Tough spiders*

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Abstract

Spider graphs are the intersection graphs of subtrees of subdivisions of stars. Thus, spider graphs are chordal graphs that form a common superclass of interval and split graphs. Motivated by previous results on the existence of Hamilton cycles in interval, split and chordal graphs, we show that every 3/2-tough spider graph is hamiltonian. The obtained bound is best possible since there are $(3/2-\varepsilon)$-tough spider graphs that do not contain a Hamilton cycle.

1 Introduction

We study the existence of Hamilton cycles in a special class of graphs with an additional toughness assumption. The notion of toughness is well-established and closely related to hamiltonian graphs [8, 9]. A graph G is *β-tough* if for every set A of its vertices, $G \setminus A$ is connected or the number $\kappa(G \setminus A)$ of its components does not exceed $|A|/\beta$. Clearly, if G is hamiltonian, then G is 1-tough (but the converse does not hold). A famous conjecture of Chvátal [6] from 1973 asserts that the converse holds at least in an approximate sense:

Conjecture 1. *There exists a constant β such that every β-tough graph is hamiltonian.*

*The research was conducted while the second and the third author were visiting Institute for Theoretical Computer Science (ITI) at University of West Bohemia in Pilsen in May 2005.

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Conjecture 1 was believed to be true with $\beta = 2$, but in 2000, Bauer, Broersma and Veldman [1] provided a construction of $(9/4 - \varepsilon)$-tough graphs that are not hamiltonian (for every $\varepsilon > 0$). The close connection between toughness and hamiltonicity can be demonstrated on the concept of 2-*walks*, closed spanning walks visiting each vertex at most twice: every 4-tough graph has a 2-walk and there exist $(17/24 - \varepsilon)$-tough graphs with no 2-walk [7]. Let us remark that a conjecture of Jackson and Wormald [10] asserts that every 1-tough graph has a 2-walk. A result of Win [15] in this direction asserts that every 1-tough graph contains a spanning tree with maximum degree three. In particular, it has a closed spanning walk visiting each vertex at most three times.

Conjecture 1 has been established for several special classes of graphs including interval graphs, split graphs and chordal graphs. Interval graphs are the intersection graphs of intervals on a line, i.e., a graph is *interval* if there exists a family of intervals that correspond to its vertices and two intervals intersect if and only if the corresponding vertices are adjacent. The paper of Keil [11] contains an implicit proof that every 1-tough interval graph is hamiltonian. Since every hamiltonian graph must be 1-tough, this result is the best possible.

Split graphs are graphs whose vertex set can be partitioned into an independent set and a clique. Kratsch, Lehel and Muller [12] showed that every 3/2-tough split graph is hamiltonian. Since there exists a $(3/2 - \varepsilon)$-tough split graph that is not hamiltonian [6, p. 223] (for every $\varepsilon > 0$), their result is the best possible. Finally, *chordal graphs* are graphs that do not contain an induced cycle of length four or more. The best known result for this class of graphs is the result of Chen et al. [5] that every 18-tough chordal graph is hamiltonian. On the other hand, a construction of $(7/4 - \varepsilon)$-tough non-hamiltonian chordal graphs was given in [1]. It was conjectured in [2] that every 2-tough chordal graph is hamiltonian. Let us notice at this point that chordal graphs form a superclass of both interval graphs and split graphs.

All the three classes of graphs mentioned above (interval graphs, split graphs and chordal graphs) have nice characterizations as intersection graphs of connected subgraphs of special classes of graphs. A graph G is the *intersection graph* of subgraphs $H_1, \ldots, H_n$ of a graph H if the vertices of G one-to-one correspond to the subgraphs $H_1, \ldots, H_n$ and two vertices of G are adjacent if and only if the corresponding subgraphs intersect. Note that every graph is an intersection graph of (connected) subgraphs of a graph.

In this paper, we study a subclass of chordal graphs that is a proper superclass of interval and split graphs. Its definition is motivated by characterizations of interval, split and chordal graphs as intersection graphs of special families of trees (see, e.g., [3]): a graph is an *interval graph* if and only if it is an intersection graph of subpaths of a path. A graph is a *split graph* if and only if it is an intersection graph of subtrees of a star, i.e., a graph $K_{1,n}$. Finally, a graph is *chordal* if and only if it is an intersection graph of subtrees of a tree.

Spiders form a class of graphs that contain both paths and stars. A graph is a *spider* if it is a subdivision of a star. The vertex of the spider of degree greater than two (if any) is called the *central vertex* and the paths from its leaves to the central vertex are *legs*. A graph G is a *spider graph* if it is an intersection graph of subtrees of a spider. Clearly, every interval graph and every split graph is a spider graph, and spider graphs are chordal.

We show that every 3/2-tough spider graph is hamiltonian, matching the bound obtained for the split graphs in [12]. Since there are $(3/2-\varepsilon)$-tough non-hamiltonian split graphs, our bound is the best possible. We obtain our result for spider graphs using the result on the existence of Hamilton cycles in 1-tough interval graphs as a black box. Our argument employs Hall's theorem and the matroid intersection theorem. In this way, we provide an alternative proof that 3/2-tough split graphs are hamiltonian. In order to make the paper more accessible to the reader, we first show how our arguments apply to split graphs, and later we generalize the proof to the class of spider graphs. We believe that our work might be useful in improving the bound on the toughness that guarantees the existence of a Hamilton cycle in chordal graphs.

2 Hamilton cycles in split graphs

We first provide a short proof that 3/2-tough split graphs are hamiltonian. We believe that presenting our arguments first for split graphs will help the reader follow the proof of the general result. The reader is also invited to consult Figure 1 where the steps of the proof presented in this section are visualized. We start with establishing the following auxiliary lemma:

Lemma 1. *Let G be a split graph with the parts A and B where A is the independent set and B is the clique. Let G^* be the multigraph obtained from G by replacing each edge of G by a pair of parallel edges. If G is 3/2-tough, then G^* contains a spanning bipartite subgraph G' with the parts A and B such that the degree of every vertex of A in G' is three and the degree of every vertex of B in G' is at most two.*

Proof. We first form an auxiliary bipartite graph H as follows: for every vertex $a \in A$, H contains three vertices a_1, a_2 and a_3, and for every vertex $b \in B$, H contains two vertices b_1 and b_2. Vertices a_i and b_j are joined by an edge if the vertices a and b are adjacent in G. Let A' and B' be the parts of G' comprised of the vertices corresponding to A and B, respectively.

We show that H contains a matching M covering the vertices of A'. If this is not the case, then (by Hall's theorem, see e.g. [4] if necessary) there exists a subset $A'_0 \subseteq A'$ such that $|N_H(A'_0)| < |A'_0|$. Let A_0 be the set of the vertices a such that A'_0 contains at least one vertex of the triple corresponding to a. Clearly, $|A_0| \geq |A'_0|/3$. Let $B_0 = N_G(A_0)$. By the construction of H, $|B_0| = |N_H(A'_0)|/2$.

We infer the following:

$$|N_G(A_0)| = |B_0| = |N_H(A_0')|/2 < |A_0'|/2 \leq 3|A_0|/2 \; .$$

Observe that the vertices of A_0 are isolated in the graph $G \setminus B_0$. Since $|B_0| < 3|A_0|/2$, the graph G is not 3/2-tough.

We have shown that H contains a matching M that covers the vertices of A'. Consider the following subgraph G' of G^*: two vertices $a \in A$ and $b \in B$ are joined by an edge if M contains an edge $a_i b_j$ for some $i \in \{1,2,3\}$ and $j \in \{1,2\}$. If there are two such edges in M, G' contains the pair of parallel edges between a and b. The graph G' contains no edges between vertices of B. Since M covers the vertices of A', the degree of each vertex $a \in A$ is three in G'. On the other hand, by the construction of H, the degree of each vertex $b \in B$ is at most two in G'. □

The core of our argument is the matroid intersection theorem. We refer the reader to the monographs [13, 14] for an introduction to matroid theory.

Theorem 2 (Matroid intersection theorem). *Let M_1 and M_2 be two matroids on the same support set X. There exists a subset of X of size N that is independent in both M_1 and M_2 if and only if the following holds for all subsets Y of X:*

$$r_{M_1}(Y) + r_{M_2}(X \setminus Y) \geq N$$

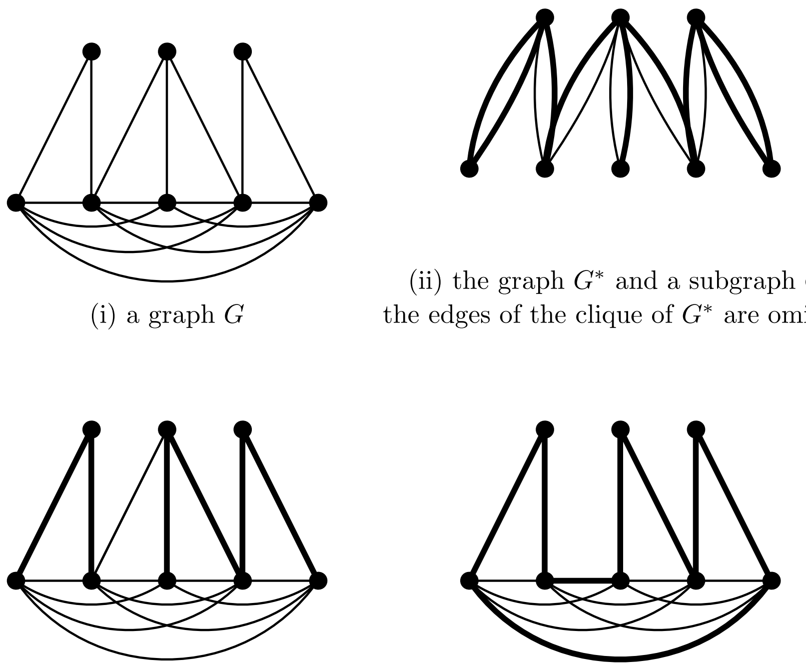
where r_{M_i} is the rank function of the matroid M_i.

We now establish another auxiliary lemma:

Lemma 3. *Let G be a split graph with the parts A and C where A is the independent set and C is the clique. If G is 3/2-tough, then G contains a subgraph G'' comprised of disjoint paths such that each vertex of A is an internal vertex of a path of G''.*

Proof. Let G' be a subgraph of G^* with the properties described in the statement of Lemma 1. We define two matroids M_1 and M_2 on the set $E(G')$. The matroid M_1 is the cycle matroid of G', i.e., a set $Y \subseteq E(G')$ is independent in M_1 if and only if it is acyclic. In particular, the set comprised of two parallel edges is dependent. The second matroid M_2 is a special type of transversal matroid defined as follows: a set $Y \subseteq E(G')$ is independent in M_2 if and only if each vertex of A is incident with at most two edges of Y.

If there exists a set $Y \subseteq E(G')$ of size $2|A|$ that is independent in both M_1 and M_2, then the edges of Y form a subgraph G'' of G that is comprised of disjoint paths such that each vertex of A is an internal vertex of a path of G''. Indeed, since Y is independent in M_1, G'' is acyclic (and simple). Since the size of Y is $2|A|$ and Y is independent in M_2, each vertex of A is incident with exactly two



(i) a graph G

(ii) the graph G^* and a subgraph G'
the edges of the clique of G^* are omitted

(iii) a subgraph G''

(iv) a Hamilton cycle

Figure 1: Main steps of the construction of a Hamilton cycle in a split graph.

edges of Y. Finally, since the degrees of vertices of C in G^* are at most two, the subgraph G'' is comprised of one or more disjoint paths.

We now verify the condition of Theorem 2 for the existence of a set of size $2|A|$ independent in both M_1 and M_2. Assume that the condition is violated for a set $Y \subseteq E(G')$, i.e., $r_{M_1}(Y) + r_{M_2}(E(G') \setminus Y) < 2|A|$. First, we show that we can assume that each vertex v of A is incident to none or three edges of Y. If v is incident with one edge, then remove this edge from Y. This removal decreases the rank of Y in M_1 by one and preserves the rank of its complement in M_2. Hence, the new set also violates the condition of Theorem 2. If v is incident with two edges of Y, include the remaining edge incident with v into the set Y. This decreases the rank of $E(G') \setminus Y$ in M_2 by one and increases the rank of Y in M_1 by at most one. Again, the new set also violates the inequality.

Let $C_1, \ldots, C_m$ be the components of the subgraph spanned by the edges of Y. Let k be the number of vertices of A incident with three edges of Y and let k_i be the number of such vertices contained in the component C_i, $i = 1, \ldots, m$. The rank of $E(G') \setminus Y$ in M_2 is $2(|A| - k)$. We bound the rank of Y in M_1, which equals to the sum of sizes of spanning trees of the components $C_1, \ldots, C_m$. The component C_i contains k_i vertices of A and at least $3k_i/2$ vertices of B (because of the degree condition). Hence, the size of its spanning tree is at least

$\lceil 5k_i/2 \rceil - 1 \geq 2k_i$. We can now conclude that:

$$r_{M_1}(Y) + r_{M_2}(E(G') \setminus Y) \geq \sum_{i=1}^{m} k_i + 2(|A| - k) = 2|A| \ .$$

This contradicts our assumption that Y violates the condition of Theorem 2 and consequently the existence of the desired subgraph G'' is established. □

We are now ready to provide a proof that 3/2-tough split graphs are hamiltonian:

Theorem 4. *Every 3/2-tough split graph G contains a Hamilton cycle.*

Proof. Let A be the part of G corresponding to the independent set. By Lemma 3, G contains a subgraph G'' comprised of disjoint paths such that every vertex of A is an internal vertex of a path G''. Since the end vertices of the paths are all contained in the part corresponding to the clique of G, they can be joined together. We conclude that G is hamiltonian. □

3 Hamilton cycles in interval graphs

When proving our main result on the existence of Hamilton cycles in spider graphs, we will be facing two obstacles while extending the proof presented in Section 2 to spider graphs. The first is that the vertices not contained in the "central clique" do not form an independent set but rather an union of interval graphs. In this section, we study hamiltonicity of interval graphs in order to construct such cycles with additional properties needed for arguments later. We start by recalling a result from [11] on the existence of Hamilton cycles in interval graphs that was mentioned in Section 1:

Theorem 5. *An interval graph G is hamiltonian if and only if G is 1-tough.*

Using Theorem 5, we show that there exist Hamilton cycles of a special type in certain 1-tough interval graphs:

Lemma 6. *Let G be the intersection graph of subpaths of a path $P = w_1 \cdots w_\ell$, $\ell \geq 2$. Let $v_1, \ldots, v_n$, $n \geq 3$, be the vertices of G and $P_1, \ldots, P_n$ subpaths of P corresponding to the vertices $v_1, \ldots, v_n$. Assume that the subpath P_1 consists of the vertex w_1 alone and P_1 is the only such subpath among $P_1, \ldots, P_n$. Let P_α be the shortest subpath containing w_1 that is different from P_1, and P_β be the longest such subpath. If G is 1-tough, then G contains a Hamilton cycle on which the vertices v_α, v_1 and v_β are consecutive.*

Proof. Note first that $\alpha \neq \beta$ since G is 1-tough. We modify G into a different interval graph G'. Let us define a new collection of subpaths of P:

$$P'_i = \begin{cases} P_i & \text{if } i \in \{1, \alpha, \beta\}, \\ P_i & \text{if } w_1 \notin P_i, \text{ and} \\ P_i \setminus w_1 & \text{otherwise.} \end{cases}$$

Let G' be the intersection graph of subpaths $P'_1, \ldots, P'_n$ of the path P. The vertices of G' can be identified with the vertices of G, and G' then forms a subgraph of G. Note that the vertex v_1 has degree two in G' and its only two neighbors are the vertices v_α and v_β.

We show that G' is 1-tough. Let A be a subset of $V(G')$. Next, we distinguish two cases based on whether A contains the vertex v_α or not. If $v_\alpha \notin A$, then the components of $G \setminus A$ and $G' \setminus A$ are precisely the same: each path from which w_1 was removed can be prolonged back to w_1 without changing the structure of $G' \setminus A$. Since G is 1-tough, $\kappa(G' \setminus A) = \kappa(G \setminus A) \leq |A|$.

The other case to consider is that $v_\alpha \in A$. Let $A' = A \setminus \{v_\alpha\}$ and let B be the set of all vertices of G corresponding to the paths of length at least two that contain the vertex w_1. We have already established that $\kappa(G' \setminus A') \leq |A'|$. Assume that $\kappa(G' \setminus A) > |A|$. Then, the vertex v_α is adjacent to at least three different components of $G' \setminus A$. Let C_1 and C_2 be two of the components of $G' \setminus A$ such that neither C_1 nor C_2 contains the vertex v_1.

We claim that $B \subseteq A$: otherwise, consider a vertex $b \in B \setminus A$. Since P_α is the shortest path containing w_1 among $P_2, \ldots, P_n$, it holds that $N_G(v_\alpha) \subseteq N_G(b)$. Consequently, $N_{G'}(v_\alpha) \setminus \{v_1\} \subseteq N_{G'}(b)$, b is adjacent to a vertex of C_1 as well as to a vertex of C_2 in G' and C_1 and C_2 are not two distinct components of $G' \setminus A$. We conclude that A contains all vertices v_i such that P_i contains w_1. Hence, $B \subseteq A$ and the components of $G \setminus A$ and $G' \setminus A$ are the same. The assumption that G is 1-tough yields that $\kappa(G' \setminus A) = \kappa(G \setminus A) \leq |A|$.

Since G' is 1-tough, it has a Hamilton cycle by Theorem 5. Such a Hamilton cycle is also a Hamilton cycle of G and the vertices v_α, v_1 and v_β are consecutive in the cycle since the only two neighbors of v_1 in G' are the vertices v_α and v_β. The lemma has been established. ◻

In our considerations in Section 4, we will be adding new vertices to interval graphs in order to increase their toughness. If G is the intersection graph of subpaths $P_1, \ldots, P_n$ of a path $P = w_1 \cdots w_\ell$, there are several different ways of adding new paths such that the resulting intersection graph is 1-tough. However, if we assume that all the new paths contain the vertex w_1, then there is a unique minimal set of paths with this property. We state more precisely and prove this claim in what follows. In order to simplify our notation, we write $[w_i, w_j]$ for the subpath of P between the vertices w_i and w_j (inclusively) and $\langle P_1, \ldots, P_n \rangle$ denotes the intersection graph of subpaths $P_1, \ldots, P_n$.

Lemma 7. *Let G be the graph $\langle P_1, \ldots, P_n \rangle$ for a family of subpaths $P_1, \ldots, P_n$ of a path $P = w_1 \cdots w_\ell$, $\ell \geq 2$, such that G is not 1-tough and assume that $P_1 = [w_1, w_1]$. There exist integers $\ell \geq k_1 \geq \cdots \geq k_r \geq 2$ with the following property: the graph $\langle P_1, \ldots, P_n, Q_1, \ldots, Q_r \rangle$ for $Q_1 \supseteq \cdots \supseteq Q_r \supseteq [w_1, w_1]$ is 1-tough if and only if $[w_1, w_{k_i}] \subseteq Q_i$ for every $i = 1, \ldots, r$.*

Note that it can be assumed that $P_1 = [w_1, w_1]$ without loss of generality since every intersection graph of subpaths of a path can be modified to an intersection graph satisfying this condition.

The integers $k_1, \ldots, k_r$ can be computed in the following way: for $i = 2, \ldots, \ell$, let τ_i be the following maximum:

$$\tau_i = \begin{cases} \max\limits_{A \subseteq V(G)} \kappa(\langle P_1, \ldots, P_n, [w_1, w_{i-1}] \rangle \setminus A) - |A| \\ \qquad\qquad\qquad \text{if } \langle P_1, \ldots, P_n, [w_1, w_{i-1}] \rangle \text{ is disconnected, and} \\ \max\limits_{A \subseteq V(G), A \neq \emptyset} \kappa(\langle P_1, \ldots, P_n, [w_1, w_{i-1}] \rangle \setminus A) - |A| \qquad \text{otherwise.} \end{cases}$$

It is important that the maximum is taken over subsets of the vertices of G, not of the vertices of $\langle P_1, \ldots, P_n, [w_1, w_{i-1}] \rangle$. It is easy to see that all τ_i are non-negative integers and $\tau_2 \geq \tau_3 \geq \cdots \geq \tau_\ell \geq 0$. Moreover, if the graph $\langle P_1, \ldots, P_n, Q_1, \ldots, Q_r \rangle$ from Lemma 7 is 1-tough, then at least τ_i of the paths $Q_1, \ldots, Q_r$ must contain the subpath $[w_1, w_i]$, i.e., it must hold that $k_{\tau_i} \geq i$. In fact, this fully determines the integers $k_1, \ldots, k_r$ (see Figures 2 and 3 for an example) as stated in the next lemma. Let us recall that two sequences $a_1, \ldots, a_A$ and $b_1, \ldots, b_B$ of integers are conjugate if $A = b_1$, $B = a_1$, a_i is the number of b_j's greater or equal to i and b_i is the number of a_j's greater or equal to i.

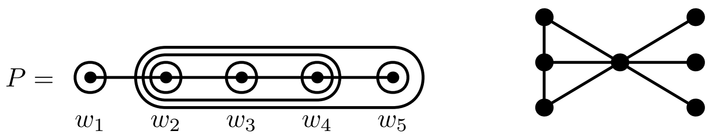
Lemma 8. *Let $G = \langle P_1, \ldots, P_n \rangle$ as in Lemma 7 and let $\tau_2, \ldots, \tau_\ell$ be the numbers defined as above. The sequence $k_1, \ldots, k_r$ from Lemma 7 is conjugate to the sequence $\tau_2, \tau_2, \ldots, \tau_\ell$, i.e., the number r is equal to τ_2 and exactly $\tau_i - \tau_{i+1}$ of the numbers k_j are equal to i.*

Since the sequences $k_1, \ldots, k_r$ and $\tau_2, \tau_2, \ldots, \tau_\ell$ are conjugate, the following lemma immediately follows:

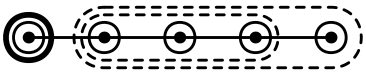
Lemma 9. *It holds that $\tau_{k_i+1} \leq i - 1$ for every $i = 1, \ldots, r$ (setting $\tau_{\ell+1} = 0$).*

In order to further simplify the notation used in the proof of Lemmas 7 and 8, we set $\mathcal{P} = \{P_1, \ldots, P_n\}$ and $\mathcal{Q} = \{Q_1, \ldots, Q_r\}$ and write, e.g., $\langle \mathcal{P}, \mathcal{Q} \rangle$ for the intersection graphs of paths contained in the sets $\mathcal{P}$ and $\mathcal{Q}$. We now proceed with the postponed proof:

Proof of Lemmas 7 and 8. First, we establish that if the condition of Lemma 7 is violated, then $\langle \mathcal{P}, \mathcal{Q} \rangle$ is not 1-tough. Assume that the path Q_i does not contain $[w_1, w_{k_i}]$. Since the sequences $\tau_2, \tau_2, \ldots, \tau_\ell$ and $k_1, \ldots, k_r$ are conjugate, less than

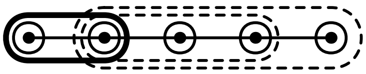


(i) the graph $G = \langle P_1, \ldots, P_7 \rangle$ and its intersection representation



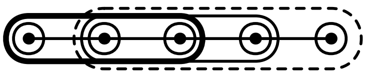
$\tau_2 = 5 - 2 = 3$

(ii) the representaion of the graph $\langle P_1, \ldots, P_7, [w_1, w_1] \rangle$



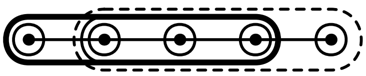
$\tau_3 = 4 - 2 = 2$

(iii) the representaion of the graph $\langle P_1, \ldots, P_7, [w_1, w_2] \rangle$



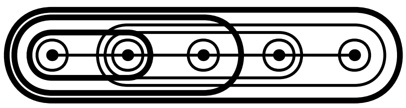
$\tau_4 = 2 - 1 = 1$

(iv) the representaion of the graph $\langle P_1, \ldots, P_7, [w_1, w_3] \rangle$



$\tau_5 = 2 - 1 = 1$

(v) the representaion of the graph $\langle P_1, \ldots, P_7, [w_1, w_4] \rangle$



$k_1 = 5$, $k_2 = 3$ and $k_3 = 2$

(vi) the representation of the "minimal" 1-tough supergraph of G

Figure 2: An example of computation of the integers $k_1, \ldots, k_r$ and $\tau_2, \tau_2, \ldots, \tau_\ell$ from Lemmas 7 and 8 (also see Figure 3). The subpaths $P_1, \ldots, P_7$ are depicted by ovals, the added subpaths by bold ovals and the elements of sets witnessing the equalities in the maximum in the definition of τ_i by dashed ovals.

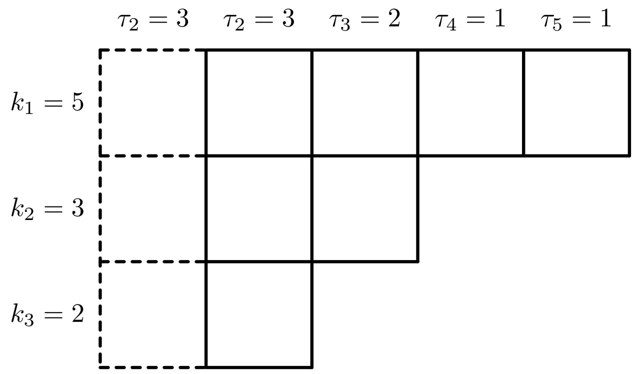


Figure 3: The conjugation of the sequences $k_1, \ldots, k_r$ and $\tau_2, \tau_2, \ldots, \tau_\ell$ from the example in Figure 2.

τ_{k_i} of the paths $Q_1, \ldots, Q_r$ contain $[w_1, w_{k_i}]$. Let A' be the set of the vertices corresponding to the subpaths Q_j that contain the vertex w_{k_i}. Note that $|A'| < \tau_{k_i}$.

By the definition of τ_{k_i}, there exists a set $A \subseteq V(G)$ such that the graph $\langle \mathcal{P}, [w_1, w_{k_i-1}] \rangle \setminus A$ has at least $|A| + \tau_{k_i}$ components. Note that it must hold that $|A| + \tau_{k_i} \geq 2$: if the graph $\langle \mathcal{P}, [w_1, w_{k_i-1}] \rangle$ is disconnected, then $\tau_{k_i} \geq 2$ follows directly from the definition, and if the graph is connected, then $|A| \geq 1$ (the maximum in the definition is taken only over non-empty sets A) and $\tau_{k_i} \geq 1$ (if $\tau_{k_i} = 0$, all the numbers $k_1, \ldots, k_r$ would be at least $k_i + 1$). Since A' contains all the paths among $Q_1, \ldots, Q_r$ that are superpaths of $[w_1, w_{k_i}]$, it follows that

$$\kappa(\langle \mathcal{P}, \mathcal{Q} \rangle \setminus (A \cup A')) \geq \kappa(\langle \mathcal{P}, [w_1, w_{k_i-1}] \rangle \setminus A).$$

The inequalities $\kappa(\langle \mathcal{P}, [w_1, w_{k_i-1}] \rangle \setminus A) \geq |A| + \tau_{k_i}$ and $|A \cup A'| < |A| + \tau_{k_i}$ together yield that the graph $\langle \mathcal{P}, \mathcal{Q} \rangle$ is not 1-tough.

Assume now that the condition of Lemma 7 is satisfied. Clearly, it is enough to show that the graph $\langle \mathcal{P}, [w_1, w_{k_1}], \ldots, [w_1, w_{k_r}] \rangle$ is 1-tough. Let us consider a subset A of the vertices of the graph $\langle \mathcal{P}, [w_1, w_{k_1}], \ldots, [w_1, w_{k_r}] \rangle$. Let i be the smallest index such that the vertex corresponding to the path $[w_1, w_{k_i}]$ is not contained in A. If there is no such i, set $i = r + 1$ (and for completeness, define $k_{r+1} = 1$). If $k_i = \ell$, then $\langle \mathcal{P}, [w_1, w_{k_1}], \ldots, [w_1, w_{k_r}] \rangle \setminus A$ is connected and there is nothing to prove. By the choice of i, the set A contains at least $i - 1$ vertices corresponding to the subpaths $[w_1, w_{k_j}]$.

Let $A' = A \cap V(G)$. By the choice of i, we have the following:

$$\kappa(\langle \mathcal{P}, [w_1, w_{k_1}], \ldots, [w_1, w_{k_r}] \rangle \setminus A) = \kappa(\langle \mathcal{P}, [w_1, w_{k_i}] \rangle \setminus A).$$

If the graph $\langle \mathcal{P}, [w_1, w_{k_i}] \rangle$ is connected and the set A' is empty, then the graph $\langle \mathcal{P}, [w_1, w_{k_1}], \ldots, [w_1, w_{k_r}] \rangle \setminus A$ is connected and there is nothing to prove. If

$\langle \mathcal{P}, [w_1, w_{k_i}] \rangle$ is disconnected or A' is non-empty, then the following holds by Lemma 9:

$$\kappa(\langle \mathcal{P}, [w_1, w_{k_i}] \rangle \setminus A') \leq |A'| + \tau_{k_i+1} \leq |A'| + i - 1 \leq |A|.$$

We conclude that the graph $\langle \mathcal{P}, [w_1, w_{k_1}], \ldots, [w_1, w_{k_r}] \rangle$ is 1-tough. □

4 Hamilton cycles in spider graphs

The proof of our main result consists of three steps as in the case of the proof presented in Section 2. In the first step (that corresponds to Lemma 1), we assign each leg of a spider vertices corresponding to subtrees containing the central vertex in such a way that each leg is assigned a sufficient number of such vertices but each such vertex is assigned to at most two legs. In the second step (the counterpart of Lemma 3), we use the assignment to build paths covering all the vertices corresponding to subtrees not containing the central vertex of the spider. In the final step, we connect the paths to obtain a Hamilton cycle. The next lemma describes the first step of the proof:

Lemma 10. *Let G be the intersection graph of subtrees $T_1, \ldots, T_n$ of a spider S with m legs of lengths $\ell_1, \ldots, \ell_m$. Let $L_i = w_1^i \cdots w_{\ell_i}^i$ be the legs of S where $w_1^1 = \cdots = w_1^m$ is the central vertex of the spider. Assume that exactly the first n_0 subtrees among $T_1, \ldots, T_n$ contain the central vertex of S. Let G_i, $i = 1, \ldots, m$, be the intersection graph formed by the subtrees of S fully contained in the path $w_2^i \cdots w_{\ell_i}^i$ and the single-vertex subtree formed by the vertex w_1^i. Finally, let $2 \leq k_1^i \leq \cdots \leq k_{r_i}^i$ be the numbers from the statement of Lemma 7 for the graph G_i.*

If the graph G is $3/2$-tough and each G_i consists of precisely two components, then there exists an assignment of subtrees $T_1, \ldots, T_{n_0}$ to the legs of S with the following properties (it is allowed to assign the same subtree twice to the same leg):

1. *each leg L_i is assigned at least $3j - 2$ subtrees containing the vertex $w_{k_j^i}^i$ for $j = 1, \ldots, r_i - 1$,*
2. *each leg L_i is assigned at least $3r_i - 3$ subtrees containing the vertex $w_2^i = w_{k_{r_i}^i}^i$, and*
3. *each subtree T_i is assigned at most twice.*

At the first sight, one would expect that a leg L_i should be assigned $3j$ subtrees containing the vertex $w_{k_j^i}^i$, i.e., three subtrees per each vertex $w_{k_j^i}^i$. However, this requirement is too strong and there are 3/2-tough spider graphs that do not admit such an assignment. We will overcome the deficit in the number of assigned subtrees using Lemma 6 later.

Proof of Lemma 10. As in the proof of Lemma 1, the core of our argument is Hall's theorem. We first construct an auxiliary bipartite graph H with parts A and B. The part A consists of $\sum_{i=1}^{m}(3r_i - 3)$ vertices u^i_j where $i = 1, \ldots, m$ and $j = 1, \ldots, 3r_i - 3$ (they correspond to the demands of the legs). The part B consists of n_0 vertices $u'_1, \ldots, u'_{n_0}$ corresponding to the subtrees $T_1, \ldots, T_{n_0}$. A vertex u^i_j is joined by an edge to a vertex $u'_{i'}$ if the subtree $T_{i'}$ contains the vertex $w^i_{k^i_{\lceil \frac{j+2}{3} \rceil}}$, i.e., the subtree $T_{i'}$ can fulfil the demand corresponding to the integer $k^i_{\lceil \frac{j+2}{3} \rceil}$. Clearly, the desired assignment exists if and only if H contains a subgraph H' such that the degree of each vertex of A is one in H' and the degree of each vertex of B is at most two (assign each leg L_i the subtrees matched to the vertices $u^i_1, \ldots, u^i_{3r_i-3}$). By Hall's theorem, the existence of the subgraph H' is equivalent to the following:

$$|A'|/2 \leq |N_H(A')| \text{ for every subset } A' \subseteq A.$$

Assume that the inequality is violated for a subset A' of A. Under this assumption, we construct a non-empty subset V of vertices of G such that $\kappa(G \setminus V) > 3|V|/2 > 1$.

Let V_0 be the set of vertices of G corresponding to the vertices of H contained in $N_H(A')$. In the following paragraphs, for each $i = 1, \ldots, k$, we construct a subset V_i of the vertices corresponding to subpaths of the path $w^i_2 \cdots w^i_{\ell_i}$ and define a number d_i. Consider an integer $i = 1, \ldots, m$. If A' contains none of the vertices $u^i_1, \ldots, u^i_{3r_i-3}$, set $V_i = \emptyset$ and $d_i = 0$. Otherwise, let $d_i = |A' \cap \{u^i_1, \ldots, u^i_{3r_i-3}\}|$. By the construction of H, we may assume without loss of generality that $A' \cap \{u^i_1, \ldots, u^i_{3r_i-3}\} = \{u^i_1, \ldots, u^i_{d_i}\}$. Moreover, d_i can be assumed to be equal to either $3j - 2$ for some j or $3r_i - 3$. We deal with these two cases separately.

We first consider the case when $d_i = 3j - 2$ for some $j \in \{1, \ldots, r_i - 1\}$. It can be assumed without loss of generality that $k^i_j > 2$ (otherwise, add the vertices $u^i_{d_i+1}, \ldots, u^i_{3r_i-3}$ to A'; since this preserves $N_H(A')$, we can proceed with the enlarged set A'). Let G'_i be the intersection graph of subpaths of the path $w^i_2 \cdots w^i_{\ell_i}$ (fully contained in this path) and the paths $[w^i_1, w^i_1]$ and $[w^i_1, w^i_{k^i_j - 1}]$. Since $k^i_j > 2$ and G'_i is comprised of two components (one of them being the isolated vertex corresponding to the subpath $[w^i_1, w^i_1]$), G'_i is connected. By Lemma 8, there exists a non-empty subset V_i of vertices of G_i such that $j \leq \kappa(G'_i \setminus V_i) - |V_i|$. Clearly, V_i does not contain the added vertex corresponding to the trivial path $[w^i_1, w^i_1]$. The graph $G \setminus (V_0 \cup V_i)$ contains at least $j + |V_i| - 1$ components comprised only of the vertices corresponding to the subpaths of the path $w^i_2 \cdots w^i_{\ell_i}$. A simple calculation yields that the number of such components is at least the following (recall that V_i is non-empty):

$$j + |V_i| - 1 = d_i/3 + 2/3 + |V_i| - 1 \geq d_i/3 + 2|V_i|/3 .$$

In the latter case, $d_i = 3r_i - 3$. Since G_i is disconnected, $r_i \geq 2$. By Lemma 8, there exists a set V_i of vertices of G_i such that $r_i = \kappa(G_i \setminus V_i) - |V_i|$. Clearly, V_i does not contain the added vertex corresponding to the trivial path $[w_1^i, w_1^i]$. Note that the graph $G \setminus (V_0 \cup V_i)$ contains at least $r_i + |V_i| - 1 \geq d_i/3 + |V_i|$ components comprised only of the vertices corresponding to the subpaths of the path $w_2^i \cdots w_{\ell_i}^i$.

Let us summarize the properties of $V_0, \ldots, V_m$ and the numbers d_i:

- $|V_0| < |A'|/2 = \sum_{i=1}^m d_i/2$ (by the definitions of V_0 and d_i), and
- the graph $G \setminus (V_0 \cup V_i)$ contains at least $2|V_i|/3 + d_i/3$ components comprised only of vertices corresponding to the subpaths of the path $w_2^i \cdots w_{\ell_i}^i$.

Set now $V = V_0 \cup \cdots \cup V_m$. The size of V is bounded by the following:

$$|V| < \sum_{i=1}^{m} (|V_i| + d_i/2) \tag{1}$$

On the other hand, the graph $G \setminus V$ consists of at least the following number of components:

$$\sum_{i=1}^{m} (2|V_i|/3 + d_i/3) \tag{2}$$

Comparing (1) and (2) yields an immediate contradiction to the fact that G is 3/2-tough. □

In the second step of the proof, we connect Hamilton cycles to the vertices of the central clique to obtain a set of disjoint paths as in the proof for split graphs. The general idea behind this step of the proof is the following: we remove the vertices of the central clique from the resulting Hamilton cycles and obtain several paths (whose end-vertices are adjacent to the vertices of the central clique). However, we need to be careful to enter and leave the resulting paths at their different ends. In order to achieve this, we will proceed similarly as in Lemma 3 but we require in addition that a certain matching of H is contained in the paths:

Lemma 11. *Let H be a bipartite multigraph with parts A and B and let $F \subseteq E(H)$ be a matching covering the vertices of A. If each vertex of A has degree three and each vertex of B has degree at most two, then there exists a subgraph H' of H comprised of disjoint paths such that $F \subseteq E(H')$ and the degree of every vertex of A in H' is two.*

Proof. As in the proof of Lemma 3, we define two matroids M_1 and M_2 on the set $E(H)$. The matroid M_1 is again the cycle matroid H. The second matroid M_2 is a special type of transversal matroid defined as follows: a set $Y \subseteq E(H)$ is independent in M_2 if and only if each vertex of A is incident with at most one edge of $Y \setminus F$. If $Y \subseteq E(H)$ is a set of size $2|A|$ that is independent in both M_1 and M_2, then each vertex of A is incident with two edges of Y. In particular (see the proof of Lemma 3), the edges of Y form the desired subgraph H'.

We now verify the condition of Theorem 2 for the existence of a set of size $2|A|$ that is independent in both M_1 and M_2. Assume that the condition is violated for a set $Y \subseteq E(H)$, i.e., $r_{M_1}(Y) + r_{M_2}(E(H) \setminus Y) < 2|A|$. First, we show that we can assume that each vertex v of A is incident to none or three edges of Y. If v is incident with one edge, then remove this edge from Y. This decreases the rank of Y in M_1 by one and increases the rank of its complement in M_2 by at most one. If v is incident with two edges of Y, include the remaining edge incident with v into the set Y. This decreases the rank of $E(H) \setminus Y$ in M_2 by one and increases the rank of Y in M_1 by at most one. In both the cases, the new set violates the condition.

The rest of the proof is the same as the proof of Lemma 3: we consider the components $C_1, \ldots, C_m$ of the subgraph spanned by Y. Let k_i be the number of vertices of A contained in the component C_i, $i = 1, \ldots, m$. The rank of Y in M_1 is the sum of sizes of spanning trees of the components $C_1, \ldots, C_m$ that are at least $2k_i$ each (see the proof of Lemma 3). The rank of $E(H) \setminus Y$ in M_2 is $2(|A| - \sum_{i=1}^{m} k_i)$. We can now conclude that:

$$r_{M_1}(Y) + r_{M_2}(E(H) \setminus Y) \geq \left(\sum_{i=1}^{m} 2k_i \right) + 2(|A| - \sum_{i=1}^{m} k_i) = 2|A| \; .$$

Hence, the condition of Theorem 2 is fulfilled and the existence of a subgraph H' is established. ☐

We are now ready to prove our main theorem:

Theorem 12. *Every $3/2$-tough spider graph G is hamiltonian.*

Proof. We use the notation of Lemma 1: G is the intersection graph of subtrees $T_1, \ldots, T_n$ of a spider S with m legs $L_i = w_1^i \cdots w_{\ell_m}^i$ where $w_1^1 = \cdots = w_1^m$ is the central vertex of S. Assume that exactly the first n_0 subtrees contain the central vertex and let V_c be the set of the vertices of G corresponding to these subtrees. We can also assume without loss of generality that the subgraph of G that is the intersection graph of the subtrees that are subpaths of the path $w_2^i \cdots w_{\ell_k}^i$ is connected for every $i = 1, \ldots, k$: if this is not the case, the spider S can be modified to a spider with more legs representing each connected component in a single leg. In addition, we can also assume that there is a subtree T_j for each leg L_i that is a subpath of L_i and T_j does not contain the central vertex (if this is not the case, the leg L_i can be cut without changing the structure of the graph G).

Let G_i be the interval graph defined in Lemma 10. Observe that each G_i consists of exactly two components, and one of them is the isolated vertex corresponding to the single-vertex path $[w_1^i, w_1^i]$. Let r_i and $k_1^i, \ldots, k_{r_i}^i$ be the integers as in Lemma 10. Since G is 3/2-tough, each leg of S can be assigned a non-empty set of the subtrees $T_1, \ldots, T_{n_0}$ in the way that is described in Lemma 10. Since $r_i \geq 2$ for every i (this follows from the fact that each L_i contains a subpath T_j that does not contain the central vertex), each leg is assigned some of the subtrees.

Next, we construct another auxiliary interval graph G'_i for each $i = 1, \ldots, m$. G'_i is the intersection graph of the subpaths that form the graph G_i and the paths $[w^i_1, w^i_{k^i_1}], \ldots, [w^i_1, w^i_{k^i_{r_i}}]$. By Lemma 7, G'_i is 1-tough. Let $u^i_1, \ldots, u^i_{r_i}$ be the vertices of G'_i corresponding to the paths $[w^i_1, w^i_{k^i_1}], \ldots, [w^i_1, w^i_{k^i_{r_i}}]$ and u^i_0 be the vertex corresponding to the single-vertex path $[w^i_1, w^i_1]$. By Lemma 6, G'_i contains a Hamilton cycle C_i in which the vertices $u^i_{r_i}$, u^i_0 and u^i_1 are consecutive.

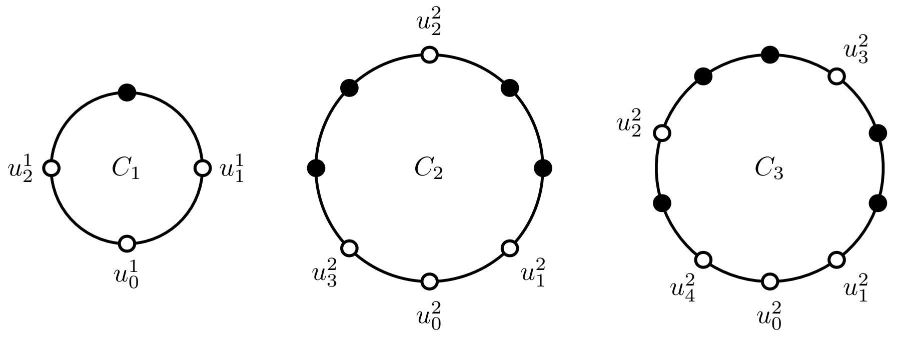
The vertices corresponding to the subtrees assigned to the leg L_i are partitioned into (multi)sets $U^i_1, \ldots, U^i_{r_i}$ as follows (the reader is invited to consult Figure 4): The set U^i_1 consists of the vertex corresponding to the subtree containing the vertex $w^i_{k_1}$, each U^i_j for $j = 2, \ldots, r_i - 1$ contains three vertices corresponding to subtrees that contain the vertex $w^i_{k_j}$ and $U^i_{r_i}$ consists of the two vertices containing $w^i_{k_{r_i}}$. The partitioning is always possible by the properties stated in Lemma 10. Note that the partitioning does not need to be unique.

Next, we modify the cycles C_i. Let us orient the edges of each C_i in such a way that the arc from $u^i_{r_i}$ leads to u^i_0 (and thus the arc from u^i_0 leads to u^i_1). The cycle C_i is modified as follows:

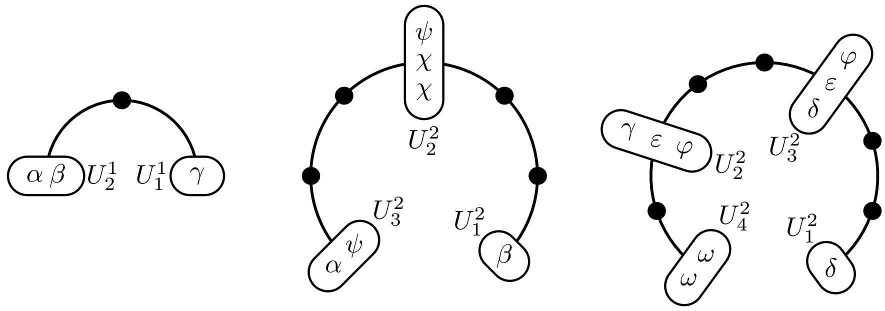
- the vertices u^i_0, u^i_1 and $u^i_{r_i}$ are removed from the cycle,
- a blue edge is added from the predecessor of $u^i_{r_i}$ in C_i to each of the two vertices contained in $U^i_{r_i}$ (in case the same vertex is contained twice in $U^i_{r_i}$, add two parallel edges),
- a red edge is added from the (single) vertex contained in U^i_1 to the successor of u^i_1 in C_i,
- each of the vertices u^i_j, $j = 2, \ldots, r_i - 1$, is removed from C_i, a red edge is added from one of the vertices of U^i_j to the successor of u^i_j in C_i and two blue edges are added from the predecessor of u^i_j to the remaining two vertices contained in U^i_j (adding parallel edges if appropriate).

We have obtained a collection of paths in G such that the internal vertices of the paths are all the vertices of $V(G) \setminus V_c$. Each of the paths starts with a red edge leading from a vertex of V_c and ends with a fork of two blue edges leading to two vertices of V_c. Since each subtree was assigned to at most two legs, each vertex of V_c is incident with at most two colored edges. Let N be the number of these paths. Note that $N \geq m$.

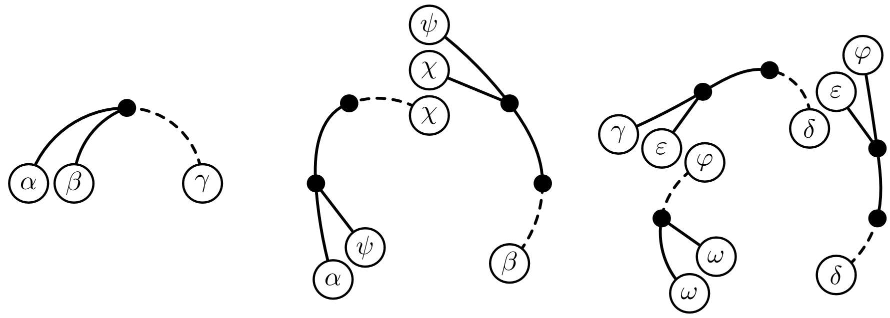
We now construct an auxiliary graph H. Consider the subgraph of G formed by the N constructed paths. Contract all the inner vertices of each path to a



(i) Hamilton cycles in graphs G'_1, G'_2 and G'_3



(ii) partitioning V_c into sets U_j^i



(iii) the collection of paths (red edges are dashed)

Figure 4: An example of the construction of paths in the proof of Theorem 12. Vertices contained in the central clique (i.e., those of the set V_c) are represented by Greek letters. Vertices corresponding to subtrees fully contained in the legs of the spider are drawn as solid circles.

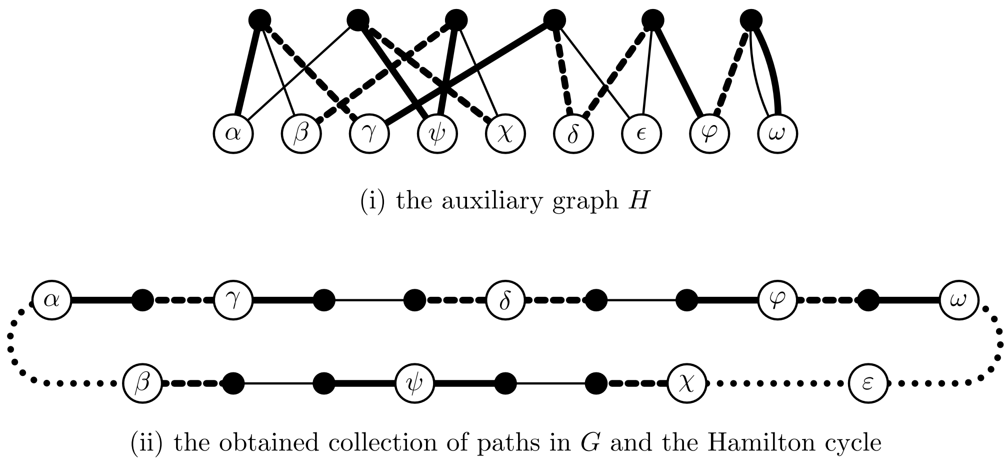


Figure 5: The auxiliary graph H from the proof of Theorem 12 constructed from the paths depicted in the example in Figure 4. The edges of the subgraph H' are drawn bold, the red edges are dashed, and the edges interconnecting resulting paths are drawn with dots.

single vertex and remove all the edges that are neither red nor blue. The resulting graph is H (see Figure 5). Note that H is a bipartite graph with a part A formed by the vertices corresponding to the N paths and a part B formed by the vertices of V_c. Each vertex of A is incident with a single red edge and two blue edges. Let F be the set of red edges. By Lemma 11, there exists a subgraph H' of H that is comprised of paths such that every vertex of A has degree two and is incident with a single red and a single blue edge. After decontracting the paths, we obtain a collection of (disjoint) paths in G with the following property:

- each vertex of $V(G) \setminus V_c$ is contained in a path, and
- every path starts and ends in a vertex of V_c.

Since the vertices of V_c form a clique in G (the subtrees corresponding to them contain the central vertex of the spider), it is trivial to connect the resulting paths into a Hamilton cycle of G. □

Acknowledgement

The authors would like to thank Vít Jelínek, Moshe Rosenfeld, Zdeněk Ryjáček and Florian Zickfeld for interesting discussions on the subject.

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