Homomorphisms of triangle-free graphs without a K_5-minor

Reza Naserasr [a], Yared Nigussie [b] and Riste Škrekovski [c]

[a] *Department of Combinatorics and Optimization, Faculty of Mathematics, University of Waterloo, Waterloo, Ontario, N2L 3G1, Canada. naserasr@math.uwaterloo.ca*

[b] *Department of Applied Mathematics and Institute for Theoretical Computer Science(ITI) Charles University Malostranské nám.25, 11800 Praha 1 Czech Republic. yared@kam.mff.cuni.cz.*

[c] *Department of Mathematics, University of Ljubljana, Jadranska 19, 1111 Ljubljana, Slovenia. skreko@kam.mff.cuni.cz.*

Abstract

In the course of extending Grötzsch's theorem, we prove that every triangle-free graph without a K_5-minor is 3-colorable. It has been recently proved that every triangle-free planar graph admits a homomorphism to the Clebsch graph. We also extend this result to the class of triangle-free graphs without a K_5-minor. This is related to some conjectures which generalize the Four-Color Theorem. While we show that our results cannot be extended directly, we conjecture that every K_6-minor-free graph of girth at least 5 is 3-colorable.

1 Introduction

Graphs in this paper are finite and loopless, they might have multiple edges in which case we rather use the term multigraph. We will use the standard notations of graph theory mainly following Diestel [2] and Hell and Nešetřil [8]. A graph H is said to be a minor of G if it can be constructed from G by deleting vertices, deleting edges and contracting edges. Given a finite set $\mathcal{M}$ of graphs we define $Forb_m(\mathcal{M})$ to be the set of all graphs which have no minor from $\mathcal{M}$. If $\mathcal{M} = \{H\}$, then we simply write $Forb_m(H)$.

Let G, G' be two graphs. We say there is a *homomorphism* of G to G', and write $G \preceq G'$, if there exists a map $f : V(G) \to V(G')$ such that $uv \in E(G)$ implies $f(u)f(v) \in E(G')$. This binary relation is a quasi order on the class

Preprint submitted to ... *1 September*

of all graphs and with this order naturally comes the concepts of bounds, maximums and cuts. To be precise, given a class $\mathcal{C}$ of graphs and a subset A of $\mathcal{C}$ we say A is a *cut* of $\mathcal{C}$ if for every $G \in \mathcal{C}$ there is an H in A such that either $G \preceq H$ or $H \preceq G$. A 1-*cut* of $\mathcal{C}$ is a graph $H \in \mathcal{C}$ such that $A = \{H\}$ is a cut of $\mathcal{C}$. A graph B is a *bound* for $\mathcal{C}$ if for every graph $G \in \mathcal{C}$ we have $G \preceq B$. A bound M for $\mathcal{C}$ is called a *maximum* if it is also an element of $\mathcal{C}$.

The study of cuts and bounds in the homomorphism order of graphs is initiated by Nešetřil and Ossona de Mendez [12], (see also Chapter 3 of [8]). Using this terminology some of the most famous theorems and conjectures in the theory of coloring of graphs can be restated quite nicely. For example, consider the following classical theorem of Grötzsch:

Theorem 1 *Every triangle-free planar graph is 3-colorable.*

Let $\mathcal{P}$ be the class of all planar graphs (i.e., $Forb_m(\{K_5, K_{3,3}\})$). Then Theorem 1 is equivalent to stating that K_3 is a 1-cut of $\mathcal{P}$. Similarly, the Four-Color Theorem is claiming that K_4 is a maximum of $\mathcal{P}$. A less obvious result is a restatement of Hadwiger's conjecture, which claims that every k-chromatic graph contains a K_k-minor. It is shown in [10,12] that this is equivalent to:

Conjecture 2 *Every minor closed family of graphs has a maximum with respect to the homomorphism order.*

Another nontrivial example is a reformulation of a conjecture of P. Seymour. In a generalization of an equivalent form of the Four-Color Theorem, introduced by Tait [17], Seymour [16] conjectured that:

Conjecture 3 *Every planar k-graph is k-edge-colorable.*

A *k-graph* is a k-regular multigraph which does not have any odd edge-cut of size smaller than k. An *odd edge-cut* is a partition (X, Y) of the vertices of G such that $|X|$ or $|Y|$ is odd. The *size* of an edge-cut is the number of edges with one end in X and the other end in Y.

For odd values of k, a reformulation of Conjecture 3 is given by Naserasr [9]. Let H_{2k+1} be a connected component of the Cayley graph $C(\mathbb{Z}_2^{2k+1}, S_{2k+1})$ where S_{2k+1} is the set of $2k+1$ vectors with exactly two consecutive 1's in a cyclic order. Then the following is shown to be equivalent to Conjecture 3 for the corresponding value of $2k+1$.

Conjecture 4 *Every planar graph of odd-girth at least $2k+1$ admits a homomorphism to H_{2k+1}.*

As it is shown in [9], the equivalence of Conjecture 3 and Conjecture 4 together with a proof of Conjecture 3 by Guenin [5] for $k = 5$ implies the following theorem. Note that H_5 is a triangle-free graph known as the Clebsch graph and also as the Greenwood-Gleason graph.

Theorem 5 *Every triangle-free planar graph admits a homomorphism to H_5.*

A generalization of Conjectures 3 and 4 has been recently introduced by Guenin [6]. While Guenin's conjecture is general, stated for both even and odd values of k and in terms of edge-colorings and homomorphisms both, for simplicity we only state the homomorphism version of his conjecture and only for odd values. For a definition of odd-minor we refer to [6]. However, we would like to mention that the class of graphs with no odd-K_5-minor strictly includes the class of K_5-minor-free graphs.

Conjecture 6 *Every graph of odd-girth at least $2k+1$ and with no odd-K_5-minor admits a homomorphism to H_{2k+1}.*

It has also been recently conjectured in [11] that:

Conjecture 7 *For $k \geq 5$ every 1-cut of the class $Forb_m(K_k)$ is a complete graph.*

In the last section of this paper we construct a graph $H \in Forb_m(K_k)$ homomorphically incomparable to K_j for each $3 \leq j \leq k-2$ and $k \geq 6$. This together with a validity of Conjectures 2 and 7 imply that K_1, K_2 and K_{k-1} are the only 1-cuts of $Forb_m(K_k)$ for $k \geq 6$.

Section 2 is about the extensions of Grötzsch's theorem. In Section 3, we extend Theorem 5 to the class $Forb_m(K_5)$. The last section is devoted to examples and open problems.

2 Homomorphism to K_3

In this section we first introduce some extensions of the Grötzch's theorem within planar graphs. Then, using these extensions, we generalize Grötzsch's theorem to the class of triangle-free graphs without a K_5-minor.

2.1 Some strengthening of Grotzsch's theorem within planar graphs

The following strengthening of Grötzsch's theorem was first introduced by Grünbaum [4] in 1973. The proof published by Grünbam turned out to be incomplete. A correct proof was given by Aksionov [1] a year later.

Theorem 8 *Every planar graph with at most three triangles is 3-colorable.*

The assumption that there are at most three triangles cannot be weakened because K_4 and H_7 (obtained by the Hajós sum of two copies of K_4) are 4-chromatic and each contains four triangles.

We now give an easy strengthening of the above result, allowing our plane graph–a planar graph with a fixed planar drawing– to have more triangles but arranged in a specific way. For this, we need the following definitions: Let G be a plane graph and let C be a cycle of G. Then, the *interior of* C, denoted by $\mathrm{Int}(C)$, is the subgraph of G which is induced by the vertices inside or on C. The $\mathrm{Out}(C)$ is defined analogously. Let $\mathcal{C}_3(G)$ be the set of all triangles of G. Let $C_1, C_2 \in \mathcal{C}_3(G)$. We say that C_1 is *smaller* than C_2 (or C_2 is *bigger* than C_1) and write $C_1 \leq C_2$, if C_1 is a subgraph of $\mathrm{Int}(C_2)$. If $C_1 \leq C_2$ and $C_2 \not\leq C_1$, then we write $C_1 < C_2$ and if $C_1 \not\leq C_2$ and $C_2 \not\leq C_1$, then we say that these two triangles are *incomparable*. Notice that $(\mathcal{C}_3(G), \leq)$ is a partial order.

Finally, we say that a planar graph G has a *nice* structure of triangles if the following two conditions are satisfied for some planar drawing of G.

- (i) G has at most three pairwise incomparable triangles, and
- (ii) for any three pairwise incomparable triangles of G, there is no other triangle of G which is bigger than all of these three triangles.

Theorem 9 *Every planar graph with a nice structure of triangles is 3-colorable.*

Proof. Let G be a planar graph embedded in the plane with a nice structure of triangles and let $\mathcal{C}_3 = \mathcal{C}_3(G)$. The proof is by induction on the number of triangles of G. If this number is at most three, then we apply Theorem 8. So we may assume that $|\mathcal{C}_3| \geq 4$.

We claim that G has a triangle C such that each of $\mathrm{Int}(C)$ and $\mathrm{Out}(C)$ contains a triangle of G distinct from C. This is easy to see, because by (i) there must be two triangles X and X' with $X > X'$. If there is a triangle bigger than X or incomparable to X, then we let $C = X$ and we are done. Otherwise, X is bigger than at least three other triangles, by (ii) there are triangles X_1 and X_2 in $\mathrm{Int}(X)$ with $X_1 \geq X_2$, now $C = X_1$ has the property.

By the choice of C, each one of $\mathrm{Int}(C)$ and $\mathrm{Out}(C)$ has less triangles than G. Moreover, each one of the plane graphs induced by $\mathrm{Int}(C)$ and $\mathrm{Out}(C)$ has a nice structure of triangles. By the induction hypothesis, we have a 3-coloring for each of the two graphs. After a permutation of the colors, if needed, these two colorings agree on C, thereby producing a 3-coloring of G. □

In the next proposition we show that almost every 3-coloring of any three vertices on a same face of a triangle-free plane graph is extendible to a 3-coloring of the graph. The only exception is when we have three pairwise non-adjacent vertices colored with three different colors. Note that in this case they may have a common neighbor, in which case the coloring is obviously not extendible.

Proposition 10 *Let G be a plane triangle-free graph and let $A = \{x, y, z\}$ be a set of three vertices on a same face of G. Let $c : A \to \{1, 2, 3\}$ be a proper coloring such that if A is an independent set, then c does not color vertices of A all differently. Then, c can be extended to a proper 3-coloring of G.*

Proof. We first connect any two vertices $a, b \in A$ with $c(a) \neq c(b)$ in a way that the new graph is also a plane graph. Next we identify any two vertices with $a, b \in A$ with $c(a) = c(b)$ and then remove any possible multiple edge. Let the resulting graph be G' and let A' be the set of vertices of G' that correspond to vertices in A. Note that G' is also a plane graph and also that $1 \leq |A'| \leq 3$ and any two vertices of A' are adjacent and colored differently.

We will prove that G' has a nice structure of triangles, therefore, proving that G' is 3-colorable by Theorem 9. A 3-coloring of G' then can easily be lifted to a 3-coloring of G. Let $H = G[A]$ be the subgraph of G induced by A and $H' = G'[A']$. In order to show that G' has a nice structure of triangles we will consider several cases regarding the number of edges of H.

(i) First suppose H has two edges, so it is a 2-path xyz. Note that $c(x) \neq c(y)$, $c(y) \neq c(z)$ and that $xz \notin E(G)$. Now, if $c(x) \neq c(z)$, then $G' = G + xz$. So every triangle of G' contains xz, hence G' has at most two pairwise incomparable triangles (one on each side of xz). If $c(x) = c(z)$, then $G' = G/xz$ and again any triangle of G' contains the vertex of the identification. Since each such a triangle corresponds to a 3-path in G, we may have again at most two pairwise incomparable triangles in G'. Thus, G' has a nice structure of triangles.

(ii) Suppose now that H has only one edge, say xy. Then, $c(x) \neq c(y)$. If $c(z) \neq c(x)$ and $c(z) \neq c(y)$, then $G' = G + xz + yz$. Notice that H' is a triangle of G' and every other triangle of G' contains precisely one of the edges xz and yz. Thus, G' may contain at most three pairwise incomparable triangles. Note that there is no other triangle bigger than all of them and, therefore, G' has a nice structure of triangles. Now, without loss of generality, we may assume that $c(y) = c(z)$. Then $c(x) \neq c(z)$. In this case, $G' = G + xz/yz = G/yz$ and the argument of the case (i) can be applied to show that G' has a nice structure of triangles.

(iii) Finally suppose that H has no edge. If $c(x) = c(y) = c(z)$, then G' is obtained from G by identifying all these three vertices. Thus, each triangle of G' contains the identification vertex and it corresponds to a 3-path joining two of the three vertices x, y and z in G. Therefore, by the planarity, there are at most three incomparable triangles in this graph and if there is any set of three incomparable triangles, then there is no other triangle bigger than all of them. Thus, also in this case G' has a nice structure of triangles.

Note that by the assumptions not all of x, y, z have distinct colors, so for the last case, we may assume that $c(y) = c(z) \neq c(x)$. Notice that in this case we add the edge xy and identify y, z into a vertex, say w. Thus, each triangle of G' contains the vertex w. Again it is easy to check that G' has a nice structure of triangles. □

The following is a special case of Proposition 10 but because of its application in extending the Götzsch theorem to the class $Forb_m(K_5)$ we state it independently.

Corollary 11 *Let G be a plane triangle-free graph and let x, y be two vertices on a same face f of G. Then, every proper coloring of x, y can be extended to a 3-coloring of G.*

2.2 Extension of Grötzsch's theorem to K_5-minor-free graphs

We will use the following fundamental theorem of Wagner [21] (see also [2]).

Theorem 12 *Let G be an edge-maximal K_5-minor-free graph on at least 4 vertices. Then, G can be constructed recursively, by pasting along K_2's and K_3's, from plane triangulations and copies of the Wagner graph.*

The *Wagner graph*, V_8, is constructed from an 8-cycle (we call it the *outer* cycle) by connecting the antipodal vertices (these edges will be called the *diagonal* edges). Note that the Wagner graph is triangle-free and 3-colorable (because it is cubic). The graph is depicted in Figure 1 in two different ways. Our definition of this graph is based on the representation on the right hand side of this figure. To prove the main theorem of this section we will need the following easy lemma about Wagner graph:

Lemma 13 *If e is an edge of V_8 then $V_8 - e$ admits a 3-coloring such that the end vertices of e receive a same color.*

In order to make our arguments easy to follow we introduce the following notations: Let $\mathcal{T} = T_1, T_2, \ldots, T_r$ be a sequence of graphs where each T_i is either a plane triangulation or a copy of V_8. We construct another sequence $\mathcal{G} = \{G_i\}_{i=1}^{i=r}$ of graphs as follows: $G_1 = T_1$, G_i is obtained from G_{i-1} and T_i by pasting T_i to G_{i-1} along a K_2 or a K_3. Given an edge-maximal K_5-minor-free

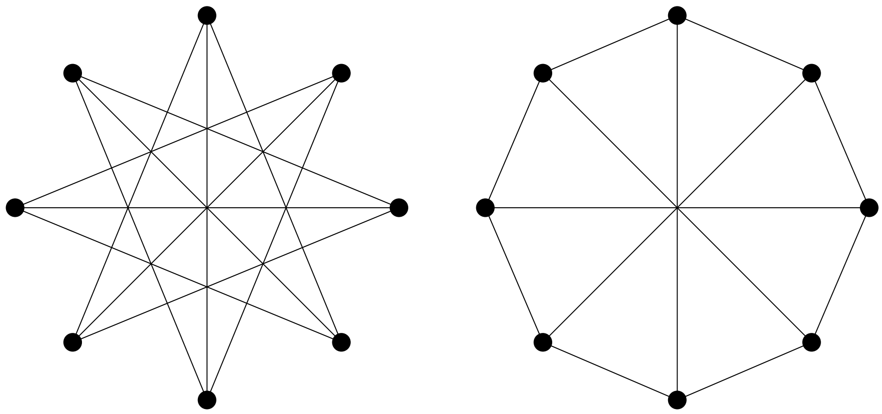

Fig. 1. Two different representations of the Wagner graph

graph G, the sequence $\mathcal{T}$ is said to be a *Wagner sequence* of a graph G, if $G = G_r$ for some sequence $\mathcal{G}$ constructed from $\mathcal{T}$.

Note that each edge-maximal K_5-minor-free graph has a Wagner sequence by Theorem 12. A member of a Wagner sequence is called a *brick*. A Wagner sequence is called a *good Wagner sequence* if every triangle xyz that is in at least two bricks is a face of each one of the bricks it belongs to. Note that for every Wagner sequence there exists a good Wagner sequence. That is because, if a brick T_i is pasted along a triangle C to G_{i-1}, where C is a separating triangle of T_i (i.e., not a face), then we can split T_i into two new bricks $\mathrm{Int}(C)$ and $\mathrm{Out}(C)$. It is also important to note that for each i, $1 \le i \le r$, the subsequence $T_1, T_2, \ldots, T_i$ is a (good) Wagner sequence of the subgraph G_i of G.

Next we extend these notations to any K_5-minor-free graph. Given K_5-minor-free graphs G and G' with $V(G) = V(G')$ and $G \subseteq G'$ any Wagner sequence of G' is also a Wagner sequence of G. A good Wagner sequence of G is defined analogously. Finally we define the *Wagner number* of a K_5-minor-free graph G to be the length of a shortest good Wagner sequence of G and we denote it by $\mathrm{wg}(G)$.

Theorem 14 *Every K_5-minor-free triangle-free graph is 3-colorable.*

Proof. The theorem is true for triangle-free graphs with no K_5-minor and $\mathrm{wg}(G) = 1$ by Grötzsch's theorem and the fact that Wagner graph is 3-colorable. Suppose G is a minimal counterexample with respect to the Wagner number and assume it has Wagner number $r \ge 2$. Let $\hat{G}$ be an edge-maximal K_5-minor-free extension of G from which the good Wagner sequence of size r for G is produced and let $T_1, T_2, \ldots, T_r$ be the corresponding good Wagner sequence. There are two type of edges in $\hat{G}$: The ones in $E(G)$, which we call them *thick* edges. The ones not in $E(G)$, which we call them *thin* edges.

Our aim is to provide a coloring c_i for each T_i (inductively) so that c_i is an extension of the already colored vertices of T_i and that it is proper with respect to thick edges. Toward this we prove a bit stronger statement. We require, moreover, that if xyz is a thin triangle in more than one brick, then its vertices are assigned at most two different colors all together. Note that we may assume G is the smallest counterexample to this stronger statement with respect to Wagner number, also that this additional condition is trivially true for $r = 1$.

By our choice of G, the subgraph G_{r-1} of G has a 3-coloring that satisfies our additional assumption as well (note that $T_1, T_2, \ldots, T_{r-1}$ is a good Wagner sequence of G_{r-1}). Now, if T_r is pasted to G_{r-1} along a K_2, then, using Corollary 11, the 3-colorability of V_8 and Lemma 13, we are done. If T_r is pasted to G_{r-1} along a triangle which is not a thin triangle then simply apply Proposition 10. So we may assume T_r is pasted to G_{r-1} along a thin triangle xyz. If this triangle is in at least two other T_i's, $1 \leq i \leq r-1$, then we are done-again using Proposition 10–because xyz has received at most two colors by our additional assumption.

Finally, let xyz be a thin triangle that is only in T_r and T_j for some j, $1 \leq j \leq r-1$. Insert a new vertex t to G_{r-1} and join it to x, y and z. Let G'_{r-1} be the new graph. Let also T'_{r-1} be a triangulation obtained from T_{r-1} by inserting t inside the xyz-face and joining it to x, y and z using thick edges. Note that $T_1, T_2, \ldots, T_{r-2}, T'_{r-1}$ is a good Wagner sequence of G_{r-1}. So, by the choice of r, G'_{r-1} admits a 3-coloring satisfying all our requirements. In this coloring x, y, z must receive at most two different colors. Therefore, if we take the induced coloring on G_{r-1}, then this coloring, by Proposition 10, will be extendible to $G_{r-1} + T_r = G$. This extended coloring satisfies our additional assumption as well. □

3 Homomorphism to the Clebsch graph

Let $k \geq 1$ and let $S_k = \{s_1, s_2, \ldots, s_k\}$ be a set of k vectors in $\mathbb{Z}_2^k$ such that $\sum_{i=1}^{k} s_i = 0$ (in $\mathbb{Z}_2^k$) and no proper subset of S sums to 0. For example, one can take the set of all vectors with two consecutive 1's in a cyclic order. Let Γ_k be the subgroup of $\mathbb{Z}_2^k$ generated by S_k. It is an elementary group theory fact that Γ_k is isomorphic to $\mathbb{Z}_2^{k-1}$ for any choice of S_k. For the example of S_k we chose, elements of Γ_k are those k-vectors in $\mathbb{Z}_2^k$ that have an even number of 1's in their coordinates.

We now define H_k to be the Cayley graph $C(\Gamma_k, S_k)$. Vertices of this Cayley graph are the elements of Γ_k and two vertices are adjacent if and only if their difference is in S_k. We will show below that H_k is independent from the choice of S_k. Note that, since S_k is a generator of Γ_k, H_k is a connected graph.

For $k = 1, 2, 3$ and 4 the graph H_k is isomorphic to K_1, K_2, K_4 and $K_{4,4}$, respectively. For $k = 5$, H_5 is isomorphic to the Clebsch graph. H_5 contains two disjoint copies of the Wagner graph, this can be observed easily in a representation as in Figure 2.

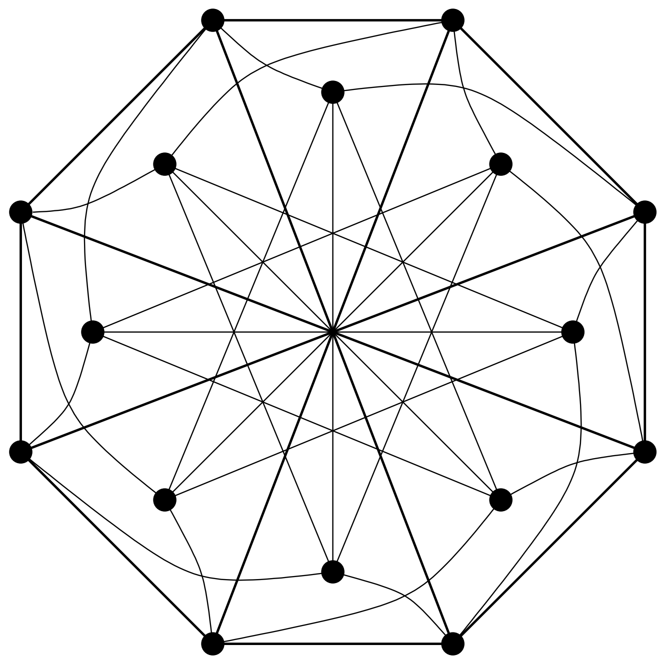

Fig. 2. Clebsch graph

It is easy to check that H_{2k} is bipartite, H_{2k+1} has odd-girth $2k + 1$ and that $\chi'(H_k) = k$, see [9]. A *canonical edge-coloring* φ of H_k is a k-edge-coloring of H_k, using the elements of S_k, obtained as follows: Each element s of S_k induces a perfect matching because it matches each vertex x of H_k to a unique vertex $x + s$. All together these perfect matchings form a k-edge-coloring of H_k.

Let $S_k^* = \Gamma_k \backslash S_k \cup \{0\}$. Note that every element of Γ_k can be represented as the sum of the elements of S_k in two different ways. When working with odd values of k we can make this representation unique by only considering the shorter term. For example $S_5^* = \{s_i + s_j | s_i, s_j \in S_5 \text{ and } i \neq j\}$ (note also that $S_5^* = \{s_i + s_j + s_l | 1 \leq i < j < l \leq 5\}$). The complement $\overline{H}_k$ of H_k is the Cayley graph $C(\Gamma_k, S_k^*)$. The canonical edge-coloring of $\overline{H}_k$ is defined analogously.

An important characteristic of the canonical edge-coloring φ of H_k is that it satisfies the following property:

Property $\mathbf{P_c}$. *For every given cycle C of G the following holds* $\sum_{e \in C} \varphi(e) = 0$.

In fact this property allows us to reconstruct the labeling of the vertices (up to an automorphism of H_k) from a given canonical edge-coloring. To do this, we label an arbitrary vertex x with 0, then for any other vertex y we choose an xy path P and label y by $\sum_{e\in P}\varphi(e)$. This proves that any permutation σ of S_k induces an automorphism of H_k, note that this induced automorphism is not unique, in fact there are 2^{k-1} such automorphisms as we have that many choices for a vertex to be labeled 0. Moreover, since an automorphism of any graph is also an automorphism of its complement, a permutation σ of S_k also induces an automorphism of $\overline{H}_k$.

The property P_c divides the set of cycles of H_k into two groups with respect to the canonical edge-coloring of H_k. Given a cycle C, either every color appears an even number of times (which might be zero) or every color appears an odd number of times (in particular they all must appear). Now, if we change our choice of the difference set from S_k to $\hat{S}_k$, then the bijection from S_k to $\hat{S}_k$ will not change the parity and, therefore, it will not affect the property P_c. The relabeling, using this new canonical edge-coloring, is an isomorphism between (Γ_k, S_k) and $(\hat{\Gamma}_k, \hat{S}_k)$. This shows that H_k is independent form the choice of S_k. It also follows from this argument that H_k is edge-transitive.

The following conjecture is the focus of this section:

Conjecture 15 *The class of K_5-minor-free graphs of odd-girth at least $2k+1$ is bounded by H_{2k+1}.*

This conjecture is closely related to some other conjectures. In particular it is a generalization of Conjecture 4 and it is a relaxation of Conjecture 6, which in turn is also a relaxation of the Cycling conjecture, see [6,15]. The first case of the Conjecture 15 (i.e., $k = 1$) is shown by Wagner [21] to be equivalent to the Four-Color Theorem. In this section, using Theorem 5, we verify the conjecture for $k = 2$ (so the Four-Color Theorem is used in our proof).

Theorem 16 *Every triangle-free graph in $Forb_m(K_5)$ admits a homomorphism to H_5.*

The proof is similar to that of Theorem 14. We will use the Wagner sequence but we first need some definitions and preliminary lemmas together with a strengthening of Theorem 5 for planar graphs.

A *mixed graph* is a pair (G, G') of graphs such that G' is a subgraph of G and has the same set of vertices as G. We can look at a mixed graph as one graph G with two different types of edges: Those in $E(G)$, we will call them thick edges. Those in $E(G)\backslash E(G')$, we will call them thin edges. A homomorphism of a mixed graph (G, G') to (H, H') is a mapping of $V(G)$ to $V(H)$, which not only preserves the adjacency but also preserves the thickness of the edges as well. An isomorphism (and an automorphism) of mixed graphs is defined analogously. For more on homomorphism of mixed graphs we refer to [14]. The canonical edge-coloring of the mixed graph (K_{16}, H_5) is the combined canonical edge-colorings of H_5 and $\overline{H}_5$.

Our first lemma of this section is about the transitivity of the Clebsch graph. Informally speaking we prove that (K_{16}, H_5) is triangle transitive and that any isomorphism between two mixed triangles of this mixed graph extends to an automorphism of the whole graph.

Lemma 17 *Let A and B be two subsets of the vertices of the mixed graph (K_{16}, H_5). Suppose $|A| = |B| \leq 3$ and that the mixed subgraph (X_A, X'_A) induced by A is isomorphic to the mixed subgraph (X_B, X'_B) induced by B. Let ϕ be such an isomorphism. Then, there is an automorphism θ of the mixed graph (K_{16}, H_5) such that $\theta|_A = \phi$.*

Proof. The case $|A| = |B| = 1$ follows from the fact that every Cayley graph, in particular H_5, is vertex transitive. For the case $|A| = |B| = 2$ note that the Clebsch graph and its complement both (and, therefore, also (K_{16}, H_5)) are edge transitive. Let θ be an automorphism of (K_{16}, H_5) which maps the edge induced by A to the edge induced by B. If ϕ and θ agree on A, then we are done. Otherwise, let s be the color of the edge induced by B in the canonical edge-coloring of (K_{16}, H_5). Note that θ_s, defined by $\theta_s(x) = x + s$, is an automorphism of (K_{16}, H_5) which switches the two vertices of B. Now $\theta_s \circ \theta$ is an automorphism of (K_{16}, H_5) that agrees with ϕ on A.

For the last case we have $|A| = |B| = 3$. Note that X_A and X_B have the same number of thick edges, moreover this number cannot be three as H_5 is a triangle-free graph. If X_A has two thick edges corresponding to s_i and s_j, then the thin edge corresponds to $s_i + s_j$. If X_A has only one thick edge corresponding to s_i, then the two thin edges correspond to $s_j + s_k$ and $s_r + s_t$ with all the five different elements of S_5 being used here. Finally, if there is no thick edge in X_A, then they are colored by $s_i + s_j$, $s_i + s_k$ and $s_j + s_k$.

It is easy to check that in either one of these cases there exists a permutation σ of S_5 which changes the color of the edge xy of X_A to the color of the edge $\phi(x)\phi(y)$ of X_B. Let θ_σ be an induced automorphism of (K_{16}, H_5) by σ. Let θ' be the automorphism of (K_{16}, H_5) defined by $\theta'(t) = t + \phi(x_0) - \theta_\sigma(x_0)$ where x_0 is a fixed element of A. It is now easy to check that $\theta' \circ \theta_\sigma$ is an automorphism of H_5 which agrees with ϕ on A. $\square$

The following result is a generalization of Theorem 5.

Theorem 18 *Let (G, G') be a mixed graph such that G is planar and G' is triangle-free. Then, there is a homomorphism of (G, G') to (K_{16}, H_5).*

Proof. Let (G, G') be a mixed graph such that G is planar and G' is triangle-free. For every thin edge uv first we add a new copy (of uv) so that there are two (multiple) edges uv and then we subdivide one of them once the other one twice. In this way, the edge uv is replaced by a 5-cycle in which u and v are nonadjacent vertices. Let G'' be the simple graph obtained from G in this way.

Note that every vertex of G is also a vertex of G'' and that an edge of G is an edge of G'' if and only if it is a thick edge. It follows from construction of G'' that it is a triangle-free planar graph, therefore, by Theorem 5, it maps to H_5. The restriction of this homomorphism to the vertices of G is a homomorphism of (G, G') to (K_{16}, H_5). To see this, note that thick edges of G are also edges of G'' and, therefore, are mapped to the thick edges of (K_{16}, H_5). For a thin edge uv of G, note that G'' contains a 5-cycle having u and v as non adjacent vertices. Since H_5 is a triangle-free graph, image of every 5-cycle must be a 5-cycle and, therefore, u and v are mapped to a pair of nonadjacent vertices of H_5, that is a thin edge in (K_{16}, H_5). □

To prove the main theorem of this section we require another lemma which is about homomorphisms of (mixed) Wagner graph to the Clebsch graph. To prove this lemma we will use the following interpretation of homomorphisms to H_5 and (K_{16}, H_5).

Let f be a homomorphism of a given graph G to H_5. Then f induces a (not necessarily proper) edge-coloring of G using the canonical edge-coloring of H_5. We denote this edge-coloring by f'. It is not hard to check that f' satisfies the property P_c. Using this property once again we see that f' also uniquely determines f (up to an automorphism of H_5). Suppose G is a connected graph and let f' be an edge-coloring of G, using the five elements of S_5, that satisfies the property P_c. For a fixed vertex x of G define $f(x) = 0$ and then for any other vertex y choose an xy-path P and define $f(y) = \sum_{e \in P} f'(e)$. If a graph has more than one component, then repeat this on each component. For simplicity, an edge-coloring using elements of S_5 which satisfies the property P_c will be called an S_5-*edge-coloring.*

Analogously, an (S_5^*, S_5)-*edge-coloring* f' of a mixed graph (G', G) is a (not necessarily proper) edge-coloring of G such that thin edges receive their colors from S_5^*, thick edges receive their colors from S_5 and f satisfies the property P_c. Again it is easily seen that a mixed graph (G, G') admits a homomorphism to (K_{16}, H_5) if and only if it admits an (S_5^*, S_5)-edge-coloring.

Lemma 19 *For every subgraph V_8' of V_8 (on the same set of vertices) the mixed graph (V_8, V_8') admits a homomorphism to (K_{16}, H_5).*

Proof. We will show that (V_8, V_8') admits an (S_5^*, S_5)-edge-coloring. We start with a reference S_5-edge-coloring c of V_8. There is a homomorphism of V_8 to H_5 because V_8 is in fact a subgraph of H_5. The canonical edge-coloring of H_5 now induces an S_5-edge-coloring on V_8. Note that this coloring is unique up to a permutation of S_5. The four diagonal edges are colored by a same color and every pair of parallel edges of the 8-cycle receive a same color but distinct from the color of the other edges. It is also not difficult to find a homomorphism of V_8 to $\overline{H}_5$, in fact $\overline{H}_5$ consists of two disjoint copies of H_5 (see [3]) and, therefore, contains V_8 as a subgraph. So for the rest of the proof we will assume that V_8' has at least one thick edge.

Let A be the set of edges in an edge-cut of V_8. Note that if we change the color of every edge in A from $c(e)$ to $\gamma + c(e)$, with a fixed $\gamma \in \Gamma_k$, then the new edge-coloring (we call it c') still satisfies the property P_c. However $c'(e)$ does not necessarily belong to S_5 anymore. In fact $c'(e)$ may even be zero based on the choice of γ.

To prove the lemma, we will show that by a careful choice of γ and by repeated applications of the above edge-cut operation we can change the color of every edge not in V_8' to an element of S_5^* while keeping the color of other edges in S_5. To simplify the proof we introduce two local operators.

Claim 1 (Single operator). Suppose e_1, e_2 and e_3 are the three edges of V_8 being incident to a vertex v. Then, there is a $\gamma \in \Gamma_k$ such that $\gamma + c(e_i)$ is in S_5 if e_i is a thick edge and is in S_5^* otherwise.

Proof. Let $c(e_1) = x$, $c(e_2) = y$ and $c(e_3) = z$. Note that x, y and z are distinct elements of S_5. Let t be one of the two other elements of S_5. Based on the number of e_i's in V_8' we have four different cases. If they are all in V_8', then we do nothing (i.e., $\gamma = 0$). If there are two of them in V_8', say e_1 and e_2, then let $\gamma = x + y$. We now have $c'(e_1) = y$, $c'(e_2) = x$ and $c'(e_3) = x + y + z$. If there is only one of them in V_8', say e_1, then let $\gamma = t + x$. The new colors are $c'(e_1) = t$, $c'(e_2) = t + x + y$ and $c'(e_3) = t + x + z$. Finally, if none of them is in V_8', then let $\gamma = t$. e_i's are now colored by $t + x$, $t + y$ and $t + z$. $\diamond$

Claim 2 (Double operator). Let $e_0 = uv$ be an edge of the outer cycle of V_8 and let e_1 and e_2 be the two other edges incident to v and e_3 and e_4 the two other edges incident to u. Then, there are $\gamma_1, \gamma_2 \in \Gamma$ such that, by adding γ_1 to $c(e_1)$ and $c(e_2)$, γ_2 to $c(e_3)$ and $c(e_4)$ and $\gamma_1 + \gamma_2$ to $c(e_0)$, the color of each thick e_i remains in S_5 but each thin e_i receives its color from S_5^*.

Proof. We assume $c(e_0) = x$, $c(e_1) = y$, $c(e_3) = z$ and $c(e_2) = c(e_4) = t$ (therefore e_2 and e_4 are the diagonal edges). Hence, x, y, z and t are distinct elements of S_5. Let r be the remaining element of S_5.

If both e_1 and e_2 are in V_8', then we are done by applying Single operator at u. So we may assume at least one of e_1 or e_2 (similarly at least one of e_3 or e_4) is a thin edge. Suppose there are exactly two thin edges among e_1, e_2, e_3 and e_4. We assume e_0 is a thick edge and we give a detailed proof of how Double operator works on each possible case. For the corresponding cases of when e_0

is a thin edge we only give the value for γ_1 and γ_2 and leave the details to the reader.

- *e_1 and e_3 are not in V_8'.* We let $\gamma_1 = \gamma_2 = t + x$. Then new colors are: $c'(e_0) = c'(e_2) = c'(e_4) = x$, $c'(e_1) = t + x + y$ and $c'(e_3) = t + x + z$. (If e_0 is a thin edge, then we will let $\gamma_1 = t + z$ and $\gamma_2 = t + y$.)
- *e_1 and e_4 are not in V_8'.* We let $\gamma_1 = t + z$ and $\gamma_2 = x + z$. This changes the colors as follows: $c'(e_0) = t$, $c'(e_1) = y + t + z$, $c'(e_2) = z$, $c'(e_3) = x$ and $c'(e_4) = t + z + x$. (If e_0 is a thin edge, then we will let $\gamma_1 = t + r$ and $\gamma_2 = z + r$.) Note that the case when e_2 and e_3 are not in V_8' is symmetric to this case.
- *e_2 and e_4 are not in V_8'.* We let $\gamma_1 = x + y$ and $\gamma_2 = y + z$. The final colors are $c'(e_0) = z$, $c'(e_1) = x$, $c'(e_2) = t+x+y$, $c'(e_3) = y$ and $c'(e_4) = t+y+z$. (If e_0 is a thin edge, then we will let $\gamma_1 = r + y$ and $\gamma_2 = y + z$.)

If there are three or more thin edges among e_1, e_2, e_3 and e_4, then we may assume, without loss of generality, that both e_1 and e_2 are thin. In this case, we first let $\gamma_1 = x + r$. Therefore, changing the color of e_0 to r, e_1 to $y + x + r$ and e_2 to $t + x + r$. Now, since the edges incident to v are colored from S_5 using three different colors, γ_2 can be find by applying Single operator at u. $\diamond$

Single and Double operators are like local changes. In order to complete our proof, we need to show that these changes can be done globally without conflicting each other. For this purpose, assume there is at least one edge e of the outer cycle of V_8 that is also in V_8'. Let v_1, v_2, v_3 and v_4 be the four vertices at distance 1 from e, assuming that v_3v_4 is the edge parallel to e. Notice that these four vertices cover all the edges of V_8 except e. Moreover, v_3v_4 is the only edge incident to two of these vertices. Now, we apply Single operator at v_1 and at v_2 and Double operator at $\{v_3, v_4\}$.

The remaining cases are when none of the edges of the outer cycle are in V_8'. In this case, in the clockwise order (of the outer cycle) we add the color of every edge of the outer cycle to the color of the next edge. This way all the edges of the outer cycle have colors from S_5^* and it is again easy to check that this new coloring satisfies the property P_c. Since we have assumed (V_8, V_8') has at least one thick edge, there are at most three diagonal thin edges. We choose a set A of independent vertices of V_8 that covers the thin diagonal edges, making sure each selected vertex is incident to one such diagonal edge. For each vertex u in A we add the sum of the colors of two edges of the outer cycle incident to u to all three edges incident to u. This only exchanges the colors of the two edges of the outer cycle while it changes the color of the corresponding diagonal edge from an element of S_5 to an element of S_5^*. This proves the final case of the lemma. $\square$

We are now ready to prove the following stronger form of Theorem 16.

Theorem 20 *Let (G, G') be a mixed graph where G is a maximal K_5-minor-free graph and G' is a triangle-free graph. Then (G, G') admits a homomorphism to (K_{16}, H_5).*

Proof. Our proof is by contradiction. Assume (G, G') is the smallest counterexample with respect to Wagner number of G. Let $T_1, T_2, \ldots, T_r$ be the good Wagner sequence of G. Each member T_i of this sequence can be considered as a mixed graph (T_i, T'_i) where the edges in $E(T_i) \cap E(G')$ are the thick edges and edges in $E(T_i) \backslash E(G')$ are the thin edges. Let (G_{r-1}, G'_{r-1}) be the mixed graph induced by G_{r-1}. By the choice of r, (G_{r-1}, G'_{r-1}) admits a homomorphism to (K_{16}, H_5), call this homomorphism f_1.

Now, if T_r is isomorphic to V_8, then, by Lemma 19, there is a homomorphism f_2 of (T_r, T'_r) to (K_{16}, H_5). Since T_r and G_{r-1} have only an edge in common, using Lemma 17 we may choose f_2 so that the image of the end vertices of this edge under f_2 is the same as their image under f_1. A homomorphism of (G, G') can now be obtained from combining f_1 and f_2.

If T_r is a triangulation, then we can find a homomorphism f_3 of (T_r, T'_r) to (K_{16}, H_5) using Theorem 18. Moreover, since T_r and G_{r-1} have at most 3 vertices in common, using Lemma 17, we may choose f_3 so that it agrees with f_1 on these common vertices. The homomorphism of (G, G') is again obtained from combining f_1 and f_3. □

4 Examples and Remarks

In the first part of this section we show that our results from Sections 2 and 3 cannot extend to the class $Forb_m(K_6)$. We have some remarks and a conjecture in the second part.

4.1 Examples

A graph is called *apex* if by removing at most one vertex it becomes planar. It is clear that an apex graph is K_6-minor-free. Our first example is a triangle-free apex graph which is not 3-colorable. It was conjectured by Thomas [18] that every triangle-free apex graph is 3-colorable. This conjecture was disproved by Hare [7]. Here we provide an smaller counterexample to this conjecture.

Proposition 21 *There exists a triangle-free 4-chromatic apex graph.*

Proof. The graph G, depicted in Figure 3, is our example. G is clearly an apex graph as by removing vertex y it becomes planar. It is also easily seen that G is triangle-free. G is 4-colorable because $G - y$ is a triangle-free planar graph and therefore, by Grötzsch's theorem, is 3-colorable. It is only left to prove that G is not 3-colorable. Assume contrary, that G is 3-colorable. Let c be a 3-coloring of G.

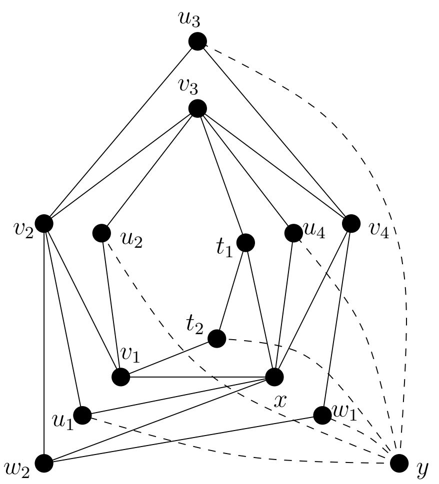

Fig. 3. A triangle-free 4-chromatic apex graph.

We first claim that $c(x) = c(y) = 1$. Let G' be the subgraph of G induced by v_i's, u_i''s, x and y. Note that G' is obtained from the Grötzsch's graph by removing a vertex of degree 3. We show that in any 3-coloring of G' vertices x and y must receive a same color. This will prove our claim because c also induces a 3-coloring on G'. Let c_1 be a 3-coloring of G' such that $c_1(y) = 1$ and $c_1(x) \neq 1$. For each v_i if $c_1(v_i) = 1$, then we change the color of v_i to $c_1(u_i)$. This new coloring is still a proper 3-coloring of G' but the color 1 does not appear on the 5-cycle induced by x and v_i's, a contradiction.

To complete our proof we notice that the set of vertices not colored 1 must induce a bipartite graph. Since v_2 is the only vertex of the 5-cycle induced by $\{w_1, w_2, v_2, u_3, v_4\}$ that is not adjacent to x or y, we must have $c(v_2) = 1$. But then every vertex of the 5-cycle induced by $\{v_1, u_2, v_3, t_1, t_2\}$ is adjacent to a vertex of color 1. This contradicts the fact that c was a 3-coloring. □

Our next example is also an apex graph, but this one is K_4-free and 5-chromatic.

Proposition 22 *There exists a k_4-free 5-chromatic graph in $Forb_m(K_6)$.*

Proof. An example of such a graph is depicted in Figure 4, we call this graph H. Note that H is also an apex graph and, therefore, K_6-minor-free. The fact that it is K_4-free can be checked easily. We prove that it is 5-chromatic.

Let H' be the subgraph of H which is obtained by removing two doted edges (i.e., x_0y_3 and x_0t_3). It is easy to check that H' is a 4-chromatic graph (it is

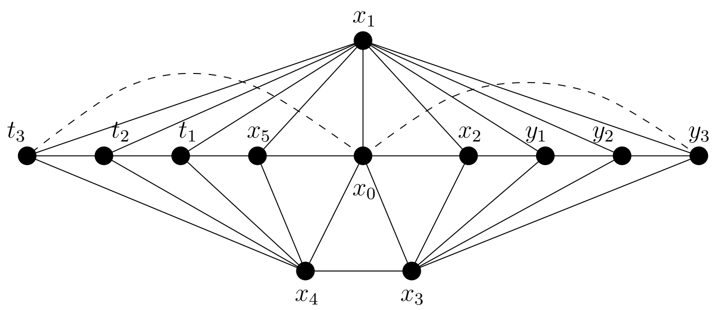

Fig. 4. A K_4-free 5-chromatic apex graph.

planar and contains the odd wheel W_5). We claim that in any 4-coloring of this graph either y_3 or t_3 receives the same color as x_0. To see this, note that in a 4-coloring of H' each x_i, $i = 1, 2, \ldots, 5$ receives a color different from that of x_0. Furthermore, at least one of x_3 and x_4 (by symmetry, say x_3) receives a color different from that of x_1. Now it is easy to see that y_1 must be colored the same as x_0, y_2 must be colored the same as x_2 and finally y_3 gets the color of x_0. This proves that H is not 4-colorable. Since H is an apex graph, it is 5-colorable. Therefore, it is 5-chromatic. □

Using Propositions 21 and 22 we can generally state that:

Corollary 23 *For every $k \geq 6$ and $3 \leq j \leq k-2$, the class $Forb_m(K_k)$ contains a graph homomorphically incomparable to K_j.*

Proof. For $k = 6$, the graph homomorphically incomparable to K_j is constructed in Proposition 21 (for $j = 3$) and Proposition 22 (for $j = 4$). For a general k and j all we need to do is to add a disjoint copy of a complete graph of an appropriate size to G (from Proposition 21) or H (from Proposition 22) and join all its vertices to all the vertices of G (or H). □

The following proposition proves that Theorem 15 cannot be extended to the class of triangle-free graphs in $Forb_m(K_6)$ either.

Proposition 24 *Let F be the graph of Figure 5. Then F is a triangle-free graph in $Forb_m(K_6)$ that does not admit a homomorphism to H_5.*

Proof. The fact that F is triangle-free is clear from the picture. To see that F does contain a K_6-minor let F' be the graph obtained from F by contracting u_1x and u_2y. Note that the size of the largest clique-minor of F and F' are the same. Also that F' is of maximum degree 4. If K_6-which is a 5-regular graph-is a minor of F', then every vertex of this minor must be formed from identifying at least 2 vertices of F'. But F' has only 10 vertices. This proves that K_6 is not a minor of F, i.e., $F \in Forb_m(K_6)$.

To prove that F does not admit a homomorphism to H_5 we show that it does not admit an S_5-edge-coloring. By contradiction, assume c' is an S_5-edge-

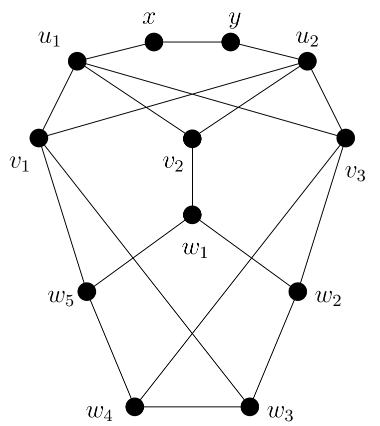

Fig. 5. A Δ-free graph in $Forb_m(K_6)$ which does not map to the Clebsch graph.

coloring of F. We first claim that $c'(v_iu_1) \neq c'(v_iu_2)$. Because if $c'(v_iu_1) = c'(v_iu_2)$, then the induced homomorphism by c' maps u_1 and u_2 to a same vertex. This is not possible because if u_1 and u_2 are mapped to a same vertex, then the image of the 3-path joining u_1 and u_2 must contain a triangle or a loop.

Next we claim that for each $1 \leq i < j \leq 3$ the set $\{c'(v_iu_1), c'(v_iu_2)\}$ is the same as the set $\{c'(v_ju_1), c'(v_ju_2)\}$. This is followed from the fact that $\{v_i, v_j, u_1, u_2\}$ induces a 4-cycle, and that every cycle, in particular this 4-cycle, must satisfy the property P_c. Then, it follows that from the three edges joining u_1 to v_i, $i = 1, 2, 3$ there are at least two being colored the same by c'. But every pair of these edges belong to a 5-cycle and no two edges of a 5-cycle can receive a same color in an S_5-edge-coloring. □

4.2 Remarks

A short proof of Grötzsch theorem is published by Thomassen [19]. In fact Thomassen proved a stronger result that every planar graph of girth at least 5 is 3-choosable. We do not know whether this can be extended to the class of K_5-minor-free graphs. (Thomassen's proof is based on the planar representations of these graphs). Notice that girth 5 is needed here. A triangle-free planar graph which is not 3-choosable is constructed by Voigt [20].

It follows from a general result of Nešetřil and Ossona de Mendez [13] that the class of K_5-minor-free graphs of odd-girth at least $2k + 1$ is bounded by a graph B of odd-grith $2k + 1$. To our knowledge this is the best supportive result for Conjecture 15.

While we showed, in this section, that our results cannot be extended to the class $Forb_m(K_6)$, we conjecture that absence of both triangles and 4-cycles still implies a similar result on the class of K_6-minor-free graphs.

Conjecture 25 *Every K_6-minor-free graph of girth at least five is 3-colorable.*

We would also like to remark that Thomas [18] has conjectured that every triangle-free graph in $Forb_m(K_6)$ is 4-colorable.

Acknowledgment Parts of this research was conducted at Charles University, Prague, we are grateful for their hospitality and we would like to acknowledge the support by ITI, Institute for Theoretical Computer Science. The first author was a Post doctoral fellow at Royal Military College of Canada for parts of this research. The second author was partially supported by DIMATIA. The third author acknowledges a partial support by Ministry of Science of Slovakia, Research program P1-0297.

References

[1] V. A. Aksionov, *Concerning the extension of the 3-coloring of planar graphs* (in Russian), Diskrete Anal. Issled. Oper. **16**, (1974) 3–19.

[2] R. Diestel, **Graph Theory**, Springer-Verlag, New York, (2000).

[3] R. E. Greenwood, A. M. Gleason, *Combinatorial relations and chromatic graphs*, Canad. J. Math. **7**, (1955) 1–7.

[4] B. Grünbaum, *Acyclic colorings of planar graphs*, Israel J. Math. **14**, (1973) 390–408.

[5] B. Guenin, *Edge coloring plane regular multigraphs*, to appear.

[6] B. Guenin, *Packing odd circut covers: A conjecture*, manuscript.

[7] D. Hare, personal communication.

[8] P. Hell, J. Nešetřil, **Graphs and Homomorphisms**, Oxford University Press, (2004).

[9] R. Naserasr, *Homomrphism and edge colourings of planar graphs*, submited.

[10] R. Naserasr, Y. Niguusie, *On a new reformulation of Hadwiger's conjecture*, (to appear in Discrete Math.)

[11] J. Nešetřil, Y. Nigussie, *On the 1-cuts of clique-minor-free graphs*, manuscript.

[12] J. Nešetřil, P. Ossona de Mendez, *Cuts and bounds*, KAM-DIMATIA Series, **592** (2002).

[13] J. Nešetřil, P. Ossona de Mendez, *Tree depth, subgraph coloring and homomorphism bounds*, (to appear in European J. of Combin.)

[14] J. Nešetřil, A. Raspaud, *Colored homomorphisms of colored mixed graphs.* J. Combin. Theory Ser. B **80** (2000) 147–155.

[15] A. Schrijver, **Combinatorial Optimization-Polyhedra and Efficiency**, volume C, Springer, (2003) PP. 1408–1409.

[16] P. D. Seymour, *On Tutte's extension of the four-colour problem*, J. Combin. Theory Ser. B **31** (1981) 82–94.

[17] P. G. Tait, *Remarks on the previous communication*, Proceeding of the Royal Society of Edinbrugh **10** (1878–1880) 729.

[18] R. Thomas, Personal communication.

[19] C. Thomassen, *A short list color proof of Grötzsch's theorem*, J. Combin. Theory Ser. B **88** (2003) 189–192.

[20] M. Voigt, *A not* 3*-choosable planar graph without* 3*-cycles*, Discrete Math. **146** (1995) 325–328.

[21] K. Wagner, *Über eine Eigenschaft der ebenen Komplexe*, Math. Ann. **144** (1937) 570–590.