T-joins intersecting small edge-cuts in graphs

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Abstract

In an earlier paper [3], we studied cycles in graphs that intersect all edge-cuts of prescribed sizes. Passing to a more general setting, we examine the existence of T-joins in grafts that intersect all edge-cuts whose size is in a given set $A \subseteq \{1, 2, 3\}$. In particular, we characterize all the contraction-minimal grafts admitting no T-joins that intersect all edge-cuts of size 1 and 2. We also show that every 3-edge-connected graft admits a T-join intersecting all 3-edge-cuts.

1 Introduction

In [3], we investigated the existence of cycles intersecting each edge-cut whose size is in a given set A of positive integers, and showed that this question is related to several fundamental problems, including Tutte's 4-flow conjecture and the Dominating cycle conjecture. As in [3], we define an *edge-cut* (in short, *cut*) to be an inclusionwise minimal set of edges whose removal increases the number of components, and define a *cycle* as a (possibly disconnected) graph in which all vertices have even degrees. Our graphs are undirected, loopless and may contain multiple edges.

We now take up the above approach and generalize it by replacing cycles with structures known as T-joins. A *graft* is a pair (G, T) where G is a graph (possibly with multiple edges but no loops) and T is a set of vertices of G such that $|T|$ is even. A *T-join* in a graft (G, T) is a spanning subgraph H such that the vertices whose degree in H is odd are precisely those in T. Note that if $T = \emptyset$, then

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T-joins are the cycles of G. A *T-cut* is an edge-cut C such that $G \setminus C$ has more components K with $|T \cap V(K)|$ odd than G does. If (G, T) is a graft, then we refer to T-joins simply as *joins*. This is not the case with cuts.

Given a set $A \subseteq \mathbb{N}$ (where $\mathbb{N}$ is the set of positive integers), when does a graft (G, T) possess a join intersecting all cuts C with $|C| \in A$ (we call such a join *A-covering*)? This is the question studied in the present paper for $A \subseteq \{1, 2, 3\}$. It turns out that the situation is interesting even in this rather specific case.

The property of having an A-covering join is monotone with respect to naturally defined edge contraction. In detail, let (G, T) be a graft and e be an edge with endvertices x and y. The *contraction* of e is the operation that identifies x and y (preserving multiple edges but discarding loops) and whose action on the set T is as follows. Let the new vertex corresponding to e be denoted by v_e. In the resulting graft (G', T'), the set T' is defined as

$$T' = \begin{cases} T \setminus \{x, y\} \cup \{v_e\} & \text{if precisely one of } x \text{ and } y \text{ belongs to } T, \\ T \setminus \{x, y\} & \text{otherwise.} \end{cases}$$

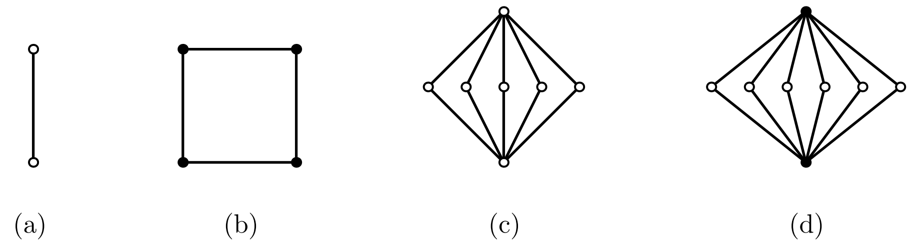
It is easy to check that $|T'|$ is even, and hence that (G', T') is a graft. If H is a subgraph of G, then the contracted graft $(G, T)/H$ is obtained by contracting all the edges of H. Observe that the result is independent of the order in which the edges are contracted.

Note that if (G, T) has an A-covering join, then (G', T') has a naturally obtained A-covering join. This monotonicity property suggests that one may try to characterize the grafts (G, T) with no A-covering join such that (G, T) is minimal with respect to edge contraction. In the present paper, we determine these grafts for $A = \{1, 2\}$. Before stating the result, we introduce names for the minimal grafts. The symbols K_n and $K_{m,n}$ refer to complete graphs and complete bipartite graphs, respectively. The grafts are shown in Figure 1; black dots represent the vertices in T. Correspondingly, we sometimes speak of the vertices in T as *black*, and of the other vertices as *white*.

- $\mathcal{K}_2 = (K_2, \emptyset)$, where K_2 is the complete graph on two vertices,
- $\mathcal{C}_4 = (C_4, V)$, where V is the vertex set of the 4-cycle C_4,
- $\mathcal{K}_{2,2k+1} = (K_{2,2k+1}, \emptyset)$, where $k \geq 1$,
- $\mathcal{K}_{2,2k} = (K_{2,2k}, L)$, where $k \geq 1$ and L is the set of vertices in the color class of size 2.

Observe that none of the above grafts have a $\{1, 2\}$-covering join.

A graft (G_1, T_1) is *contractible* to another graft (G_2, T_2) if the latter can be obtained from (G_1, T_1) by a (possibly void) series of edge contractions. We write $(G_2, T_2) \preceq (G_1, T_1)$. Note that the relation $\preceq$ is transitive (in fact, a partial order).



(a) (b) (c) (d)

Figure 1: The minimal grafts (G, T) with no $\{1, 2\}$-covering joins: (a) $\mathcal{K}_2$; (b) $\mathcal{C}_4$; (c) $\mathcal{K}_{2,2k+1}$ (for $k = 2$); (d) $\mathcal{K}_{2,2k}$ (for $k = 3$). The vertices in T are shown black.

Theorem 1.1. *A graft (G, T) has a $\{1, 2\}$-covering join if and only if it is not contractible to any of the grafts $\mathcal{K}_2$, $\mathcal{C}_4$, $\mathcal{K}_{2,2k+1}$ and $\mathcal{K}_{2,2k}$, where $k \geq 1$.*

For $\{3\}$-covering joins, we have no characterization in the style of Theorem 1.1. However, we proved the following:

Theorem 1.2. *If (G, T) is a graft and G is 3-edge-connected, then (G, T) has a $\{3\}$-covering join.*

We decided to restrict our inquiry to A-covering joins with $A \subseteq \{1, 2, 3\}$. However, most of the problems and conjectures in [3] involving A-covering cycles (for other sets A) might have an extension to A-covering T-joins. As an example, we mention the conjecture from [3] that every graph has an A-covering cycle for $A = \{4, 5, 6, \ldots\}$, which is an extension of the well-known Dominating cycle conjecture [1].

We conclude this section with several definitions. Recall that we refer to (inclusionwise minimal) edge-cuts simply as *cuts*. A cut is *trivial* if it consists of all edges incident with a vertex. For a graft (G, T), a subgraph of G is *T-odd* if it contains an odd number of vertices in T; otherwise, it is *T-even*.

2 Cuts of size 1 and 2

In this section, we prove a characterization of the grafts admitting $\{1, 2\}$-covering joins. The corresponding result for $\{1\}$-covering and $\{2\}$-covering joins is also easy to derive; it is given below as Proposition 2.2.

The following lemma is well known:

Lemma 2.1. *A graft (G, T) contains a join if and only if each of its components is T-even.*

Proof. Necessity follows from the fact that the number of odd degree vertices in any graph is even. We prove sufficiency. Consider a component K of G and choose an arbitrary partition $\mathcal{P}_K$ of the vertices in $T \cap V(K)$ into pairs. For each pair $\{x, y\} \in \mathcal{P}_K$, choose a path P_{xy} joining x to y. Let J_K be the symmetric difference of edge sets of the paths P_{xy} over all pairs $\{x, y\} \in \mathcal{P}_K$. It is routine to check that the union of the sets $\mathcal{P}_K$ over all components K is the edge set of a join in (G, T). $\square$

The main theorem of this section was stated already in Section 1:

Theorem 1.1. *A graft (G, T) has a $\{1, 2\}$-covering join if and only if it is not contractible to any of the grafts $\mathcal{K}_2$, $\mathcal{C}_4$, $\mathcal{K}_{2,2k+1}$ and $\mathcal{K}_{2,2k}$ where $k \geq 1$.*

Proof. Let (G, T) be a counterexample with $|E(G)|$ as small as possible. Throughout the proof, we freely use the fact that no graft $(G', T') \preceq (G, T)$ is contractible to any of the obstructions.

Claim 1. *The graph G is 2-edge-connected.*

Let $e = x_1x_2$ be a bridge in G. For $i \in \{1, 2\}$, let H_i be the component of $G \setminus e$ containing x_i. Since $\mathcal{K}_2 \not\preceq (G, T)$, each H_i is T-odd. Thus, if we set T_i to be the symmetric difference of $T \cap V(H_i)$ and $\{x_i\}$, each (H_i, T_i) is a graft. Note that $(H_i, T_i) \preceq (G, T)$ and hence (H_i, T_i) is not contractible to any obstructions. By the minimality of (G, T), each (H_i, T_i) has a $\{1, 2\}$-covering join J_i. Clearly, $J_1 \cup J_2 \cup \{e\}$ is a join in (G, T) with the same property.

Claim 2. *For every nontrivial 2-cut C in G, each component of $G \setminus C$ is T-odd.*

Assume that C is a 2-cut contradicting the claim. Let the components of $G \setminus C$ be denoted by H_1 and H_2. Since C is nontrivial, both $(G, T)/H_1$ and $(G, T)/H_2$ have fewer edges than (G, T) does. By the minimality of (G, T), we can find a $\{1, 2\}$-covering join J_i in $(G, T)/H_i$ ($i = 1, 2$). For $i = 1, 2$, let v_i denote the vertex obtained by contracting all of H_i in $(G, T)/H_i$. Our assumption implies that the vertex v_i in each $(G, T)/H_i$ is white, and so J_i must use both edges incident with v_i. It follows that J_1 and J_2 may be combined to produce a join J in (G, T).

We claim that J is $\{1, 2\}$-covering. If not, then G contains a 2-cut C' not intersected by J. If both edges of C' are contained in some H_i (in H_1, say), then C' is a 2-cut in G/H_2, a contradiction since C' would have been intersected by J_1. Consequently, each of H_1 and H_2 contain one edge of C'. Let the four components of $G \setminus (C \cup C')$ be denoted by $A_1, \ldots, A_4$. For each $i = 1, \ldots, 4$, exactly one edge of the join J leaves A_i. This shows that the degree sum $D_i = \sum_{x \in A_i} d_J(x)$ is odd, since the edges with both ends in A_i do not change the parity of D_i. On the other hand, D_i has the same parity as the number of black vertices in A_i, since the modulo 2 contribution of a vertex to D_i is 0 or 1 according to whether the vertex is white or black, respectively. In summary, the number of black vertices in each

A_i is odd. But this implies that (G,T) can be contracted to $\mathcal{C}_4$, a contradiction. We conclude that J is $\{1,2\}$-covering as required.

We proceed with the proof of Theorem 1.1. Let (G',T') be the graft obtained by suppressing all white vertices of degree 2 in G and removing any resulting loops. Notice that $T' = T$. Let R be the set of all edges of G' that are not present in G (i.e., the edges created by the suppression).

Claim 3. *The set R can be extended to a join in (G',T').*

Form a graft (G'',T'') by removing, one by one, each edge in R and inverting the colors of its endvertices. (Thus, the color of a vertex will change precisely when it is incident with an odd number of edges in R.) Clearly, it suffices to show that (G'',T'') has a join. By Lemma 2.1, it suffices to show that each component of G'' is T''-even.

Let K be a T''-odd component of G''. Let x be the number of edges of R with precisely one endvertex in K and let $y = |T' \cap V(K)|$. Since K is T''-odd, $x+y$ is an odd number.

Enumerate the components of $G'' \setminus V(K)$ as $L_1,\dots,L_t$. For $i = 1,\dots,t$, let x_i denote the number of edges in R between K and L_i, and let $y_i = |T' \cap V(L_i)|$. Note that x_i sum up to x, and the sum of all y_i has the same parity as y. Consequently, there is some j such that $x_j + y_j$ is odd.

Let $S \subseteq V(G)$ be the set of degree 2 vertices corresponding to the edges of R joining K to L_j. Let G_1 and G_2 be the two components of $G \setminus S$. Contracting each of G_1 and G_2 to a single vertex, we obtain the graph K_{2,x_j}. As a graft, this is (for $k = \lfloor x_j/2 \rfloor$) $\mathcal{K}_{2,2k}$ if x_j is even, or $\mathcal{K}_{2,2k+1}$ if x_j is odd. In either case, we obtain a contradiction.

The proof of Theorem 1.1 can now be finished. Using Claim 3, we find a join $J' \supseteq R$ in G'. Let J be the set of edges of G corresponding to those in J' (in particular, for each edge of R, take all of the corresponding edges in G). Claim 2 implies that every 2-cut C, such that the sides of C (components of $G \setminus C$) are T-even, is trivial. By the construction, J intersects each such 2-cut. Furthermore, J automatically intersects each 2-cut with T-odd sides since such a cut is a T-cut. By Claim 1, G has no bridges, so J is a $\{1,2\}$-covering join in (G,T). The proof is complete. □

Theorem 1.1 deals with $\{1,2\}$-covering joins. It is easy to modify the proof to the case of $\{1\}$-covering and $\{2\}$-covering joins:

Proposition 2.2. *(i) A graft (G,T) has a $\{1\}$-covering join if and only if it is not contractible to $\mathcal{K}_2$.*

(ii) A graft (G,T) has a $\{2\}$-covering join if and only if it is not contractible to any of the grafts $\mathcal{C}_4$, $\mathcal{K}_{2,2k+1}$ and $\mathcal{K}_{2,2k}$, where $k \geq 1$.

Figure 2: Two minimal grafts with no $\{3\}$-covering joins.

3 Cuts of size 3

As mentioned in Section 1, for $\{3\}$-covering joins, we have no characterization similar to that proved for A-covering joins with $A = \{1\}$, $A = \{2\}$ or $A = \{1, 2\}$. We only have two examples of minimal grafts with no $\{3\}$-covering joins; these are shown in Figure 2.

Although a complete characterization remains an open problem, we prove that 3-edge-connected grafts do admit $\{3\}$-covering joins. The proof is based on a nice relation to the T-path packing problem in cubic graphs.

Let G be a graph and $e_1, e_2 \in E(G)$ two edges incident with a vertex z of degree $d_G(z) \geq 4$. Assume that the endvertices of each e_i $(i = 1, 2)$ are z and v_i. To *split off* e_1 and e_2 from z, remove e_1 and e_2 and add a new vertex v' adjacent to v_1 and v_2. The resulting graph is denoted by $G(e_1, e_2)$. We let $\lambda_G(x, y)$ denote the maximum number of edge-disjoint paths joining vertices x and y of the graph G. The following is a well-known theorem of Mader [4]:

Theorem 3.1 (Mader). *Let G be a graph and $z \in V(G)$ a vertex such that $d(z) \geq 4$, z has at least 2 distinct neighbors and it is not a cut-vertex of G. There are two edges e_1, e_2 incident with z such that for every $x, y \in V(G) \setminus \{z\}$,*

$$\lambda_{G(e_1,e_2)}(x, y) = \lambda_G(x, y).$$

Given a graph H, let $S(H)$ be the graph obtained from H by suppressing all degree 2 vertices and removing the resulting loops (if any). To any set $F \subseteq E(H)$, there naturally corresponds a set $S(F) \subseteq E(S(H))$. To get $S(F)$, replace every edge in F that is incident with a degree 2 vertex w by the edge created by suppressing w; if the new edge is a loop, delete it from $S(F)$.

The following lemma is a corollary of Mader's theorem:

Lemma 3.2. *Let G be a 3-edge-connected graph and $z \in V(G)$ a vertex such that $d(z) \geq 4$, z has at least two distinct neighbors and it is not a cut-vertex of G. There are edges e_1, e_2 incident with z such that the graph $G^- = S(G(e_1, e_2))$ is 3-edge-connected. Moreover, if C is a 3-cut in G, then $S(C)$ is a 3-cut in G^-.*

Proof. Let e_1 and e_2 be edges satisfying the conclusion of Theorem 3.1. We begin with the first assertion. Note that if $d(z) = 4$, then $V(G^-) = V(G) \setminus \{z\}$

and the claim follows from Theorem 3.1. Hence, we may assume that $d(z) \geq 5$. It must be shown that for all $x \in V(G) \setminus \{z\}$,

$$\lambda_{G^-}(x, z) \geq 3. \tag{1}$$

Thus, let $x \in V(G) \setminus \{z\}$ and assume, for the sake of a contradiction, that (1) is violated. By Menger's theorem, there is a set $F \subset E(G^-)$ such that $|F| \leq 2$ and x, z are in different components of $G^- \setminus F$. Since z is incident with at least 3 edges in G^-, there is an edge $e^* \in E(G^-)$ with endvertices z and y such that $e^* \notin F$. Clearly, z and y are in the same component K of $G^- \setminus F$. Since $\lambda_{G^-}(x, y) \geq 3$, x is also in K. Thus, x and z are in the same component of $G^- \setminus F$, a contradiction.

To prove the second assertion of the lemma, assume that C is a 3-cut in G. Observe that the graph $G^- \setminus S(C)$ is disconnected, which implies that $|S(C)| = 3$ as G^- is 3-edge-connected. By the same token, $S(C)$ is a minimal set of edges disconnecting G^-. Thus, $S(C)$ is a 3-cut. □

In a graft (G, T), a *T-path* is any path with both ends in T. A *T-path covering* is a system of vertex-disjoint T-paths spanning T. The proof of Theorem 1.2 makes use of the following result on T-path coverings, proved in [2]:

Theorem 3.3. *Suppose that G is a k-regular k-edge-connected graph, where $k \geq 2$, and (G, T) is a graft. Then every edge of G is contained in a T-path covering.*

The main theorem of this section was already stated in Section 1:

Theorem 1.2. *If (G, T) is a graft and G is 3-edge-connected, then (G, T) has a $\{3\}$-covering join.*

Proof. We first show that it suffices to prove the theorem for cubic graphs. If G is not cubic, then by repeated use of Lemma 3.2, we produce a graft (G', T'), where G' is 3-edge-connected and cubic. Lemma 3.2 also implies that any 3-cut in G is also a 3-cut in G'. It follows that by reversing the splitting process (i.e., contracting the edges created by the splitting), we turn any $\{3\}$-covering join J' in (G', T') into a $\{3\}$-covering join in (G, T). Thus, G may be assumed to be cubic as claimed.

We prove the following stronger assertion:

Claim 1. *If (G, T) is a graft, G is 3-edge-connected and cubic, and $e, e' \in E(G)$ are edges incident with a vertex $v \notin T$, then (G, T) has a $\{3\}$-covering join containing e and e'.*

We proceed by induction on the order of G. Assume that G has a nontrivial 3-cut F that is not a T-cut. Denoting the components of $G \setminus E(F)$ by C_1, C_2, let (G_i, T_i) be the graft resulting from (G, T) by contracting C_{3-i} ($i = 1, 2$) to a vertex c_{3-i}. Note that since F is not a T-cut, both c_1 and c_2 are white. The cut F is a matching, for otherwise we would obtain a 2-cut in G. Consequently, exactly one of G_1 and G_2 (say, G_1) contains both e and e'. Applying the inductive hypothesis to G_1, we find a $\{3\}$-covering join J_1. Since the vertex c_2 is of degree 3 and $c_2 \notin T_1$, the join J_1 contains exactly 2 edges (say, f and f') incident with c_2. Using the inductive hypothesis on G_2, we find a $\{3\}$-covering join in (G_2, T_2) containing the two edges (incident with c_1) that correspond to f and f'. Since J_1 and J_2 agree on the edges corresponding to F, we may combine them to obtain a (clearly $\{3\}$-covering) join in (G, T).

We may thus assume that every 3-cut of G is either trivial or a T-cut. Since a T-cut is automatically intersected by any T-join, it is sufficient to find a join J in (G, T) such that $e, e' \in E(J)$ and every vertex of G is incident with an edge of J (let us call such a join *suitable*). Assuming for the moment that we have found a suitable join J, let us set $\bar{J} = G \setminus E(J)$ and $\bar{T} = V(G) \setminus T$. Observe that the degree in $\bar{J}$ of a vertex $x \in V(G)$ is 1 if $x \in \bar{T}$, and 0 or 2 otherwise. Hence, $\bar{J}$ is a $\bar{T}$-path covering. Moreover, since $v \in \bar{T}$, $\bar{J}$ contains the third edge e'' incident with v. Conversely, any $\bar{T}$-path covering containing e'' determines a suitable join in (G, T). However, the existence of a $\bar{T}$-path covering containing e'' is guaranteed by Theorem 3.3 (applied to $\bar{T}$ in place of T). This concludes the proof of Claim 1 as well as that of Theorem 1.2. □

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