Disjoint T-paths in tough graphs

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Abstract

Let G be a graph and T a set of vertices. A *T-path* in G is a path that begins and ends in T, and none of its internal vertices are contained in T. We define a *T-path covering* to be a union of vertex-disjoint T-paths spanning all of T. Concentrating on graphs that are *tough* (the removal of any nonempty set X of vertices yields at most $|X|$ components), we completely characterize the edges that are contained in some T-path covering. Our main tool is Mader's $\mathcal{S}$-paths theorem. A corollary of our result is that each edge of a k-regular k-edge-connected graph ($k \geq 2$) is contained in a T-path covering. This is, in a sense, a best possible counterpart of the result of Plesník that every edge of a k-regular $(k-1)$-edge-connected graph of even order is contained in a 1-factor.

1 Introduction

Possibly the most important result in all of matching theory is Tutte's theorem [11] that gives a necessary and sufficient condition for the existence of a 1-factor in a graph G (we let $V(G)$ denote its vertex set and note that our graphs may contain multiple edges):

Theorem 1 (Tutte) *A graph G has a 1-factor if and only if, for every $X \subseteq V(G)$, the graph $G - X$ has at most $|X|$ components with an odd number of vertices.*

(For background on matching theory, we refer the reader to [4] or [10]. A good reference for graphs in general is [1].)

Tutte's theorem has a number of generalizations, including a result of Gallai [2] on T-paths. For $T \subseteq V(G)$, a path in G is a *T-path* if it begins and ends in T, and none of its internal vertices are contained in T. Gallai's result (which yields Theorem 1 upon setting $T = V(G)$) is the following:

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Theorem 2 (Gallai) *Let G be a graph and $T \subset V(G)$. The maximum number of vertex-disjoint T-paths in G is equal to the minimum, over all $X \subseteq V(G)$, of*

$$|X| + \sum_K \left\lfloor \frac{|T \cap K|}{2} \right\rfloor,$$

where the sum is over all components K of $G - X$.

A further extension of Theorem 2 is due to Mader [5] (see also [9]). Let $\mathcal{S}$ be a partition of $T \subseteq V(G)$. We define an *$\mathcal{S}$-path* to be any T-path whose endvertices are contained in different sets of $\mathcal{S}$. If $F \subseteq E(G)$, we write $\langle F \rangle$ for the subgraph of G edge-induced by F, and abbreviate $V(\langle F \rangle)$ as $V(F)$.

Theorem 3 (Mader) *Let G be a graph and $\mathcal{S}$ be a partition of $T \subset V(G)$. The maximum number of vertex-disjoint $\mathcal{S}$-paths in G is equal to the minimum, over all $X \subseteq V(G)$ and $F \subseteq E(G - X)$ such that $\langle F \rangle$ contains no $\mathcal{S}$-path, of the quantity*

$$|X| + \sum_K \left\lfloor \frac{|(T \cup V(F)) \cap K|}{2} \right\rfloor,$$

where the sum is over all components K of $G - X - F$.

Although our main concern in this paper is with T-paths, we will need to work with $\mathcal{S}$-paths and use Theorem 3 as a tool.

Let us return to Theorem 1. By a simple counting argument, Theorem 1 implies the well-known theorem of Petersen [6]:

Theorem 4 (Petersen) *Every bridgeless cubic graph has a 1-factor.*

In fact, one can easily generalize the argument to regular graphs of arbitrary degree and show that every k-regular $(k-1)$-edge-connected graph on an even number of vertices has a 1-factor. Extending a result of Schönberger [8] for $k = 3$, Plesník [7] proved that more can be said about 1-factors of such graphs:

Theorem 5 (Plesník) *Every edge of a k-regular $(k-1)$-edge-connected graph $(k \geq 2)$ on an even number of vertices is contained in a 1-factor.*

A question we study in the present paper is to what extent Theorem 5 carries over to the context of T-paths. For a graph G and $T \subseteq V(G)$, we define a *T-path covering* to be a union of vertex-disjoint T-paths that spans all of T. (Observe that for $T = V(G)$, a T-path covering is just a 1-factor of G.) A T-path covering can only exist if $|T|$ is even. For this reason, it is natural to work with *grafts*, i.e., pairs (G, T), where G is a graph and $T \subseteq V(G)$ is a set of even size.

In view of Theorem 5, one might ask the following:

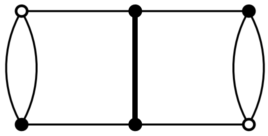

Figure 1: A graft (G, T); a vertex is black if it is in T and white otherwise. G is cubic and bridgeless, but the thick edge is not contained in any T-path covering.

Question 6 *Let (G, T) be a graft, where G is k-regular and $(k-1)$-edge-connected and $|V(G)|$ is even. Is it true that every edge of G is contained in a T-path covering?*

If k is odd, the answer is 'no'. For $k = 3$, this is shown by the graph in Figure 1, and counterexamples for larger odd k are easy to find. However, the real question that motivated the present paper, and originated in our work on intersections of 'T-joins' and edge-cuts in [3], is slightly different: it is what we obtain from Question 6 upon replacing the number $k-1$ with k. As it turned out, with this slightly stronger connectivity assumption, the answer is affirmative:

Theorem 7 *Suppose that G is a k-regular k-edge-connected graph, where $k \geq 2$, and (G, T) is a graft. Then every edge of G is contained in a T-path covering.*

It is worth noting that for even k, Theorem 7 implies an affirmative answer to Question 6, because a k-regular graph (k even) is $(k-1)$-edge-connected if and only if it is k-edge-connected. Thus, Theorem 7 is, in a sense, a best possible counterpart of Theorem 5 for T-path coverings.

It can be shown (cf. the proof of Theorem 7 in Section 5) that every k-regular k-edge-connected graph G is *tough*, i.e., the removal of any nonempty set $Y \subseteq V(G)$ produces a graph of at most $|Y|$ components. (A related concept, the toughness of a graph, is defined at the end of Section 5.) We managed to extend our analysis to the class of tough graphs, proving a complete characterization of the edges contained in a T-path covering:

Theorem 8 *Let G be a tough graph with $|V(G)| \geq 3$ and let (G, T) be a graft. An edge e of G with ends u and v is contained in a T-path covering if and only if there is no set X such that*

(i) $\{u, v\} \subseteq X \subseteq T$,

(ii) *$G - X$ has precisely $|X|$ components, and*

(iii) *each of these components contains an odd number of vertices in T.*

We conclude this section with several definitions. Let (G, T) be a graft and H a subgraph of G. We say that H is *T-odd* if $|T \cap V(H)|$ is odd; otherwise, H is *T-even*. The number of components of H is denoted by $\omega(H)$. The symbols $\omega_T(H)$ and $\tilde{\omega}_T(H)$ denote the number of T-odd and T-even components of H, respectively.

2 Excess

Let us extend the definition of a T-path covering to $\mathcal{S}$-paths. Throughout this section, let (G, T) be a graft and $\mathcal{S}$ be a partition of T. We define an *$\mathcal{S}$-path covering* to be a union of vertex-disjoint $\mathcal{S}$-paths in G that spans T. Mader's min-max theorem (Theorem 3) directly implies a necessary and sufficient condition for the existence of an $\mathcal{S}$-path covering. To state it concisely, we introduce the following parameter. The *excess* $\mathrm{exc}_{G,T}(X, F)$ of a pair (X, F), where $X \subseteq V(G)$ and $F \subseteq E(G - X)$, is defined as

$$\mathrm{exc}_{G,T}(X, F) = |X| + |X - T| + |V(F) - T| - \omega_{T \cup V(F)}(G - X - F).$$

We abbreviate $\mathrm{exc}_{G,T}(X, \emptyset)$ as $\mathrm{exc}_{G,T}(X)$.

While the excess parameter may seem rather mysterious at first, Proposition 10 below gives the definition some support. However, we begin with a parity lemma that will be useful on several occasions.

Lemma 9 *If (G, T) is a graft, $X \subseteq V(G)$ and $F \subseteq E(G - X)$, then $\mathrm{exc}_{G,T}(X, F)$ is even.*

Proof. Observe that

$$\omega_{T \cup V(F)}(G - X - F) \equiv |(T \cup V(F)) - X| \pmod 2$$

Thus, the sum

$$|X| + |X - T| + |V(F) - T| + |(T \cup V(F)) - X| \tag{1}$$

has the same parity as $\mathrm{exc}_{G,T}(X, F)$. Interpreting the cardinalities in (1) as sums of contributions from the vertices of $V(G)$ (e.g., $|X| = \sum_{x \in X} 1$), it is easy to check that the total contribution of a vertex x is odd if and only if $x \in T \cup (V(F) \cap X)$. Since $V(F) \cap X = \emptyset$, it follows that (1) has the same parity as $|T|$, which is even by the definition of a graft. Thus, $\mathrm{exc}_{G,T}(X, F)$ is even as well. $\square$

Let us define a set $F \subseteq E(G)$ to be *$\mathcal{S}$-admissible* if $\langle F \rangle$ contains no $\mathcal{S}$-path and each component of $\langle F \rangle$ contains at least two vertices of T.

Proposition 10 *Let (G,T) be a graft and $\mathcal{S}$ be a partition of T. There exists an $\mathcal{S}$-path covering in G if and only if for all $X \subseteq V(G)$ and all $\mathcal{S}$-admissible $F \subseteq E(G-X)$, it holds that*

$$\mathrm{exc}_{G,T}(X,F) \geq 0. \tag{2}$$

Proof. By Theorem 3, an $\mathcal{S}$-path covering exists if and only if for each $X \subseteq V(G)$ and $F \subseteq E(G-X)$ such that $\langle F\rangle$ contains no $\mathcal{S}$-path, one has

$$|X| + \sum_K \left\lfloor \frac{|(T\cup V(F))\cap K|}{2} \right\rfloor \geq \frac{|T|}{2}, \tag{3}$$

where K ranges over components of $G-X-F$. Noting that the effect of rounding is to subtract $1/2$ for each $(T\cup V(F))$-odd component of $G-X-F$, and multiplying by two, we can rewrite (3) as

$$2\,|X| + \sum_K |(T\cup V(F))\cap K| - \omega_{T\cup V(F)}(G-X-F) \geq |T|\,.$$

The sum of $|(T\cup V(F))\cap K|$ just equals $|(T\cup V(F))-X|$. Furthermore, since

$$2\,|X| + |(T\cup V(F))-X| - |T| = |X| + |X-T| + |V(F)-T|\,,$$

we finally obtain that (3) is equivalent to

$$|X| + |X-T| + |V(F)-T| \geq \omega_{T\cup V(F)}(G-X-F),$$

or in other words, to (2). So far, we have proved that an $\mathcal{S}$-path covering exists if and only if (2) holds for each $X \subseteq V(G)$ and $F \subseteq E(G-X)$ such that $\langle F\rangle$ contains no $\mathcal{S}$-path.

We need to prove a little more: namely that the validity of (2) for just the $\mathcal{S}$-admissible sets F ensures the existence of an $\mathcal{S}$-path covering. Thus, let $X \subseteq V(G)$ and $F \subseteq E(G-X)$ be such that (2) fails, $\langle F\rangle$ contains no $\mathcal{S}$-path, and F is an inclusionwise minimal set with these properties. By the definition of $\mathcal{S}$-admissible sets, $\langle F\rangle$ has a component with edge set F_0 such that

$$|V(F_0)\cap T| \leq 1. \tag{4}$$

We show that X and $F' = F - F_0$ still violate (2), contradicting the minimality of F.

Observe that, for trivial reasons,

$$|V(F')-T| = |V(F)-T| - |V(F_0)-T|$$

and

$$\omega_{T\cup V(F')}(G-X-F') \geq \omega_{T\cup V(F)}(G-X-F) - |V(F_0)|\,.$$

Thus, we can estimate $\mathrm{exc}_{G,T}(X, F')$ as

$$\begin{aligned}\mathrm{exc}_{G,T}(X, F') &= |X| + |X - T| + |V(F') - T| - \omega_{T\cup V(F')}(G - X - F') \\ &\le |X| + |X - T| + (|V(F) - T| - |V(F_0) - T|) \\ &\quad - (\omega_{T\cup V(F)}(G - X - F) - |V(F_0)|) \\ &= \mathrm{exc}_{G,T}(X, F) + |T \cap V(F_0)| \,.\end{aligned}$$

By (4) and Lemma 9, we conclude that

$$\mathrm{exc}_{G,T}(X, F') = \mathrm{exc}_{G,T}(X, F) < 0,$$

a contradiction with the minimality of F. The proof is complete. □

Observe that if $\mathcal{S}$ is a partition into singleton sets, then the only $\mathcal{S}$-admissible set $F \subseteq E(G)$ is $F = \emptyset$. Hence the following corollary (which also follows directly from Theorem 2):

Corollary 11 *A graft (G, T) admits a T-path covering if and only if for all $X \subseteq V(G)$,*

$$\mathrm{exc}_{G,T}(X) \ge 0. \tag{5}$$

3 Case I: e is incident with T

In this section, we prove the 'if' part of Theorem 8 for edges e with at least one end in T. The other case is considered in Section 4. The results are used in the proof of Theorem 8 in Section 5.

Lemma 12 *Let e be an edge of a graft (G, T) with endvertices u and v, at least one of which is contained in T. Consider the graft (G', T') given by*

$$\begin{aligned}G' &= G - e - (T \cap \{u, v\}), \\ T' &= T \oplus \{u, v\} \,,\end{aligned}$$

where $\oplus$ denotes the symmetric difference. Then e is contained in a T-path covering in G if and only if G' admits a T'-path covering.

Proof. It is straightforward to check that if P is any T-path covering in G and $e \in E(P)$, then by removing e and the vertices of $T \cap \{u, v\}$, we obtain a T'-path covering in G'. Conversely, adding e and its endvertices to a T'-path covering in G' produces a T-path covering in G containing e. □

The main result of this section is the following:

Proposition 13 *Let e be an edge of a graft (G, T) with endvertices u and v, at least one of which is contained in T. If G is tough and there is no set $X \subseteq V(G)$ with the properties (i)–(iii) from Theorem 8, then e is contained in a T-path covering.*

Proof. Assume that there is no set X satisfying (i)–(iii). Let the graft (G', T') be defined as in Lemma 12; by the lemma, it suffices to show that G' admits a T'-path covering. Corollary 11 implies that it is enough to show that each set $X' \subseteq V(G')$ satisfies the following inequality:

$$\mathrm{exc}_{G',T'}(X') \geq 0. \tag{6}$$

We proceed by contradiction. Suppose that (6) fails for a certain set X'. In a series of claims, we show that the set $X \subseteq V(G)$, defined by

$$X = X' \cup (T \cap \{u, v\}),$$

has properties (i)–(iii) from the theorem, contradicting our assumption. The end of the proof of each claim is marked by $\triangle$.

Claim 1 *We have* $\mathrm{exc}_{G,T}(X) \leq \mathrm{exc}_{G',T'}(X') + 2$.

Let us examine the effect of passing from the triple (G, T, X) to (G', T', X') on the excess parameter. Roughly speaking, the difference

$$\begin{aligned} \mathrm{exc}_{G,T}(X) - \mathrm{exc}_{G',T'}(X') = {} & (|X| - |X'|) + (|X - T| - |X' - T'|) \\ & - (\omega_T(G - X) - \omega_{T'}(G' - X')) \end{aligned} \tag{7}$$

can be interpreted as a sum of contributions, from u and v, to the three terms on the right hand side. For instance, u contributes 1 to the first term if u is in X but not in X'. Note that by the definition of X, this happens if and only if $u \in T$. In fact, it is easy to see that

$$|X'| = |X| - |\{u, v\} \cap T|.$$

Similarly, we have

$$|X' - T'| = |X - T| - |\{u, v\} \cap (X - T)|.$$

Substituting into (7) and noting that $\{u, v\} \cap T \subseteq X$, we obtain

$$\begin{aligned} \mathrm{exc}_{G,T}(X) - \mathrm{exc}_{G',T'}(X') = {} & |\{u, v\} \cap X| \\ & - (\omega_T(G - X) - \omega_{T'}(G' - X')). \end{aligned} \tag{8}$$

(We remark that this equality will also be useful in the proof of a later claim.)

Consider now the last term on the right hand side of (8). Every T'-odd component K of $G' - X'$ is also a T-odd component of $G - X$, unless K contains u or v. Thus:

$$\omega_{T'}(G' - X') \leq \omega_T(G - X) + |\{u, v\} - X| \,.$$

Combining with (8), we obtain that

$$\begin{aligned} \mathrm{exc}_{G,T}(X) - \mathrm{exc}_{G',T'}(X') &\leq |\{u, v\} \cap X| + |\{u, v\} - X| \\ &= |\{u, v\}| = 2. \end{aligned}$$

This proves the claim. △

Claim 2 *The number* $\mathrm{exc}_{G,T}(X)$ *is zero.*

Since $\mathrm{exc}_{G',T'}(X') < 0$, Claim 1 implies that $\mathrm{exc}_{G,T}(X) \leq 1$. By Lemma 9, $\mathrm{exc}_{G,T}(X)$ is even and hence $\mathrm{exc}_{G,T}(X) \leq 0$. Since G is tough, we have

$$\omega(G - X) \leq |X| \,,$$

and so in particular $\mathrm{exc}_{G,T}(X) \geq 0$. The claim follows. △

Claim 3 *We have* $\tilde{\omega}_T(G - X) = |X - T| = 0$.

By Claim 2, $\omega_T(G - X) = |X| + |X - T|$ and therefore

$$\omega(G - X) = |X| + |X - T| + \tilde{\omega}_T(G - X).$$

Suppose that $X \neq \emptyset$. Since G is tough, we have $\omega(G - X) \leq |X|$, which yields

$$|X - T| + \tilde{\omega}_T(G - X) \leq 0$$

which proves the assertion.

It remains to handle the case that X is empty. By the definition of X, we have $X' = \emptyset$ and $u, v \notin T$. Thus, $\mathrm{exc}_{G',T'}(X') < 0$ implies that $\omega_{T'}(G') > 0$; in particular, G' is disconnected. On the other hand, G, being tough, is connected. However, $V(G') = V(G)$ and the only edge of G missing in G' is e. It follows that e is a bridge in G. Since $|V(G)| \geq 3$, one of u and v (say, u) has degree at least 2 in G. But then $\omega(G - \{u\}) \geq 2$ and G is not tough, a contradiction. Thus, the case $X = \emptyset$ cannot occur. △

Claim 4 *Both* u *and* v *are in* X.

Observe first that

$$\omega_{T'}(G' - X') \leq \omega(G - X) = \omega_T(G - X),$$

where the last equality follows from Claim 3. Using equation (8) from the proof of Claim 1, we get:

$$\mathrm{exc}_{G,T}(X) - \mathrm{exc}_{G',T'}(X') \leq |\{u, v\} \cap X| .$$

Assuming that u or v is not in X and recalling that by Lemma 9, the excess is an even number, we conclude that

$$\mathrm{exc}_{G,T}(X) \leq \mathrm{exc}_{G',T'}(X') < 0,$$

a contradiction with Claim 2. $\triangle$

We can now finish the proof of Proposition 13. By Claims 3 and 4, $\{u, v\} \subseteq X \subseteq T$, i.e., the set X has property (i) from the theorem. Claims 2 and 3 imply that $\omega(G - X) = |X|$ (property (ii)) and that all components of $G - X$ are T-odd (property (iii)). The proof is complete. $\square$

4 Case II: e is not incident with T

We now turn to the case that neither end of the edge e in Theorem 8 is in T. The lemma below is the reason why we need to use Mader's theorem (Theorem 3) rather than the more specific Gallai's theorem (Theorem 2).

Lemma 14 *Let e be an edge of a graft (G, T) with endvertices u and v, where $T \cap \{u, v\} = \emptyset$. Define a partition $\mathcal{S}'$ of $T' = T \cup \{u, v\}$ by*

$$\mathcal{S}' = \{\{t\} : t \in T\} \cup \{\{u, v\}\} .$$

The edge e is contained in a T-path covering in G if and only if G admits an $\mathcal{S}'$-path covering.

Proof. Given a T-path covering in G that contains e, we can construct an $\mathcal{S}'$-path covering by removing e. Conversely, if P is an $\mathcal{S}'$-path covering, then u and v are on different paths of P, so we may add e to obtain a T-path covering containing e. $\square$

The following proposition is an analogue of Proposition 13 for the present case:

Proposition 15 *Let e be an edge of a graft (G, T) with endvertices u and v, where $T \cap \{u, v\} = \emptyset$ and G is tough. The edge e is contained in a T-path covering.*

Proof. Let $T' = T \cup \{u, v\}$ and let $\mathcal{S}'$ be the partition of T' defined in Lemma 14. We need to prove that G admits an $\mathcal{S}'$-path covering. For this, we use Proposition 10. Let $X \subseteq V(G)$ and let $F \subseteq E(G - X)$ be an $\mathcal{S}'$-admissible set. Assume, for the sake of a contradiction, that

$$\mathrm{exc}_{G,T'}(X, F) < 0 \tag{9}$$

and that F is inclusionwise minimal with this property.

If $F = \emptyset$, then (9) implies

$$\omega_{T'}(G - X) > |X| + |X - T'| \, ,$$

contrary to the toughness assumption which asserts that $\omega(G - X) \leq |X|$. Thus, we may assume that F is nonempty.

Claim 1 *$\langle F \rangle$ is connected and $V(F) \cap T' = \{u, v\}$.*

Since F is an $\mathcal{S}'$-admissible set, every component of $\langle F \rangle$ contains at least two vertices of T'. On the other hand, $\langle F \rangle$ contains no $\mathcal{S}'$-path, and every path joining two vertices of T' other than u and v is an $\mathcal{S}'$-path. It follows that $\langle F \rangle$ has exactly one component, contains u and v, and does not contain any other vertices of T'. △

Claim 2 *Each vertex of $\langle F \rangle$ is in a different component of $G - X - F$. Furthermore, all the components of $G - X - F$ intersected by $\langle F \rangle$ are $(T' \cup V(F))$-odd.*

Let $k = |V(F)|$ and c be the number of $(T' \cup V(F))$-odd components of $G - X - F$ intersected by $\langle F \rangle$. By the minimality of F,

$$\mathrm{exc}_{G,T'}(X, \emptyset) \geq 0. \tag{10}$$

On the other hand, we may estimate $\mathrm{exc}_{G,T'}(X, \emptyset)$ as

$$\mathrm{exc}_{G,T'}(X, \emptyset) \leq \mathrm{exc}_{G,T'}(X, F) + c - (k - 2), \tag{11}$$

since, clearly,

$$\omega_{T' \cup V(F)}(G - X - F) \leq \omega_{T'}(G - X) + c$$

and by Claim 1,

$$|V(F) - T'| = k - 2.$$

By (9) and Lemma 9, $\mathrm{exc}_{G,T'}(X, F)$ is at most -2, while by (10) and (11), it is at least $k - 2 - c$. Consequently, $c \geq k$ (hence $c = k$) and the claim follows. For later use, we infer from (11) that $\mathrm{exc}_{G,T'}(X) \leq \mathrm{exc}_{G,T'}(X, F) + 2$, and hence by (10),

$$\mathrm{exc}_{G,T'}(X) = 0. \tag{12}$$

△

Claim 3 *The component K of $G - X$ containing $\langle F \rangle$ is T'-even.*

Again, let $k = |V(F)|$. By Claim 2, all the k components of $K - F$ are $(T' \cup V(F))$-odd. Exactly $k - 2$ of the components contain one vertex from $V(F) - T'$, and hence are T'-even. The remaining two components of $K - F$ contain no vertex from $V(F) - T'$, and hence are T'-odd. In total, the number of vertices of T' in K is even. $\triangle$

We are now in a position to finish the proof of Proposition 15. Using (12) and the definition of excess, we conclude that $|X| \leq \omega_T(G - X)$. On the other hand, Claim 3 implies that $\omega_T(G - X) < \omega(G - X)$. Putting the inequalities together, we get

$$|X| < \omega(G - X),$$

which contradicts the assumption that G is tough. This contradiction shows that (9) is never satisfied, so G indeed admits an $\mathcal{S}'$-path covering. $\square$

5 Proof of the characterization

In this section, we prove our main results. Theorem 7 will be obtained as a corollary of Theorem 8 which is proved first. The basic ingredients of the proof are Propositions 13 and 15 of the preceding sections.

Proof of Theorem 8. Let (G, T) be a graft, G a tough graph, and let e be an edge of G with ends u and v.

We prove the 'only if' part first. Consider a T-path covering in G containing e. For the contradiction, assume that there is a set $X \subseteq T$ with properties (i)–(iii). Set $X' = X - \{u, v\}$. By Lemma 12, the graft (G', T') (as defined in the lemma) has a T'-path covering, whence by Corollary 11,

$$\mathrm{exc}_{G',T'}(X') \geq 0. \tag{13}$$

Note that by (i), $G' = G - \{u, v\}$ and $T' = T - \{u, v\}$. Clearly, $G' - X' = G - X$, and so $\omega_{T'}(G' - X') = \omega_T(G - X)$. We compute:

$$\begin{aligned} \mathrm{exc}_{G',T'}(X') &= |X'| + |X' - T'| - \omega_{T'}(G' - X') \\ &= (|X| - 2) + 0 - \omega_T(G - X) = |X| - 2 - |X| \\ &= -2, \end{aligned}$$

a contradiction to (13).

The 'if' part follows from the combination of Proposition 13 and Proposition 15. $\square$

It is now easy to derive the result on k-regular k-edge-connected graphs:

Proof of Theorem 7. Let (G, T) be a graft with G k-regular and k-edge-connected, and let e be an edge of G. Assuming that e is not contained in any T-path covering, we aim to use Theorem 8 to reach a contradiction.

We first prove that G is tough. Given a set $Y \subseteq V(G)$, let m be the number of edges with one end in Y and the other end in a component of $G - Y$. On the one hand, $m \geq k \cdot \omega(G - Y)$ since each component of $G - Y$ is joined to the rest of the graph (hence to Y) by at least k edges. On the other hand, since G is k-regular, $m \leq k \cdot |Y|$, with equality if and only if Y is an independent set. Putting the two inequalities together, we obtain that

$$\omega(G - Y) \leq |Y| \tag{14}$$

and equality can only hold if Y is independent. Since (14) holds for each Y, G is tough.

Thus, Theorem 8 implies that there is a set $X \subseteq V(G)$ with properties (i)–(iii) listed in the theorem. By (i), X includes both ends of e; therefore, it is not independent. Consequently, we get a strict inequality in (14) for $Y = X$. This contradiction with property (ii) finishes the proof. □

We conclude the paper with another corollary of Theorem 8. To state it, we need to recall the definition of the toughness parameter. A graph G is *t-tough* (where $t > 0$ is a real number) if for every set $X \subset V(G)$, the number of components of $G - X$ is either 1 or at most $|X|/t$. Observe that a graph is 1-tough if and only if it is tough as defined in Section 1. The *toughness* $\tau(G)$ of a connected graph G is the largest t such that G is t-tough (or ∞ if G is complete, in which case it is t-tough for all $t > 0$).

Corollary 16 *Let (G, T) be a graft with G tough. If any of the following conditions holds, then every edge of G is contained in a T-path covering:*

(a) T is an independent set,

(b) G has toughness $\tau(G) > 1$.

Proof. By Theorem 8, if an edge e with ends u and v is not contained in a T-path covering, then there is a set X with the properties given in the theorem. To prove part (a), note that $\{u, v\} \subseteq X \subseteq T$ is impossible if T is independent. Part (b) follows from the fact that the equality $\omega(G - X) = |X|$ implies $\tau(G) \leq 1$, contrary to the assumption. □

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