}) = c$ there exists a function $f : V(G) \times \{1, \dots, p\} \to \{1, \dots, c\}$ such that if $x \in V_i \cap V_j$ then $f(x, i) \neq f(x, j)$.

For $1 \leq i' \leq p$, define $V_i' = \{(x, f(x, i)) : x \in V_i\}$. Then $\mathcal{P}' = \{V_1', \dots, V_p'\}$ has radius r and complexity 1. Moreover, $G/\mathcal{P}$ is obviously isomorphic to a subgraph of $(G \bullet K_c)/\mathcal{P}'$. It follows that $\nabla_r(G \bullet K_c) \geq \overset{c}{\nabla}_r(G)$.

Conversely, let $\mathcal{P}' = \{V_1', \dots, V_q'\}$ be a ball family of $G \bullet K_c$, define the ball family $\mathcal{P} = \{V_1, \dots, V_q\}$ of G by $x \in V_i$ if there exists $\alpha \in \{1, \dots, c\}$ such that $(x, \alpha) \in V_i'$. Then $\rho(\mathcal{P}) \leq \rho(\mathcal{P}')$ and $\zeta(\mathcal{P}) \leq c$. It follows that $\overset{c}{\nabla}_r(G) \geq \nabla_r(G \bullet K_c)$. □

The remaining of the section will be dedicated to the proof of the following key lemma:

Lemma 5.2. *There exist polynomials P_i ($i \geq 0$) such that for any graph G and integers r and c:*

$$\overset{c}{\nabla}_r(G) \leq P_r(c, \nabla_r(G)) \tag{5}$$

In the following, a directed graph $\vec{G}$ may not have a loop and for any two of its vertices x and y, $\vec{G}$ includes at most one arc from x to y and at most one arc from y to x.

If a directed path $\vec{P}$ has starting vertex x and end vertex y, we note $x \overset{\vec{P}}{\rightsquigarrow} y$.

If $x \overset{\vec{P}_1}{\rightsquigarrow} z$, $y \overset{\vec{P}_2}{\rightsquigarrow} z$ and if no internal vertex or edges of $\vec{P}_1$ belongs to $\vec{P}_2$ nor the converse, we note $x \overset{\vec{P}_1}{\rightsquigarrow} z \overset{\vec{P}_2}{\leftsquigarrow} y$. In such a case, either $\vec{P}_1 \cup \vec{P}_2$ is a path, or $\vec{P}_1 \cup \vec{P}_2$ is a cycle and $x = y$. Moreover, if $x \neq y$, $|\vec{P}_1| \leq a$ and $|\vec{P}_2| \leq b$, we say that y is (a, b)-*reachable* from x.

Definition 5.1. Let $\vec{G}$ be a directed graph, let a, b be integers. A set $\vec{\Lambda}$ of arcs with endpoints in $V(\vec{G})$ is an (a, b)-*augmentation* of $\vec{G}$ if, for any $x, y \in V(\vec{G})$ with y (a, b)-reachable from x, either (x, y) or (y, x) belongs to $\vec{\Lambda}$.

The *maximum indegree* of $\vec{\Lambda}$ is

$$\Delta^-(\vec{\Lambda}) = \max_{y \in V(\vec{G})} |\{x \in V(\vec{G}) : (x, y) \in \vec{\Lambda}\}|$$

Notice that if a or b is at least 1, $E(\vec{G})$ is obviously included in any (a, b)-augmentation of $\vec{G}$.

Lemma 5.3. *Let $\vec{G}$ be a directed graph, let a, b be integers and let $\vec{\Lambda}$ be an (a, b)-augmentation of $\vec{G}$.*

Then there exists a vertex coloring $\gamma_{\vec{\Lambda}}$ using at most $2\,\Delta^-(\vec{\Lambda})+1$ colors such that for any vertex x, $\gamma_{\vec{\Lambda}}(y)\neq\gamma_{\vec{\Lambda}}(x)$ for any vertex y (a,b)-reachable from x.

Proof. Let $\vec{H}$ be the directed graph with vertex set $\vec{G}$ and arc set $\vec{\Lambda}$. If y is (a,b)-reachable from x in $\vec{G}$ then (x,y) or (y,x) belongs to $E(\vec{H})$. As $\vec{H}$ has maximum indegree $\Delta^-(\vec{\Lambda})$, it is $(2\,\Delta^-(\vec{\Lambda})+1)$-choosable. Any proper coloration of $\vec{H}$ will do. □

Lemma 5.4. *Let $\vec{G}$ be a directed graph with maximum indegree $\Delta^-(\vec{G})$, let a,b be integers and let $\vec{\Lambda}$ be an (a,b)-augmentation of $\vec{G}$.*

Then there exists an edge coloring $\Upsilon_{\vec{\Lambda}}$ using at most $(2\,\Delta^-(\vec{\Lambda})+1)\,\Delta^-(\vec{G})$ colors such that for any $x \overset{\vec{P}_1}{\rightsquigarrow} z \overset{\vec{P}_2}{\leftsquigarrow} y$ with $|\vec{P}_1|\leq a+1$ and $|\vec{P}_2|\leq b+1$, all the edges of $\vec{P}_1\cup\vec{P}_2$ get different colors.

Proof. Consider an edge coloring c_0 such that two edges having the same end vertex have different colors (this is achieved with $\Delta^-(\vec{G})$ colors) and the vertex coloring $\gamma_{\vec{\Lambda}}$ defined in Lemma 5.3. Then for any arc $e=(x,y)$ define $\Upsilon_{\vec{\Lambda}}(e)=(c_0(e),\gamma_{\vec{\Lambda}}(y))$. Then if $e=(x,y)$ and $f=(x',y')$ are two different arcs in $\vec{P}_1\cup\vec{P}_2$ where either $y\neq y'$ thus y' is (a,b)-reachable from y or y is (a,b)-reachable from y' hence $\gamma_{\vec{\Lambda}}(y')\neq\gamma_{\vec{\Lambda}}(y)$, or $y=y'$ hence $c_0(e)\neq c_0(f)$. □

Notation 5.2. Let Υ be an arc-coloring of a directed graph $\vec{G}$ and let $\vec{P}$ be a directed path of $\vec{G}$ of length l. We note $\Upsilon(\vec{P})=\vec{\alpha}=(\alpha_1,\ldots,\alpha_l)$ the sequence of the colors $\Upsilon(e)$ of the arcs of $\vec{P}$, taken in the order in which they appear on $\vec{P}$.

Lemma 5.5. *Let $\vec{G}$ be a directed graph with maximum indegree $\Delta^-(\vec{G})$, let a,b be integers and let $\vec{\Lambda}$ be an (a,b)-augmentation of $\vec{G}$. Let $\Upsilon_{\vec{\Lambda}}$ be the edge coloring defined in Lemma 5.4.*

Let $\vec{P}_1,\vec{P}_2$ be two directed paths of length $l\leq\max(a,b)+1$, such that the initial vertex of one of them is different from the end vertex of the other one. If $\Upsilon_{\vec{\Lambda}}(\vec{P}_1)=\Upsilon_{\vec{\Lambda}}(\vec{P}_2)$ then either $\vec{P}_1$ and $\vec{P}_2$ do not intersect, or they share the same initial vertex and there exists $0\leq a\leq l$ such that $\vec{P}_1$ and $\vec{P}_2$ share their a first edges and do not intersect thereafter.

Proof. Without loss of generality, we may assume $a\geq b$. Let $\vec{\alpha}=\Upsilon_{\vec{\Lambda}}(\vec{P}_1)$. Assume there exists a vertex v having one incoming edge in $\vec{P}_1$ (the ith of $\vec{P}_1$, hence colored α_i) and one (different) incoming edge in $\vec{P}_2$ (the jth of $\vec{P}_2$, hence colored α_j). Without loss of generality, we may assume $i\geq j$. Then the $(j+1)$th vertex u of $\vec{P}_1$ has an incoming edge in $\vec{P}_1$ colored α_j and belong to the initial subpath of $\vec{P}_1$ ending at v. It follows that v is $(a,0)$ reachable from u. Hence an incoming edge of u may not have the same color of an incoming edge of v, contradiction.

Similarly, the initial vertex of one of the path may not be internal to the second one. As the case where the initial vertex of one of the path is the end vertex of the other one, we conclude that either the two paths do not intersect or they share their a first edges. □

Lemma 5.6. *Let $\vec{G}$ be a directed graph with maximum indegree $\Delta^-(\vec{G})$, let a, b be integers and let $\vec{\Lambda}$ be an (a, b)-augmentation of $\vec{G}$. Let $\Upsilon_{\vec{\Lambda}}$ be the edge coloring defined in Lemma 5.4. Let $\vec{\alpha}$ be a sequence of $l \leq \max(a, b) + 1$ distinct edge colors. Then the union $T_{\vec{\Lambda}}(\vec{\alpha})$ of all the directed paths $\vec{P}$ such that $\Upsilon_{\vec{\Lambda}}(\vec{P}) = \vec{\alpha}$ is a directed rooted forest.*

Proof. This is a direct consequence of Lemma 5.5. □

Lemma 5.7. *Let $\vec{G}$ be a directed graph with maximum indegree $\Delta^-(\vec{G})$, let $a \geq b$ be integers and let $\vec{\Lambda}$ be an (a, b)-augmentation of $\vec{G}$. Let $\Upsilon_{\vec{\Lambda}}$ be the edge coloring defined in Lemma 5.4. Let $\vec{\alpha}$ and $\vec{\beta}$ be sequences of respective lengths $p \leq a+1$ and $q \leq b+1$. Let $\Pi_{\vec{\Lambda}}(\vec{\alpha}, \vec{\beta})$ be the union of all the $\vec{P}_1 \cup \vec{P}_2$ where $\Upsilon_{\vec{\Lambda}}(\vec{P}_1) = \vec{\alpha}$, $\Upsilon_{\vec{\Lambda}}(\vec{P}_2) = \vec{\beta}$ and there exists three distinct vertices x, y, z so that $x \overset{\vec{P}_1}{\rightsquigarrow} z \overset{\vec{P}_2}{\leftsquigarrow} y$.*

Then a directed tree Y_1 in $\Pi_{\vec{\Lambda}}(\vec{\alpha}, \vec{\beta}) \cap T_{\vec{\Lambda}}(\vec{\alpha})$ and a directed tree Y_2 in $\Pi_{\vec{\Lambda}}(\vec{\alpha}, \vec{\beta}) \cap T_{\vec{\Lambda}}(\vec{\beta})$ with different roots may only intersect at a leaf of both of them.

Proof. Let r_1, r_2 be the roots of Y_1 and Y_2. If Y_1 and Y_2 intersects, there exists $r_1 \overset{\vec{P}_1}{\rightsquigarrow} z \overset{\vec{P}_2}{\leftsquigarrow} y$ and $x' \overset{\vec{P}'_1}{\rightsquigarrow} z' \overset{\vec{P}'_2}{\leftsquigarrow} r_2$, so that $\Upsilon_{\vec{\Lambda}}(\vec{P}_1) = \Upsilon_{\vec{\Lambda}}(\vec{P}'_1) = \vec{\alpha}$, $\Upsilon_{\vec{\Lambda}}(\vec{P}_2) = \Upsilon_{\vec{\Lambda}}(\vec{P}'_2) = \vec{\beta}$, and $\vec{P}'_2$ intersects $\vec{P}_1$ at a vertex v (up to an exchange of Y_1 and Y_2). As $r_1 \neq r_2$, v has in $\vec{P}_2$ an incoming edge e of color β_i for some $1 \leq i \leq b+1$. Let w be the vertex of $\vec{P}_2$ having in $\vec{P}_2$ an incoming edge of color β_i. If $w \neq v$, we are led to a contradiction, according to Lemma 5.4, as w is then (p, q)-reachable from v. Hence $v = w$ and v is the end vertex of $\vec{P}_1$ and $\vec{P}_2$. Thus v is also the end vertex of $\vec{P}'_1$ and $\vec{P}'_2$. It follows that v is a leaf of both Y_1 and Y_2. □

Lemma 5.8. *Let $\vec{G}$ be a directed graph with maximum indegree $\Delta^-(\vec{G})$, let r be an integer and let $\vec{\Lambda}$ be an $(r, r-1)$-augmentation of $\vec{G}$.*

Then $\vec{\Lambda}$ may be extended into an $(r+1, r)$-augmentation $\vec{\Lambda}'$ such that $\Delta^-(\vec{\Lambda}') \leq \Delta^-(\vec{\Lambda}) + ((2\,\Delta^-(\vec{\Lambda}) + 1)\,\Delta^-(\vec{G}))^{2r+1} \nabla_r(G)$.

Proof. Let $\Upsilon_{\vec{\Lambda}}$ be the edge coloring defined in Lemma 5.4.

For two sequences $\vec{\alpha}$ and $\vec{\beta}$ of respective lengths $p \leq r+1$ and $q \leq r$, let $\Pi_{\vec{\Lambda}}(\vec{\alpha}, \vec{\beta})$ be the union of all the $\vec{P}_1 \cup \vec{P}_2$ where $\Upsilon_{\vec{\Lambda}}(\vec{P}_1) = \vec{\alpha}$, $\Upsilon_{\vec{\Lambda}}(\vec{P}_2) = \vec{\beta}$ and there exists three distinct vertices x, y, z so that $x \overset{\vec{P}_1}{\rightsquigarrow} z \overset{\vec{P}_2}{\leftsquigarrow} y$. Also, let $G_{\vec{\alpha}, \vec{\beta}}$ be the graph obtained from G by contracting all the edges of $\Pi_{\vec{\Lambda}}(\vec{\alpha}, \vec{\beta})$ but those colored α_p.

Let x, y be vertices of G so that y is $(r+1,r)$-reachable from x, as witnessed by $x \overset{\vec{P}_1}{\rightsquigarrow} z \overset{\vec{P}_2}{\leftsquigarrow} y$. Let $\vec{\alpha} = \Upsilon_{\vec{\Lambda}}(\vec{P}_1)$ and $\vec{\beta} = \Upsilon_{\vec{\Lambda}}(\vec{P}_2)$. The vertices x, y are the roots of directed trees in $\Pi_{\vec{\Lambda}}(\vec{\alpha}, \vec{\beta}) \cap T_{\vec{\Lambda}}(\vec{\alpha})$ and $\Pi_{\vec{\Lambda}}(\vec{\alpha}, \vec{\beta}) \cap T_{\vec{\Lambda}}(\vec{\beta})$, respectively, hence to two adjacent distinct vertices in $G_{\vec{\alpha},\vec{\beta}}$. Similarly, two distinct vertices of $G_{\vec{\alpha},\vec{\beta}}$ adjacent by an edge of color α_p (where $p = |\vec{\alpha}|$) correspond uniquely to the roots of a tree in $\Pi_{\vec{\Lambda}}(\vec{\alpha}, \vec{\beta}) \cap T_{\vec{\Lambda}}(\vec{\alpha})$ and $\Pi_{\vec{\Lambda}}(\vec{\alpha}, \vec{\beta}) \cap T_{\vec{\Lambda}}(\vec{\beta})$, respectively.

It follows that there exists an $(r+1,r)$-augmentation $\vec{\Lambda}'$ of $\vec{G}$ extending $\vec{\Lambda}$ such that

$$\Delta^-(\vec{\Lambda}') - \Delta^-(\vec{\Lambda}) \leq \sum_{\substack{|\vec{\alpha}| \leq r+1 \\ |\vec{\beta}| \leq r}} \nabla_0(G_{\vec{\alpha},\vec{\beta}}) \leq ((2\,\Delta^-(\vec{\Lambda}) + 1)\,\Delta^-(\vec{G}))^{2r+1} \nabla_r(G)$$

□

Lemma 5.9. *For any integer r, there exists a polynomial Φ_r such that any directed graph $\vec{G}$ has a $(r+1,r)$-augmentation $\vec{\Lambda}$, where $\Delta^-(\vec{\Lambda}) \leq \Phi_r(\Delta^-(\vec{G}), \nabla_r(G))$, where G is the underlying simple graph of $\vec{G}$.*

Proof. This is a direct consequence of Lemma 5.8. □

Proof of Lemma 5.2. Define $P_r(x,y) = \Phi_r(x+y, y)$.

Consider a family $\mathcal{P}$ of balls of G with radius at most r and complexity at most c. We construct a directed graph $\vec{G}$ with underlying undirected graph G. Recall that $\vec{G}$ may have, for each edge of G, one arc in each direction. First we orient the edges of G with indegree $\nabla_0(G)$ (thus obtaining one arc per edge). For each $X \in \mathcal{P}$, let v be the center of $G[X]$. Let Y be a minimum distance tree of $G[X]$ with root v. If $\vec{G}$ does not include the arcs corresponding to an orientation of Y from its root v, we add the missing arcs. We also add if necessary all the arcs going from a leaf of Y to a vertex out of X.

Notice that the vertices of $\vec{G}$ have indegree at most $\nabla_0(G)+c$. Moreover, if r_1, r_2 are the roots of the trees Y_1 and Y_2 corresponding to some parts $X_1, X_2 \in \mathcal{P}$ which are adjacent in $G/\mathcal{P}$ then r_2 is $(r+1,r)$-reachable from r_1 in $\vec{G}$ (by a directed path of length at most r in Y_1, possibly followed by an arc between the parts and a directed path of length at most r in Y_2 in opposite direction). Hence r_1 and r_2 are adjacent in any $(r+1,r)$-augmentation of $\vec{G}$. According to Lemma 5.9, there exists such an augmentation $\vec{\Lambda}$ with $\Delta^-(\vec{\Lambda}) \leq \Phi_r(\nabla_0(G)+c, \nabla_r(G))$. As $G/\mathcal{P}$ is isomorphic to a subgraph of the graph with vertex set $V(G)$ and edge set $\vec{\Lambda}$. As this subgraph has an orientation with indegree at most $\Delta^-(\vec{\Lambda})$ we have, according to Fact 4.1 and Lemma 5.9:

$$\overset{c}{\nabla}_r(G) = \nabla_0(G/\mathcal{P}) \leq \Delta^-(\vec{\Lambda}) \leq \Phi_r(\nabla_0(G)+c, \nabla_r(G)) \leq P_r(c, \nabla_r(G)).$$

□

6. Transitive fraternal augmentation

Definition 6.1. Let $\vec{G}$ be a directed graph. A 1-*transitive fraternal augmentation* of $\vec{G}$ is a directed graph $\vec{H}$ with the same vertex set, including all the arcs of $\vec{G}$ and such that, for any vertices x, y, z,

- if (x, z) and (z, y) are arcs of $\vec{G}$ then (x, y) is an arc of $\vec{H}$ (*transitivity*),
- if (x, z) and (y, z) are arcs of $\vec{G}$ then (x, y) or (y, x) is an arc of $\vec{H}$ (*fraternity*).

A *transitive fraternal augmentation* of a directed graph $\vec{G}$ is a sequence $\vec{G} = \vec{G}_1 \subseteq \vec{G}_2 \subseteq \cdots \subseteq \vec{G}_i \subseteq \vec{G}_{i+1} \subseteq \cdots$, such that $\vec{G}_{i+1}$ is a 1-transitive fraternal augmentation of $\vec{G}_i$ for any $i \geq 1$.

The main key lemma here is that the notion of classes of bounded expansion is stable under 1-fraternal augmentations. More precisely:

Lemma 6.1. *Let $\vec{G}$ be a directed graph and let $\vec{H}$ be a 1-transitive fraternal augmentation of $\vec{G}$. Then*

$$\overset{c}{\nabla}_r(H) \leq \overset{c(\Delta^-(\vec{G})+1)}{\nabla_{2r+1}}(G) \leq P_{2r+1}(c(\Delta^-(\vec{G}) + 1), \nabla_{2r+1}(G)). \tag{6}$$

Proof. Consider a ball family $\mathcal{P} = \{V_1, \ldots, V_p\}$ of H with radius at most r and complexity c. Let $\mathcal{P}' = \{V_1', \ldots, V_p'\}$, where $V_i' = V_i \cup \{z : \exists x \in V_i, (x, z) \in E(\vec{G})\}$. Then for any $x, y \in V_i$ which are adjacent in H, either x and y are adjacent in G or there exists $z \in V_i'$ so that $\{x, z\}$ and $\{y, z\}$ are edges of G. Hence V_i' is a ball of G with radius at most $2r + 1$. Any vertex v of G belongs to a most $c + \Delta^- \vec{G}$ balls of $\mathcal{P}'$ for v belongs to V_i' if and only if either v belongs to V_i (there are at most c such V_i) or there exists an arc from a vertex $z \in V_i$ to v in $\vec{G}$ (there are at most $\Delta^-(\vec{G})$ such z hence at most $c\,\Delta^-(\vec{G})$ such V_i). Hence the complexity of $\mathcal{P}'$ is at most $c(\Delta^-(\vec{G}) + 1)$. As $H/\mathcal{P}$ is isomorphic to a subgraph of $G/\mathcal{P}'$ $|E(H/\mathcal{P})| \leq |E(G/\mathcal{P}')|$ thus $\overset{c}{\nabla}_r(H) = \frac{|E(H/\mathcal{P})|}{|\mathcal{P}|} \leq \frac{|E(G/\mathcal{P}')|}{|\mathcal{P}'|} \leq \overset{c(\Delta^-(\vec{G})+1)}{\nabla_{2r+1}}(G)$. We conclude using Lemma 5.2. ∎

Corollary 6.2. *There exists polynomials Q_i ($i \geq 1$), such that any directed graph $\vec{G}$ has a transitive fraternal augmentation $\vec{G} = \vec{G}_1 \subseteq \vec{G}_2 \subseteq \cdots \subseteq \vec{G}_i \subseteq \cdots$ where*

$$\Delta^-(\vec{G}_i) \leq Q_i(\Delta^-(\vec{G}), \nabla_{2^{i+1}-1}(G)) \tag{7}$$

We also deduce:

Corollary 6.3. *Let $\mathcal{C}$ be a class with bounded expansion. Then there exists a function g such that each graph $G \in \mathcal{C}$ has a transitive fraternal augmentation $\vec{G} = \vec{G}_1 \subseteq \vec{G}_2 \subseteq \cdots \subseteq \vec{G}_i \subseteq \cdots$ where $\Delta^-(\vec{G}_i) \leq g(i)$.*

7. Back to p-centered colorings

The aim of this section is to prove that transitive fraternal augmentations allow us to construct p-centered colorings.

Lemma 7.1. *Let $N(p,t) = 1 + (t-1)(2 + \lceil \log_2 p \rceil)$, let $\vec{G}$ be a directed graph and let $\vec{G} = \vec{G}_1 \subseteq \vec{G}_2 \subseteq \cdots \subseteq \vec{G}_i \subseteq \cdots$ be a transitive fraternal augmentation of $\vec{G}$.*

Then $\vec{G}_{N(p,\mathrm{td}(G))}$ either includes an acyclically oriented clique of size p or a rooted directed tree $\vec{Y}$ such that $G \subseteq \mathrm{clos}(Y)$ and $\mathrm{clos}(\vec{Y}) \subseteq \vec{G}_{N(p,\mathrm{td}(G))}$.

Proof. We fix the integer p and prove the lemma by induction on $t = \mathrm{td}(\vec{G})$. The base case $t = 1$ corresponds to a graph without edges, for which the property is obvious. Assume the lemma has been proved for directed graphs with tree-depth at most t and let $\vec{G}$ be a directed graph with tree-depth $t + 1$. As we may consider each connected component of $\vec{G}$ independently, we may assume that $\vec{G}$ is connected. Then there exists a vertex $s \in V(\vec{G})$ such that the connected components $\vec{H}_1, \dots, \vec{H}_k$ of $G - s$ have tree-depth at most t. As $\vec{H}_i = \vec{G}_1[V(\vec{H}_i)] \subseteq \cdots \subseteq \vec{G}_j[V(\vec{H}_i)] \subseteq \cdots$ is a transitive fraternal augmentation of $\vec{H}_i$ we have, according to the induction hypothesis, that, for each $1 \le i \le k$, there exists in $\vec{H}_i$ either an acyclically oriented clique of size p or a rooted tree $\vec{Y}_i$ rooted at r_i such that $H_i \subseteq \mathrm{clos}(Y_i)$ and $\mathrm{clos}(\vec{Y}_i) \subseteq \vec{G}_{N(p,\mathrm{td}(G))}[V(\vec{H}_i)]$. If the first case occurs for some i, then $\vec{G}$ includes an acyclically oriented clique of size p. Hence assume it does not. As $\vec{G}$ is connected, the vertex s has at least a neighbor x_i in $\vec{H}_i$ (for each $1 \le i \le k$). Let x be any neighbor of s in $\vec{H}_i$. If y is an ancestor of x in $\vec{Y}_i$, (y, x) is an arc of $\vec{G}_{N(p,t)}$ hence s and y are adjacent in $\vec{G}_{N(p,t)+1}$. Moreover, if (x, s) is an arc of $\vec{G}_{N(p,t)}$ then (y, s) is an arc of $\vec{G}_{N(p,t)+1}$. Let D_i be the subset of $V(\vec{H}_i)$ of the vertices x such that (x, s) belongs to $\vec{G}_{N(p,t)}$ and of their ancestors in $\vec{Y}_i$ and let $D = \bigcup_{i=1}^{k} D_i$. Then D includes a clique in $\vec{G}_{N(p,t)+2}$. Thus there exists a directed Hamiltonian path $\vec{P}$ in $\vec{G}_{N(p,t)+2}[D]$.

Let r be the start vertex of $\vec{P}$. Define $\pi\colon V(G) - r \to V(G)$ as follows:

- if $x \in D$, the $\pi(x)$ is the predecessor y of x in $\vec{P}$ (the arc (y, x) belongs to $\vec{G}_{N(p,t)+2}$);
- otherwise, if $x = s$, $\pi(x)$ is the end vertex y of $\vec{P}$ (the arc (y, x) belongs to $\vec{G}_{N(p,t)+1}$);
- otherwise, if $x = r_i$ then $\pi(x) = s$ (the arc (s, r_i) belongs to $\vec{G}_{N(p,t)+2}$);

- otherwise, if the father of $x \in V(\vec{H}_i) \setminus D$ does not belong to D, then $\pi(x)$ is the father of x in $\vec{Y}_i$;
- otherwise, if no descendant of x in $\vec{Y}_i$ has an arc coming from s in $\vec{G}_{N(p,t)+1}$, $\pi(x)$ is the father of x in $\vec{Y}_i$;
- otherwise, $\pi(x) = s$ (the arc (s,x) belongs to $\vec{G}_{N(p,t)+2}$).

It is easily checked that the so defined "father mapping" π actually defines a directed rooted tree $\vec{Y}$ of $\vec{G}_{N(p,t)+2}$ with root r and that $G \subseteq \mathrm{clos}(\vec{Y})$. Moreover, either $\vec{Y}$ has height at least p and $\vec{G}_{N(p,t)+2+\lceil \log_2 p\rceil}$ includes an acyclically oriented clique of size p or

$$\mathrm{clos}(\vec{Y}) \subseteq \vec{G}_{N(p,t)+2+\lceil \log_2 p\rceil}.$$

As $N(p,t+1) = N(p,t) + 2 + \lceil \log_2 p\rceil$, the induction follows. □

Lemma 7.2. *Let p be an integer, let $\vec{G}$ be a directed graph and let $\vec{G} = \vec{G}_1 \subseteq \vec{G}_2 \subseteq \cdots \subseteq \vec{G}_i \subseteq \cdots$ be a transitive fraternal augmentation of $\vec{G}$. Then either $\vec{G}_{N(p,p)}$ includes an acyclically oriented clique of size p or $\mathrm{td}(G) \leq p-1$ and there exists in $\vec{G}_{N(p,p)}$ a rooted directed tree Y so that $G \subseteq \mathrm{clos}(Y)$ and $\mathrm{clos}(\vec{Y}) \subseteq \vec{G}_{N(p,p)}$.*

Proof. If $\mathrm{td}(G) > p$ we may consider a connected subgraph of H of tree-depth p. According to Lemma 7.1, there will exists in $\vec{G}_{N(p,p)}[V(H)]$ an acyclically oriented clique of size p or a rooted directed tree $\vec{Y}$ so that $H \subseteq \mathrm{clos}(Y)$ and $\mathrm{clos}(\vec{Y}) \subseteq \vec{G}_{N(p,p)}[V(H)]$. In the later case, if $\mathrm{td}(G) = p$ then the height of $\vec{Y}$ is at least $\mathrm{td}(H) = p$ and $\mathrm{clos}(\vec{Y})$ includes an acyclically oriented clique of size p. □

Corollary 7.3. *Let $R(p) = 1 + (p-1)(2 + \lceil \log_2 p\rceil) = O(p \log_2 p)$.*

For any graph G, for any transitive fraternal augmentation $\vec{G} = \vec{G}_1 \subseteq \vec{G}_2 \subseteq \cdots \subseteq \vec{G}_i \subseteq \cdots$ of G and for any integer p:

$$\chi_p(G) \leq 2\,\Delta^{-}(\vec{G}_{R(p)}) + 1 \tag{8}$$

And also:

Corollary 7.4. *Let $\mathcal{C}$ be a class of graphs. Assume there exists a function f such that each graph $G \in \mathcal{C}$ has a transitive fraternal augmentation $\vec{G} = \vec{G}_1 \subseteq \vec{G}_2 \subseteq \cdots \subseteq \vec{G}_i \subseteq \cdots$ such that $\Delta^{-}(\vec{G}_i) \leq f(i)$. Then, for any integer p there exists an integer $X(p)$ such that every $G \in \mathcal{C}$ has a p-centered coloring using at most $X(p)$ colors.*

8. Conclusion

All previous results are gathered in the following equivalence:

Theorem 8.1. *Let $\mathcal{C}$ be a class of graphs. The following conditions are equivalent:*

- *$\mathcal{C}$ has low tree-width colorings,*

- *$\mathcal{C}$ has low tree-depth colorings,*
- *for any integer p, $\{\chi_p(G) : G \in \mathcal{C}\}$ is bounded,*
- *for any integer p, there exists an integer $X(p)$ such that any graph $G \in \mathcal{C}$ has a p-centered colorings using at most $X(p)$ colors,*
- *$\mathcal{C}$ has bounded expansion,*
- *for any integer c, the class $\mathcal{C} \bullet K_c = \{G \bullet K_c : G \in \mathcal{C}\}$ has bounded expansion,*
- *for any integer k, the class $\mathcal{C}'$ of the 1-transitive fraternal augmentations of directed graphs $\vec{G}$ with $\Delta^-(\vec{G}) \leq k$ and $G \in \mathcal{C}$ form a class with bounded expansion,*
- *there exists a function F such that any orientation $\vec{G}$ of a graph $G \in \mathcal{C}$ has a transitive fraternal augmentation $\vec{G} = \vec{G}_1 \subseteq \vec{G}_2 \subseteq \cdots \subseteq \vec{G}_i \subseteq \cdots$ where $\Delta^-(\vec{G}_i) \leq F(\Delta^-(\vec{G}), i)$,*
- *there exists a function f such that any graph $G \in \mathcal{C}$ has a transitive fraternal augmentation $\vec{G} = \vec{G}_1 \subseteq \vec{G}_2 \subseteq \cdots \subseteq \vec{G}_i \subseteq \cdots$ where $\Delta^-(\vec{G}_i) \leq f(i)$.*

Now that we know that bounded expansion is the more general condition for low tree-depth coloring to exist and that low tree-width coloring (although seemingly weaker) does not relax this condition, we may wonder what may be the weakest coloring condition equivalent to low tree-width coloring. It appears that this is a direct consequence of Lemma 3.6 and Theorem 8.1:

Corollary 8.2. *Let $\mathcal{C}$ be a class of graphs. Then $\mathcal{C}$ has bounded expansion if, and only if, for every integer $p \geq 1$ there exists a class of graphs $\mathcal{C}_p$ and an integer $C(p)$ such that:*

- *$\mathcal{C}_p$ has bounded expansion,*
- *any graph $G \in \mathcal{C}$ has a $C(p)$ vertex-coloring such that any p parts induce a graph in $\mathcal{C}_p$.*

References

[1] M.O. Albertson, G.G. Chappell, H.A. Kierstead, A. Kündgen, and R. Ramamurthi, *Coloring with no 2-colored P_4's*, Electronic Journal of Combinatorics **11** (2004), no. 1, R26.

[2] N. Alon, B. Mohar, and D.P. Sanders, *On acyclic colorings of graphs on surfaces*, Israel J. Math. (1994), no. 94, 273–283.

[3] B. Courcelle, *Graph rewriting: an algebraic and logic approach*, Handbook of Theoretical Computer Science (J. van Leeuwen, ed.), vol. 2, Elsevier, Amsterdam, 1990, pp. 142–193.

[4] ———, *The monadic second-order logic of graphs I: recognizable sets of finite graphs*, Inform. Comput. **85** (1990), 12–75.

[5] J.S. Deogun, T. Kloks, D. Kratsch, and H. Muller, *On vertex ranking for permutation and other graphs*, Proceedings if the 11th Annual Symposium on Theoretical Aspects of Computer Science (Springer, ed.), Lecture Notes in Computer Science, vol. 775, 1994, pp. 747–758.

[6] M. DeVos, G. Ding, B. Oporowski, D.P. Sanders, B. Reed, P.D. Seymour, and D. Vertigan, *Exluding any graph as a minor allows a low tree-width 2-coloring*, Journal of Combinatorial Theory, Series B **91** (2004), 25–41.
[7] F. Gavril and J. Urrutia, *An algorithm for fraternal orientation of graphs*, Inform. Process. Lett. (1992), no. 41, 271–274.
[8] R. Halin, *S-functions for graphs*, J. Geom. **8** (1976), 171–176.
[9] A. Kostochka, *On the minimum of the hadwiger number for graphs with given average degree*, Metody Diskret. Analiz. (1982), no. 38, 37–58, in Russian, English translation: *AMS Translations (2)*, 132(1986), 15-32.
[10] W. Mader, *Homomorphiesätze für graphen*, Math. Ann. (1968), no. 178, 154–168.
[11] J. Nešetřil and P. Ossona de Mendez, *Grad and classes with bounded expansion IV. structural extentions*, in preparation.
[12] ______, *Colorings and homomorphisms of minor closed classes*, The Goodman-Pollack Festschrift (B. Aronov, S. Basu, J. Pach, and M. Sharir, eds.), Algorithms and Combinatorics, vol. 25, Discrete & Computational Geometry, 2003, pp. 651–664.
[13] ______, *Grad and classes with bounded expansion II. algorithmic aspects*, Tech. Report 2005-740, KAM-DIMATIA Series, 2005.
[14] ______, *Grad and classes with bounded expansion III. restricted dualities*, Tech. Report 2005-741, KAM-DIMATIA Series, 2005.
[15] ______, *Tree depth, subgraph coloring and homomorphism bounds*, European Journal of Combinatorics (2005), (in press).
[16] N. Robertson and P.D. Seymour, *Graph minors. I. Excluding a forest*, J. Combin. Theory Ser. B **35** (1983), 39–61.
[17] ______, *Graph minors. XVI. Excluding a non-planar graph*, Journal of Combinatorial Theory, Series B **89** (2003), no. 1, 43–76.
[18] P. Schaffer, *Optimal node ranking of trees in linear time*, Information Processing Letters (1989/90), no. 33, 91–96.
[19] D.J. Skrien, *A relationship between triangulated graphs, comparability graphs, proper interval graphs, proper circular arc graphs, and nested interval graphs*, J. Graph Theory (1982), no. 6, 309–316.
[20] R. Thomas, *Problem session of the Third Slovene Conference on Graph Theory, Bled, Slovenia*, 1995.
[21] A. Thomason, *An extremal function for contractions of graphs*, Math. Proc. Cambridge Philos. Soc., no. 95, 1984, pp. 261–265.
[22] ______, *The extremal function for complete minors*, J. Comb. Th. B **81** (2001), no. 2, 318–338.
[23] K. Wagner, *Über eine Eigenschaft der Ebenen Komplexe*, Math. Ann. **114** (1937), 570–590.

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