A note on interconnecting matchings in graphs

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Abstract

We prove a sufficient condition for a graph G to have a matching that interconnects all the components of a disconnected spanning subgraph of G. The condition is derived from a recent extension of the Matroid intersection theorem due to Aharoni and Berger. We apply the result to the problem of the existence of a (spanning) 2-walk in sufficiently tough graphs.

1 Introduction

Consider a disconnected spanning subgraph F of a connected graph G. We study the following question: When can all the components of F be interconnected by a matching outside F? More precisely, we say that a matching M in $G \setminus E(F)$ is *F-connecting* if $F \cup M$ is a connected graph, and prove a sufficient condition for the existence of an F-connecting matching in a given graph. Before we state it, we introduce some terminology.

We begin with a variant of the domination number involving paths of length 2 (which we refer to as *2-paths*). A set P of 2-paths in a graph H is *dominating* if every edge of H is incident with an edge of some 2-path in P. The *v-domination* number $\gamma^{\mathrm{v}}(H)$ of H is the minimum size of a dominating set of 2-paths in H. If H contains an isolated edge, then there is no dominating set of 2-paths; accordingly, $\gamma^{\mathrm{v}}(H)$ is defined to be infinite. (The letter 'v' in this notation is meant to symbolize a 2-path.)

If X is a set of edges of a graph G, we write G_X for the graph with vertex set $V(G)$ and edge set X. For a graph H, $\omega(H)$ stands for the number of components of H.

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Theorem 1. *Let F be a spanning subgraph of a connected graph G. If*

$$\gamma^{\mathrm{v}}(G_X) \geq \omega(G \setminus X) - 1$$

for all $X \subseteq E(G) \setminus E(F)$, then G has an F-connecting matching.

The proof of Theorem 1 is based on a recent result of Aharoni and Berger [1] (see Theorem 5 below) that extends the celebrated matroid intersection theorem [6]. From this, we first derive a corollary on spanning trees satisfying a given set of restrictions (in Section 3) and Theorem 1 itself (in Section 4).

In the last section, we apply our result to a problem concerning spanning walks in sufficiently tough graphs. Recall that the *toughness* $\tau(G)$ of a graph G is the minimum, over all sets $X \subset V(G)$ such that $G \setminus X$ is disconnected, of the ratio

$$\frac{|X|}{\omega(G \setminus X)}.$$

A graph G is said to be *t-tough* for all t such that $0 < t \leq \tau(G)$. A long-standing conjecture of Chvátal [5] states that there is a constant T such that all T-tough graphs are hamiltonian. (For some time, $T = 2$ was believed to have this property, but this was disproved by Bauer et al. [4].)

A weaker graph property than containing a Hamilton cycle is containing a *2-walk*, i.e., a closed spanning walk visiting each vertex once or twice. Jackson and Wormald [9] conjectured that every 1-tough graph has a 2-walk. This conjecture is still open, but Ellingham and Zha [7] showed that the existence of a 2-walk follows from a stronger toughness assumption:

Theorem 2. *Every 4-tough graph has a 2-walk.*

For graphs of high girth, the following improvement was established in [7] (recall that the *girth* of a graph G is the least length of a cycle contained in G):

Theorem 3. *Let $k \geq 3$. If G is a $(2 + \frac{2}{k-2})$-tough graph of girth k, then G has a 2-walk.*

In Section 5, we show that the following somewhat weaker bound follows easily from Theorem 1:

Theorem 4. *Let $k \geq 4$. If G is a $(3 + \frac{9}{k-3})$-tough graph of girth k, then G has a 2-walk.*

Although the application of Theorem 1 does not yield the best available result, we believe it is interesting as it indicates a novel connection between matroid intersection and a hamiltonian-type problem in graphs of sufficient toughness. A similar connection is used in a different context in the more recent paper [10].

2 Matroids and simplicial complexes

A *simplicial complex* on a set V is any nonempty family C of subsets of V such that if $A \subseteq B \in C$, then $A \in C$. The sets in C are called the *faces* or *simplices* of C. The *dimension* of a face A is $|A| - 1$, and the dimension $\dim(C)$ of C is the maximum dimension of a face of C. A simplicial complex C determines a topological space $\|C\|$, the *polyhedron* of C. Typically, one is interested in the topological properties of C, such as its connectivity. Let us recall this concept (see [11] for more information).

A topological space X is *k-connected* ($k \geq 0$) if any continuous mapping $f : S^\ell \to X$, where $\ell \leq k$ and S^ℓ is the ℓ-dimensional sphere, can be continuously extended to a mapping from an $(\ell + 1)$-dimensional ball $B^{\ell+1}$ to X. Thus, for instance, X is 0-connected if and only if any two points are joined by a path in X. The space X is usually considered to be (-1)-connected if $X \neq \emptyset$. The *connectivity* of X is the maximum k such that X is k-connected. A simplicial complex C is defined to be *k-connected* if its polyhedron is k-connected. The connectivity of C is defined similarly. However, we shall find it convenient to use a parameter denoted by $\eta(C)$, which is defined to be the connectivity of $\|C\|$ increased by 2.

If $X \subseteq V$ is a subset of the ground set of a simplicial complex C, then $C \restriction X$ is the *induced subcomplex* of C on X, that is, the simplicial complex with ground set X such that $A \subseteq X$ is a face of $C \restriction X$ if and only if it is a face of C.

In graph theory, a simplicial complex of special importance is the *independence complex* $I(G)$ of a graph G. Its ground set is $V(G)$, with simplices being the independent sets in G. Closely related is the *matching complex* $Match(G)$ of G, whose ground set is $E(G)$, and whose simplices are the matchings in G. In fact, $Match(G)$ is the independence complex of the line graph $L(G)$ of G.

Note that the class of simplicial complexes includes any matroid, although the latter is usually considered from a different viewpoint. (See [14] for background on matroid theory.) A *matroid* is a simplicial complex M such that if A, B are faces of M and $|A| > |B|$, then there is an element $x \in A \setminus B$ such that $B \cup \{x\}$ is a face of M. We now review some of the basic facts about matroids.

The faces of a matroid M are usually referred to as its *independent sets*. Any inclusion-maximal face of M is called a *basis* of M. All the bases have the same dimension. The *rank* of M, denoted by $\mathrm{rank}(M)$, is the cardinality of any basis (i.e., its dimension plus one). If X is a subset of the ground set of M, then $M \restriction X$ is a matroid, and one defines $\mathrm{rank}_M(X) = \mathrm{rank}(M \restriction X)$. Among the most common matroids are the ones associated to graphs in the following way. The *polygon matroid* $M(G)$ of a graph G has ground set $E(G)$ and its faces are acyclic sets of edges of G. Thus, if G is connected, then bases of $M(G)$ correspond to spanning trees of G.

The matroid intersection theorem of Edmonds [6] (see also [14, Theorem 12.3.15]) gives a necessary and sufficient condition for a matroid to have a basis that is independent in another given matroid on the same ground set. Aharoni and Berger [1] obtained a generalization of this result, where one of the matroids is replaced by an arbitrary simplicial complex. Although their condition is no longer a necessary one, it does specialize to the condition of Edmonds when used for two matroids.

Theorem 5. *[1, Theorem 4.5] Let M be a matroid and C a simplicial complex on the same ground set V. If*

$$\eta(C \restriction X) + \mathrm{rank}_M(V \setminus X) \geq \mathrm{rank}(M),$$

for every $X \subseteq V$, then M has a basis belonging to C.

We remark that Theorem 5 also generalizes known results on the existence of an *independent system of representatives* in a graph (see, e.g., [2, 3, 12, 13]).

3 Compatible spanning trees

We shall be concerned with applications of Theorem 5 in a situation where the matroid M is the polygon matroid of a connected graph H. In such a case, the bases of M are just the spanning trees of H. Any simplicial complex C with ground set $E(G)$ can be viewed as defining 'admissible' subgraphs of G. We shall say that a subgraph $H' \subset H$ is *C-compatible* if $E(H')$ is a face of C. Theorem 5 provides a condition which implies that H has a C-compatible spanning tree:

Proposition 6. *Let H be a connected graph and C be a simplicial complex with $E(H)$ as the ground set. If*

$$\eta(C \restriction X) \geq \omega(H \setminus X) - 1$$

for every $X \subseteq E(H)$, then H has a C-compatible spanning tree.

The proof is given below. As a simple illustration, let H be the complete graph on 4 vertices a, b, c, d. Let us postulate that a set of edges of H is admissible if it contains no pair of disjoint edges. The corresponding simplicial complex C on ground set $E(H)$ has 8 maximal faces, all of size 3 (for instance, $\{ab, ac, ad\}$). In fact, C is the octahedron. It is easy to check that the condition in Proposition 6 is satisfied. For example, if $X = E(H)$, then $H \setminus X$ has 4 components, while $C \restriction X = C$ is 1-connected. Accordingly, H does have C-compatible spanning trees (which is, of course, easy to see directly): one of these is $\{ab, ac, ad\}$. We now proceed to the proof of the above proposition.

Proof of Proposition 6. Let M be the polygon matroid $M(H)$ of the graph H. Thus, the bases of M are the spanning trees of H. To find a C-admissible spanning tree, we verify the condition of Theorem 5 for C and M.

Let $X \subseteq E(H)$. It is easy to interpret the ranks in Theorem 5 in the present setting. The rank of X in M is just $n - \omega(H_X)$, where $n = |V(H)|$ (see, e.g., [14, p. 26]). Thus, a C-compatible spanning tree exists whenever

$$\begin{aligned}\eta(C \upharpoonright X) &\geq \operatorname{rank}(M) - \operatorname{rank}_M(E(H) \setminus X) \\ &= (n-1) - (n - \omega(H \setminus X)) = \omega(H \setminus X) - 1.\end{aligned}$$

□

4 F-connecting matchings

The topological condition in Proposition 6 may often be hard to verify. We now consider one case where it is implied by a simpler condition, involving only the structure of the graph in question.

Consider the independence complex $I(H)$ of a graph H. There are several lower bounds for $\eta(I(H))$ in terms of various kinds of domination numbers of the graph H (see [2] for a useful overview). The one that suits our purpose best (in view of the application in Section 5) uses the *edge-domination number* $\gamma^E(H)$ of H, defined as the minimum size of a set $D \subseteq E(H)$ such that each vertex of H has a neighbor which is an endvertex of an edge from D. (We refer to such a set D as a *vertex-dominating set* of edges.) If H contains an isolated vertex, we set $\gamma^E(H) = \infty$. The following bound is implied by a result of [13] (see [1]):

Theorem 7. *For any graph H,*

$$\eta(I(H)) \geq \gamma^E(H).$$

Recall from Section 2 that the matching complex $Match(G)$ of a graph G is the independence complex $I(L(G))$ of its line graph. Thus, Theorem 7 can be applied to matching complexes.

We now proceed to prove our main result that was stated in Section 1:

Theorem 1. *Let F be a spanning subgraph of a connected graph G. If*

$$\gamma^{\mathrm{v}}(G_X) \geq \omega(G \setminus X) - 1 \qquad (1)$$

for all $X \subseteq E(G) \setminus E(F)$, then G has an F-connecting matching.

Proof. We shall abbreviate $G \setminus E(F)$ as $G - F$. Let the simplicial complex C be the matching complex $I(L(G - F))$ of the graph $G - F$. Define H to be the multigraph obtained from G by contracting each component of the subgraph F to a single vertex.

Observe that if T is a C-compatible spanning tree of H, then the edges of G corresponding to those in T form an F-connecting matching in G. Thus, it suffices to find a C-compatible spanning tree of G. We verify the condition of Proposition 6.

Let $X \subseteq E(G-F)$. Observe that

$$C \restriction X = I(L(G_X)),$$

so by Theorem 7,

$$\eta(C \restriction X) \geq \gamma^E(L(G_X)).$$

Vertex-dominating sets of edges of $L(G_X)$ correspond bijectively to dominating sets of 2-paths in G_X (as defined in Section 1). It follows that $\gamma^E(L(G_X)) = \gamma^{\mathrm{v}}(G_X)$, and so

$$\eta(C \restriction X) \geq \gamma^{\mathrm{v}}(G_X). \tag{2}$$

In view of (2), the condition of Proposition 6 holds true whenever

$$\gamma^{\mathrm{v}}(G_X) \geq \omega(G \setminus X) - 1.$$

This is precisely the hypothesis of Theorem 1. The proof is complete. □

5 Toughness and 2-walks

In this section, we apply Theorem 1 to prove the following theorem (stated in Section 1):

Theorem 4. *Let $k \geq 4$. If G is a $(3 + \frac{9}{k-3})$-tough graph of girth k, then G has a 2-walk.*

Proof. Let G be a graph of girth k and toughness $t \geq 3 + 9/(k-3)$. By a result of Enomoto et al. [8], every 2-tough graph contains a 2-factor; let F be a 2-factor in G. To prove that G has a 2-walk, it is clearly enough to find an F-connecting matching (see also [7] where a corresponding structure is called a 1-quasitree).

We now verify the hypothesis (1) of Theorem 1 for any given set $X \subseteq E(G) \setminus E(F)$. In fact, we show that

$$\gamma^{\mathrm{v}}(G_X) \geq \omega(G \setminus X) \tag{3}$$

whenever X is nonempty. If G_X contains any component consisting of a single edge, then the left hand side of (1) is infinite and the inequality holds. Thus, we may assume that no such component exists. In such a case, we can choose a smallest set P of 2-paths dominating all of $E(G_X)$.

Assume, for the sake of a contradiction, that (3) is false, i.e., $|P| < \omega(G \setminus X)$. Let Y be the set of all the vertices of the 2-paths in P. We have

$$|Y| \leq 3\,|P| < 3\omega(G \setminus X) \tag{4}$$

and Y dominates all the edges of G_X. It follows that $G \setminus Y$ is a subgraph of $G \setminus X$. We need to lower-bound the number of components of $G \setminus Y$. Clearly, each component of $G \setminus X$ that is not completely contained in Y contains at least one component of $G \setminus Y$. As for the components of $G \setminus X$ that *are* contained in Y, there are at most $|Y|/k$ of these, since each of them contains at least one cycle of F, and the girth assumption implies that the length of the cycle is at least k. We conclude:

$$\omega(G \setminus Y) \geq \omega(G \setminus X) - \frac{|Y|}{k} > \omega(G \setminus X) \cdot \frac{k-3}{k}$$

by (4). Thus

$$\frac{|Y|}{\omega(G \setminus Y)} < \frac{3\omega(G \setminus X)}{(k-3) \cdot \omega(G \setminus X)/k} = \frac{3k}{k-3}. \tag{5}$$

However, our toughness assumption implies that

$$\frac{|Y|}{\omega(G \setminus Y)} \geq 3 + \frac{9}{k-3},$$

which in conjunction with (5) gives $3k < 3k$, a contradiction. □

We remark that as in [7], the girth assumption of Theorem 4 can be relaxed: in the above argument, we only use the fact that G has a 2-factor each of whose cycles is of length at least k.

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