CHROMATIC POLYNOMIAL, Q-BINOMIAL COUNTING AND COLORED JONES FUNCTION

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ABSTRACT. We define a q-chromatic function and q-bichromate on graphs and compare it with existing graph functions. Then we study in more detail the class of general chordal graphs. This is partly motivated by the graph izomorphism problem. Finally we relate the q-chromatic function to the colored Jones function of knots. This leads to a curious expression of the colored Jones function of a knot diagram $\mathcal{K}$ as a chromatic operator applied to a power series whose coefficients are linear combinations of long chord diagrams. Chromatic operators are directly related to weight systems by the work of Chmutov, Duzhin, Lando and Noble, Welsh.

CONTENTS

1. INTRODUCTION

The purpose of this paper has been a desire to recast the complicated combinatorial construction of categorification of flows of [GL] in more common combinatorial terms.

A graph is a pair $G = (V, E)$ where V is a finite set of *vertices* and E is a set of unordered pairs of elements of V, called *edges*. If $e = xy$ is an edge then the vertices x, y are called *end-vertices* of e.

In this paper we study the following *q-chromatic function on graphs*:

Definition 1.1. Let $G = (V, E)$ be a graph. Let $V = \{1, \ldots, k\}$ and let $V(G, n)$ denote the set of all vectors $(v_1, \ldots, v_k)$ such that $0 \leq v_i \leq n-1$ for each $i \leq k$ and $v_i \neq v_j$ whenever $\{i, j\}$ is an edge of G. We let

$$M_q(G, n) = \sum_{(v_1 \ldots v_k) \in V(G,n)} q^{\sum_i v_i}.$$

Note that $M_q(G, z)|_{q=1}$ is the classic chromatic polynomial of G.

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1.1. Example: q-chromatic function of a complete graph. We first recall some notation:

For $n > 0$ let $(n)_q = \frac{q^n-1}{q-1}$ denote a *quantum integer*. We let $(n)!_q = \prod_{i=1}^n (i)_q$ and for $0 \le k \le n$ we define the *quantum binomial coefficients* by

$$\binom{n}{k}_q = \frac{(n)!_q}{(k)!_q(n-k)!_q}.$$

A simple quantum binomial formula leads to a well-known formula for the summation of the products of distinct powers.

Observation 1.

$$M_q(K_k, n) = k! \binom{n}{k}_q q^{k(k-1)/2}.$$

Next we extend the definition of the q-chromatic function to *q-bichromate*.

sub.q

1.2. q-Bichromate. A graph $G = (V, E)$ is *connected* if it has a path between any pair of vertices. If a graph is not connected then its maximum connected subgraphs are called *connectivity components*. If $G = (V, E)$ is a graph and $A \subset E$ then let $C(A)$ denote the set of the connectivity components of graph (V, A) and $c(A) = |C(A)|$. If $W \in C(A)$ then let $|W|$ denote the number of vertices of W. The following function called *bichromate* is extensively studied in combinatorics. It is equivalent to the Tutte polynomial.

$$B(G, a, b) = \sum_{A \subset E} a^{|A|} b^{c(A)}.$$

Theorem 1.

$$M_q(G, z) = \sum_{A \subset E} (-1)^{|A|} \prod_{W \in C(A)} (z)_{q^{|W|}}.$$

Proof. If $A \subset E$ then let $W(A, z) = \{v \in \{0, \dots, z-1\}^V; \text{ if } \{i, j\} \in A \text{ then } v_i = v_j\}$.

We use the principle of inclusion and exclusion (PIE): If $A_1, ..., A_n$ are finite sets, and if we let $\cap(A_i; i \in J) = A_J$ then

$$| \cup (A_i; i = 1, ..., n)| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{J \in \binom{n}{k}} |A_J|.$$

The next considerations connect the PIE with the geometric series formula.

$$M_q(G, z) = \sum_{v \in \{0,\dots,z-1\}^V} q^{\sum_i v_i} - \sum_{v \in \cup_{e \in E} J_e} q^{\sum_i v_i},$$

where $J_e, e = \{i, j\} \in E$, denotes the set of all vectors satisfying $v_i = v_j$.

By PIE this equals

$$\sum_{A \subset E} (-1)^{|A|} \sum_{v \in \cap_{e \in A} J_e} t^{\sum_i v_i} = \sum_{A \subset E} (-1)^{|A|} \sum_{v \in W(A,n)} t^{\sum_i v_i} =$$

$$\sum_{A \subset E} (-1)^{|A|} \prod_{W \in C(A)} \sum_{x \in \{0,\dots,z-1\}} q^{|W|x} = \sum_{A \subset E} (-1)^{|A|} \prod_{W \in C(A)} (z)_{q^{|W|}}.$$

□

The formula of Theorem 1 leads naturally to a definition of *q-bichromate*.

Definition 1.2. We let

$$B_q(G, x, y) = \sum_{A \subset E} x^{|A|} \prod_{W \in C(A)} (y)_{q^{|W|}}.$$

Note that $B_{q=1}(G, x, y) = B(G, x, y)$.

sub.prev

1.3. **Previous work.** Let $x_1, x_2, \dots$ be commuting indeterminates and let $G = (V, E)$ be a graph. The q-chromatic function restricted to non-negative integer y is the principal specialization of X_G, the *symmetric function generalisation of the chromatic polynomial* (see [S1], [S2]). This is defined as follows:

def.sym

Definition 1.3.

$$X_G = \sum_f \prod_{v \in V} x_{f(v)},$$

where the sum ranges over all proper colorings of G by $\{1, 2, \dots\}$.

Further we define *symmetric function generalisation of the bad colouring polynomial* (see [S1]):

def.sb

Definition 1.4.

$$XB_G(t, x_1, \dots) = \sum_f (1+t)^{b(f)} \prod_{v \in V} x_{f(v)},$$

where the sum ranges over ALL colorings of G by $\{1, 2, \dots\}$ and $b(f)$ denotes the number of monochromatic edges of f.

Noble and Welsh define in [NW] the *U-polynomial* and show that it is equivalent to XB_G. Finally I. Sarmiento proved in [Sa] that the polychromate defined by Brylawski ([Bry]) is equivalent to the U-polynomial.

def.u

Definition 1.5.

$$U_G(z, x_1 \dots) = \sum_{S \subset E(G)} x(\tau_S)(z-1)^{|S|-r(S)},$$

where $\tau_S = (n_1 \geq n_2 \geq \dots n_k)$ is a partition of $|V|$ determined by the connectivity components of S, $x(\tau_S) = x_{n_1} \dots x_{n_k}$ and $r(S) = |V| - c(S)$.

The motivation for the work of Noble and Welsh is a series of papers by Chmutov, Duzhin and Lando ([CDL]). It turns out (see [NW]) that the U-polynomial evaluated at $z = 0$ and applied to the intersection graphs of chord diagrams satisfies the $4T-$relation of the weight systems. Hence the same is true for $M_q(G, z)$ for each positive integer z since it is an evaluation of $U_G(0, x_1 \dots)$.

obs.evv

Observation 2. *Let z be a positive integer. Then*

$$M_q(G, z) = (-1)^{|V|} U_G(-1, x_1 \dots)|_{x_i := (-1)(q^{i(z-1)} + \dots + 1)}.$$

On the other hand, it seems plausible that the q-bichromate determines the U-polynomial. If true, q-bichromate provides a compact representation of the multivariate generalisations of the Tutte polynomial mentioned above. This is also interesting since our Theorem 2 states that q-bichromate equals to the partition function of the Potts model in a magmetic field.

It is not difficult to observe that the following conjecture about the partitions of $n = |V|$ implies that the q-bichromate determines the U-polynomial.

Let $\tau = (n_1 \geq n_2 \geq \dots n_k)$ be a partition of n. We let $c(\tau) = (c(\tau, y))_{y=0,1,\dots}$ be an infinite sequence of polynomials in q defined by

$$c(\tau, y) = \prod_{i=1}^{k} (q^{n_i y} + q^{n_i (y-1)} + \dots + 1).$$

c.ekv

Conjecture 1. *Only trivial rational linear combination of $c(\tau)$'s is identically zero.*

sub.main

1.4. **Main results.** We study the q-bichromate with applications in statistical physics and knot theory in mind. In section 2 we introduce a q-generalisation of the partition function of the Potts model. We prove in Theorem 2 that q-bichromate equals to the partition function of the Potts model in a magmetic field. Such an observation is important for computer simulations of the q-bichromate. Jones polynomial may be viewed via the Kauffman state sum as an evaluation of the Potts partition function. This leads to a conjecture whether our q-generalisation of the Jones polynomial is a knot invariant.

In section 3 we study q-chromatic function of chordal graphs. A graph is *chordal* if no cycle of length bigger than three is induced. We are partly motivated by a seminal conjecture of Bollobas, Pebody and Riordan ([BR]): Is it true that Tutte polynomial distinguishes almost all graphs? Replacing Tutte polynomial by q-bichromate should not be too rude generalization in view of Theorem 2. The izomorphism problem for general graphs may be naturally embedded to the class of chordal graphs: if $G = (V, E)$ is a graph then we can construct a chordal graph $ch(G)$ so that we subdivide each edge e by one new vertex v_e, and add all the edges between vertices of V. Clearly two graphs G, G' are izomorphic if and only if $ch(G), ch(G')$ are izomorphic. R. Stanley conjectures in [S1] that the symmetric function generalisation of the chromatic polynomial distinguishes all trees. We ask in Conjecture 2 whether *q-bichromate distinguishes all chordal graphs.* An important consequence of Conjecture 2 is that the chordal graphs are edge and vertex reconstructable. The edge-reconstructability has already been proved by B. D. Thatte [TH].

Finally in section 4 we derive in Theorems 9 and 10 new formulas for the colored Jones function. It may be interesting to compare them with the Kontsevich integral expression for the colored Jones function. A describtion of the theory of Vassiliev knot invariants and weight systems may be found in [BN]. The formulas of Theorems 9, 10 may be briefly described as follows:

We start with $R-$matrix state sum of the Jones polynomial (see [Tur]) and apply it to the n-cabling of a knot. We associate to each state s a triple $(f(s), S(s), v(s))$, where $f(s)$ is a non-negative integer flow; you can imagine that the flow lives on the reduced knot diagram K, eventhough it turns out to be more convenient to define it on the *arc graph of the reduced knot diagram.*

Further $S(s)$ is a set system on the set of $\sum_e f(s)(e)$ elements where the sum is over all 'jump-up' transitions e of K (which are later associated to 'red edges' of the arc graph). Each such set system is called '$f(s)$-structure'. The number of $f(s)$-structures for a flow $f(s)$ is given by a product of binomial coefficients.

Finally $v(s)$ is a non-negative integer vector of length $\sum_e f(s)(e)$, and for each i, $0 \leq v(s)_i \leq (n-1)$.

We then represent $S(s)$ as a long chord diagram $D(s)$ (by *long chord diagram* we mean a system of intervals on a line with pairwise disjoint end-points) with $\sum_e f(s)(e)$ chords. Let $G(s)$ be the intersection graph of the chords of $D(s)$, i.e. its vertices are the intervals and the edges are between pairs of intervals with a non-empty intersection. Note that the difference from the intersection graph of a chord diagram is that here we also have adges between pairs of intervals where one contains the other.

Let us denote by $w(f, S)$ the sum of the state sum contributions of all states s with $f(s) = f$ and $S(s) = S$. We observe that $w(f, S)$ may be written conveniently using a chromatic operator. For instance in Theorem 9 it is

$$w(f, S) = Z_{n,f}(t) M_t^{\mathrm{def}}(G(s), n),$$

where M_t^{def} is a 'defected' q-chromatic function and $Z_{n,f}(t)$ is a Laurent polynomial in t whose precise form is given in Theorem 9.

This leads to a curious expression of the colored Jones function of a knot diagram $\mathcal{K}$ as a power series whose coefficients are equal to a chromatic operator applied to linear combinations of long chord diagrams constructed from flows on reduced $\mathcal{K}$.

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2. Q-Potts

sec.bich

It is well known that the bichromate counts several interesting things in statistical physics. Next we discuss the Potts and Ising partition functions, and their q-extensions.

sub.p

2.1. Potts partition function.

def.P

Definition 2.1. Let $G = (V, E)$ be a graph, $k \geq 1$ integer and J_e a weight (coupling constant) associated with edge $e \in E$. The Potts model partition function is defined as

$$P^k(G, J_e) = \sum_s e^{E(P^k)(s)},$$

where the sum is over all functions s from V to $\{1, \dots, k\}$ and

$$E(P^k)(s) = \sum_{\{i,j\} \in E} J_{ij} \delta(s(i), s(j)).$$

Following [FK, KF], we may write

$$P^k(G, J_e) = \sum_s \prod_{\{i,j\} \in E} (1 + v_{ij} \delta(s(i), s(j))) = \sum_{A \subset E} k^{c(A)} \prod_{\{i,j\} \in A} v_{ij},$$

where $v_{ij} = e^{J_{ij}} - 1$. The RHS is sometimes called *multivariate Tutte polynomial*; see [S] for recent survey. If all J_{ij} are the same we get an expression of the Potts partition function in the form of the bichromate:

$$P^k(G, x) = \sum_s \prod_{\{i,j\} \in E} e^{x\delta(s(i), s(j))} = \sum_{A \subset E} k^{c(A)} (e^x - 1)^{|A|} = B(G, e^x - 1, k).$$

What happens if we replace $B(G, e^x - 1, k)$ by $B_q(G, e^x - 1, k)$? It turns out that this introduces an additional external field to the Potts model.

thm.qP

Theorem 2.

$$\sum_{A \subset E} \prod_{W \in C(A)} (k)_{q^{|W|}} \prod_{\{i,j\} \in A} v_{ij} = \sum_s q^{\sum_{v \in V} s(v)} e^{E(P^k)(s)},$$

where $v_{ij} = e^{J_{ij}} - 1$ *as above.*

Proof. Let us recall that $W(A, k)$ denotes the set of all colourings s which are monochromatic on each edge of A. We have

$$P^k_q(G, J_e) = \sum_s q^{\sum_{v \in V} s(v)} \prod_{\{i,j\} \in E} (1 + v_{ij} \delta(s(i), s(j))) =$$

$$\sum_s q^{\sum_{v \in V} s(v)} \sum_{A \subset E} \prod_{\{i,j\} \in A} v_{ij} \delta(s(i), s(j)) =$$

$$\sum_{A \subset E} \sum_{s \in W(A,k)} q^{\sum_{v \in V} s(v)} \prod_{\{i,j\} \in A} v_{ij} =$$

$$\sum_{A \subset E} \prod_{W \in C(A)} (k)_{q^{|W|}} \prod_{\{i,j\} \in A} v_{ij}.$$

□

sub.i

2.2. **Ising partition function.** The Ising partition function $Z(G)$ of a graph G is equivalent to $P^2(G)$:

$$Z(G, J_e) = \sum_s e^{E(Z)(s)},$$

where the sum is over all functions s from V to $\{1, -1\}$ and

$$E(Z)(s) = \sum_{\{i,j\}\in E} J_{ij} s(i)s(j).$$

We immediately have

$$Z(G, J_e) = e^{-\sum_{\{i,j\}\in E} J_{ij}} P^2(G, 2J_e).$$

Not surprisingly, q-bichromate again adds an external field to the Ising partition function.

cor.qZ

Corollary 2.2.

$$Z(G, x) = e^{-|E|x} B(G, e^{2x} - 1, 2)$$

and

$$\sum_s q^{\sum_{v\in V} s(v)} e^{E(Z)(s)} = q^{-3|V|x} e^{-|E|x} B_q(G, e^{2x} - 1, 2).$$

sub.m

2.3. **Van der Waerden Theorem.** A remarkable fact about the Ising partition function is a theorem of Van der Waerden which expresses it using the generating function of even subgraphs. It is not hard to formulate its q-generalisation. We use the following notation:

$$sinh(x) = \frac{e^x - e^{-x}}{2}, cosh(x) = \frac{e^x + e^{-x}}{2}, th(x) = \frac{sinh(x)}{cosh(x)}.$$

thm.w

Theorem 3.

$$\sum_s q^{\sum_{v\in V} s(v)} e^{\sum_{\{i,j\}\in E} J_{ij} s(i)s(j)} =$$

$$\prod_{\{i,j\}\in E} cosh(J_{ij}) \sum_{A\subset E} \prod_{\{i,j\}\in A} th(J_{ij})(q - q^{-1})^{o(A)}(q + q^{-1})^{|V|-o(A)},$$

where $o(E)$ denotes the number of vertices of G of an odd degree.

The proof uses a lemma:

lem.w

Lemma 2.3. *Let $G = (V, E)$ be a graph. Then*

$$\sum_s q^{\sum_{v\in V} s(v)} \prod_{\{i,j\}\in E} s(i)s(j) = (q - q^{-1})^{o(E)}(q + q^{-1})^{|V|-o(E)},$$

where the first sum is over all functions s from V to $\{-1, 1\}$ and $o(E)$ denotes the number of vertices of G of an odd degree.

Proof. (of Lemma 2.3)

First note that if E is a cycle and s arbitrary then $\prod_{\{i,j\}\in E} s(i)s(j) = 1$. Hence, we can delete from G any cycle without changing the LHS

$$\sum_s q^{\sum_{v\in V} s(v)} \prod_{\{i,j\}\in E} s(i)s(j).$$

This reduces the proof to the case that E is acyclic. If E is a path, then it follows from the observation above that for s arbitrary, $\prod_{\{i,j\}\in E} s(i)s(j) = 1$ if and only if s is constant on the end-vertices of E. Hence, we can delete from E any maximal path and replace it by the edge between its end-vertices, without changing the LHS. Hence it suffices to prove the proposition for the case that each component of G contains at most one edge. This is however simply true.

□

Proof. (of Theorem 3)
Using the identity

$$e^{x(s(i)s(j))} = cosh(x) + s(i)s(j)sinh(x),$$

we have:

$$\sum_s q^{\sum_{v\in V} s(v)} e^{\sum_{\{i,j\}\in E} J_{ij} s(i)s(j)} =$$
$$\sum_s q^{\sum_{v\in V} s(v)} \prod_{\{i,j\}\in E} [cosh(J_{ij}) + s(i)s(j)sinh(J_{ij})] =$$
$$\prod_{\{i,j\}\in E} cosh(J_{ij}) \sum_s q^{\sum_{v\in V} s(v)} \prod_{\{i,j\}\in E} [1 + s(i)s(j)th(J_{ij})] =$$
$$\prod_{\{i,j\}\in E} cosh(J_{ij}) \sum_s q^{\sum_{v\in V} s(v)} \sum_{A\subset E} \prod_{\{i,j\}\in A} s(i)s(j)th(J_{ij}) =$$
$$\prod_{\{i,j\}\in E} cosh(J_{ij}) \sum_{A\subset E} \prod_{\{i,j\}\in A} th(J_{ij})U(A),$$

where

$$U(A) = \sum_s q^{\sum_{v\in V} s(v)} \prod_{\{i,j\}\in A} s(i)s(j).$$

Theorem now follows from next Lemma 2.3. □

sec.q

2.4. **Kauffman state sum of the Jones polynomial.** Following [VN] we show how the Jones polynomial may be derived from the Potts partition function. Let $G = (V, E)$ be a planar directed graph which is a directed knot diagram; hence each vertex (crossing) v has a 'directed' sign sign(v) associated with it, and two arcs entering and leaving it.

Given G, we construct its *medial graph* $M(G) = (V(M(G)), E(M(G)))$ as follows: first color the faces of G white and black so that neighbouring faces receive a different color. Let $b(v)$ be the 'undirected' sign of crossing v, induced by this coloring (see e.g. [W] for the definition of sign$(v), b(v)$).

Assume the outer face is white; then let $V(M(G))$ be the set of the black faces, and two vertices are joined by an edge if the corresponding faces share a crossing. Note that $M(G))$ is again a planar graph. For edge e of $M(G)$ let $b(e)$ be the 'undirected' sign of the crossing shared by the end-vertices of e.

Let us now describe what will a *state* be: we can 'split' each vertex v of G so that the white faces incident with v are joined into one face and the black faces are disconnected, or vice versa. Let $|V| = n$. There are 2^n ways to split all the vertices of G: the ways are called *states.* After performing all the splittings of a state s, we are left with a set of disjoint non-self-intersecting cycles in the plane; let $S(s)$ denote their number. For vertex $v \in V$ let $\epsilon_v(s) = 1$ if s splits v so that the black faces are joined (i.e. the corresponding edge of the medial graph is not cut) and let $\epsilon_v(s) = -1$ otherwise. The following statement consists of Theorems 2.6 and 2.8 by Kauffman (see [K1]).

thm.K1

Theorem 4. *Let $G = (V, E)$ be an oriented knot diagram. The following function $f_K(G)$ is a knot invariant:*

$$f_K(G, A) = (-A)^{-3W(G)} \sum_s (-A^2 - A^{-2})^{S(s)-1} A^{\sum_{v\in V} b(v)\epsilon_v(s)},$$

where $W(G) = \sum_{v\in V}$ sign(v). Moreover the Jones polynomial equals

$$J(G, A^{-4}) = f_K(G, A).$$

Each state s determines a subset of edges $E(s)$ of $M(G)$, which are not cut by the splittings of s, and it is easy to see that this gives a natural bijection between the set of states and the subsets of edges of $M(G)$. Moreover for each state s

$$\sum_{v\in V} b(v)\epsilon_v(s) = -\sum_{e\in E(M(G))} b(e) + 2\sum_{e\in E(s)} b(e).$$

prop.mm

Proposition 2.4.
$$S(s) = 2c(E(s)) + |E(s)| - |V(M(G))|,$$
where $c(E(s))$ denotes the number of connectivity components of $(V(M(G)), E(s))$.

Proof. Note that $S(s) = f(E(s)) + c(E(s)) - 1$, where $f(E(s))$ denotes the number of faces of $(V(M(G)), E(s))$. Hence the formula follows from the Euler formula for the planar graphs. □

cor.kk

Corollary 2.5.
$$f_K(G, A) = (-A)^{-3W(G)}(-A^2 - A^{-2})^{-|V(M(G))|-1} A^{-\sum_{e \in E(M(G))} b(e)}$$
$$\sum_s (-A^2 - A^{-2})^{2c(E(s))} \prod_{e \in E(s)} (-A^2 - A^{-2}) A^{2b(e)}.$$

This obviously provides an expression for the Jones polynomial of each knot-diagram with the same undirected sign of each crossing (i.e. each alternating diagram) as a bichromate.

Question 1. Is
$$\sum_s \prod_{e \in E(s)} (-A^2 - A^{-2}) A^{2b(e)} \prod_{W \in C(E(s))} ((-A^2 - A^{-2})^2)_{q^{|W|}}$$
times an appropriate constant also a knot invariant?

3. q-Chromatic function of chordal graphs

sec.chor

Next we study chordal graphs, i.e. graphs such that each cycle of length at least four has a chord. The main result is Theorem 5 which gives a formula for $\sum_G M_q(G, z)$ where G ranges over all chordal graphs of a given 'shape'. This may be viewed as a partition function of a model with states given by chordal graphs G with the same 'shape' and energy $M_q(G, z)$.

Our motivation here partly comes from a conjecture of Bollobas, Pebody and Riordan ([BR]) which asks whether almost all graphs are determined by the Tutte polynomial, and by a conjecture of Stanley ([S1]) which asks whether X_G, the symmetric function generalisation of the chromatic polynomial, determines all trees. If the following conjecture is true, then q-bichromate provides a graph function close to Tutte polynomial (by Theorem 2, q-bichromate is the partition function of Potts model in magnetic field), which, with simple graph preprocessing, tells non-izomorphic graphs apart.

con.all

Conjecture 2. *Is it true that q-bichromate distinguishes all chordal graphs?*

Let $G = (V, E)$ be a chordal graph. We fix a linear ordering $x_1, \ldots, x_k$ of the vertices of V so that for each i, vertex x_i is a simplicial vertex, i.e. its neighbourhood is complete, in the subgraph induced by vertices $x_1, \ldots, x_i$; let $m(i)$ denote the number of vertices in that neighbourhood of x_i. It is well-known that the existence of such an order of vertices characterises the chordal graphs.

A tree is an acyclic connected graph. We consider trees *rooted*, i.e. a vertex r is distinguished in each tree. Hence we will denote trees by a triple $T = (V, E, r)$. For each vertex $x \neq r$ of T there is unique path $P(x)$ in T connecteing x to r. The neighbour of x on $P(x)$ is called *predecessor of* x and denoted by $p(x)$. We say that a subtree of a rooted tree *starts* at its unique nearest vertex to the root.

Another well known characterisation says that a graph is chordal if and only if it is an intersection graph of subtrees of a tree. Let $G = (V, E)$ be a chordal graph and $x_1, \ldots, x_k$ the specified ordering of its vertices. Let $T = (W, F)$ be the tree whose subtrees 'represent' G, i.e. there are subtrees $T_v, v \in V$ of T so that $T_u \cap T_v \neq \emptyset$ if and only if uv is an edge of G. We can choose a vertex $r \in W$ arbitrarily as a root of T and define sets $A_w, B_w, w \in W$ as follows: $A_w = \{v; T_v$ starts at $w\}$ and $B_w = \{v; T_v$ contains but does not start at $w\}$. Then these have the following properties:

1. the A_w's are disjoint and $V = \cup_i A_i$.
2. $B_w \subset \cup_{w' \in P(w)} A_{w'}$. In particular $B_r = \emptyset$.

3. if i, j, j' are different vertices of T such that $i \in P(j), j' \in P(j), i \in P(j')$ and $x \in B_j \cap A_i$ then $x \in B_{j'}$.
4. if $x \in A_i, y \in A_j$, $i \neq j$ and $x < y$ then $i \notin P(j)$.
5. $e \in E$ if and only if $e \subset A_w \cup B_w$ for some w.

This leads to the following definition of a *tree structure.*

def.str

Definition 3.1. Let $T = (W, F)$ be a tree, $V = \{x_1, \dots x_k\}$ be an ordered set and sets $A(w), B(w)$: $w \in W$ satisfy the above properties 1.,2.,3.,4. Moreover let $|B_w| = b_w$. Then $(B_w : w \in W)$ is called a $(T, V, (A_w, b_w : w \in W))-$ tree structure (tree structre for short). The set of all structures is denoted by $\Sigma(T, V, (A_w, b_w : w \in W))$.

What distinguishes tree structures are the sets B_w. B_w is an arbitrary subset of $A_{p(w)} \cup B_{p(w)}$ of b_w elements. Hence we get the following observation.

prop.numstr

Proposition 3.2. *The number of tree structures is* $\prod_{r \neq w \in W} \binom{a_{p(w)}+b_{p(w)}}{b_w}$.

rem.s

Remark 3.3. On the other hand, each tree structure on $T = (W, F, r), V = \{1, \dots, k\}$ determines a set $T_v, 1 \leq v \leq k$ of subtrees of T so that $T_u \cap T_v \neq \emptyset$ if and only if $u, v \in A_w \cup B_w$ for some $w \in W$, by reversing the construction of the tree structure described above.

def.strdef

Definition 3.4. Let S be a tree structure, $v \in \{0, \dots, z-1\}^V$ and $x \in A_w$ for some $w \in W$.

- We denote by $G(S)$ the unique chordal graph with tree structure S (see the remark above).
- We let $m(S, x)$ be the number of $y \in A_w \cup B_w$ such that $y < x$. Note that $m(S, x)$ equals b_w plus the number of elements of A_w that are smaller than x. Hence $m(S, x)$ does not depend on S and we let $m(S, x) = m(x)$.
- We let $V(S, z) = \{v \in \{0, \dots, z-1\}^V$; if $\{x, y\} \subset A_w \cup B_w$ for some w then $v_x \neq v_y\}$.
- We let $\operatorname{def}(S, v, x)$ equal to the number of $y \in A_w \cup B_w$ such that $y < x$ and $v_y < v_x$.

thm.str20

Theorem 5.

$$\sum_{S \in \Sigma(T, V, (A_w, b_w : w \in W))} M_q(G(S), z) = \prod_{w \in W} \prod_{j=1}^{|A_w|} \frac{b_w + j}{(b_w + j)_{q^{-1}}} \prod_{r \neq w \in W} \binom{a_{p(w)} + b_{p(w)}}{b_w} \prod_{x \in V} (z - m(x))_q.$$

sub.thm2

3.1. **Proof of theorem 5.** We first deduce a formula for a modified q-chromatic function. Recall the definition of a tree structure S for a chordal graph G, and note that $V(S, z) = V(G, z)$.

prop.str2

Proposition 3.5. *Let G be a chordal graph and S its tree structure. Then*

$$\sum_{v = (v_1 \dots v_k) \in V(G, z)} q^{\sum_i v_i - \operatorname{def}(S, v, i)} = \prod_{i=1}^{k} (z - m(i))_q.$$

Proof. The basis for the calculation is the following Claim.

Claim. Fix numbers $v_1, \dots, v_{k-1}$ between 0 and $z-1$ so that no edge of G receives two equal numbers. Then

- $$\sum_{v_k : v = (v_1, \dots, v_k) \in V(G, z)} t^{v_k - \operatorname{def}(v, k)} = A - B + C,$$

  where $A = \sum_{v_k : v \in V(G, z)} t^{v_k}$, $B = \sum_{i=1}^{m(k)} t^{z-i}$, and $C = \sum_{\{i,k\} \in E(G)} t^{v_i}$.
- $A + C = \sum_{0 \leq j \leq z-1} t^z$ and $A - B + C = \frac{1 - t^{z - m(k)}}{1 - t}$.

Proof of Claim. Note that the second part simply follows from the first one.

Let $v'_1 < \cdots < v'_{m(k)}$ be a reordering of $\{v_i; \{i,k\} \in E(G)\}$. We may write $v'_1 = z - i_1, \dots, v'_{m(k)} = z - i_{m(k)}, 1 \leq i_{m(k)} < \cdots < i_1$. The LHS becomes

$$(\sum_{v_k: v \in V(G,z)} t^{v_k}) - t^{z-i_1+1} - \cdots - t^{z-i_2-1} - t^{z-i_2+1} - \cdots - t^{z-i_{m(k)}-1} - t^{z-i_{m(k)}+1} - \cdots - t^{z-1} + t^{z-i_1} + \cdots +$$

$$t^{z-i_2-2} + t^{z-i_2-1} + \cdots + t^{z-m(k)-1}.$$

This equals to the RHS of the equality we wanted to show. The Proposition simply follows from the Claim. □

The proof of Proposition 3.5 yields the following

prop.str.10

Proposition 3.6. *Let S be a structure. Then*

$$\sum_{v \in V(S,z)} \prod_{x \in V} q^{v_x - \mathrm{def}(S,v,x)} = \prod_{x \in V} (z - m(x))_q.$$

Hence

$$\sum_{v \in V(S,z)} \prod_{x \in V} q^{v_x - \mathrm{def}(S,v,x)}$$

is invariant for $S \in \Sigma(T, V, (A_w, b_w : w \in W))$. Note that the same is not true for the non-defected version: path of three edges and star of three edges, with their tree being the path, provide a contraexample.

Proof. (of Theorem 5)

$$\sum_{S \in \Sigma(T,V,(A_w,b_w:w\in W))} \sum_{v \in V(S,z)} \prod_{x \in V} q^{v_x - \mathrm{def}(S,v,x)} = \sum_{S=(B_w:w\in W)} \sum_{v \in V(S,z)} \prod_{x \in V} q^{v_x - \mathrm{def}(S,v,x)} =$$

$$\sum_{v_x; x \in A_r} \sum_{B_w; p(w)=r} \prod_{x \in A_r} q^{v_x - \mathrm{def}(S,v,x)} [\sum_{(B_w: p(w) \neq r)} \sum_{v_x; x \notin A_r, v \in V(S,z)} \prod_{x \in V - A_r} q^{v_x - \mathrm{def}(S,v,x)}] =$$

$$\sum_{v(r)'} \sum_{B_w; p(w)=r} (|A_r|)!_{q^{-1}} \prod_{x \in A_r} q^{v(r)'_x} [\sum_{(B_w: p(w) \neq r)} \sum_{v_x; x \notin A_r, v \in V(S,z)} \prod_{x \in V - A_r} q^{v_x - \mathrm{def}(S,v,x)}],$$

where the first sum is over all vectors $v(r)' = (v(r)'_x; x \in A_r)$ so that $v(r)'_x > v(r)_y$ for $x < y$. In the above equality we used

$$\sum_{\pi} q^{|\{(i,j); i<j, \pi(i)<\pi(j)\}|} = (n)!_q.$$

This further equals

$$\sum_{B_w; p(w)=r} (|A_r|)!_{q^{-1}} \sum_{v(r)'} \prod_{x \in A_r} q^{v(r)'_x} [\sum_{(B_w: p(w) \neq r)} \sum_{v_x; x \notin A_r, v \in V(S,z)} \prod_{x \in V - A_r} q^{v_x - \mathrm{def}(S,v,x)}] =$$

$$\sum_{B_w; p(w)=r} \frac{(|A_r|)!_{q^{-1}}}{|A_r|!} \sum_{v_x; x \in A_r} \prod_{x \in A_r} q^{v_x} [\sum_{(B_w: p(w) \neq r)} \sum_{v_x; x \notin A_r, v \in V(S,z)} \prod_{x \in V - A_r} q^{v_x - \mathrm{def}(S,v,x)}] =$$

$$\sum_{S \in \Sigma(T,V,(A_w,b_w:w\in W))} \prod_{w \in W} \prod_{j=1}^{|A_w|} \frac{(b_w + j)_{q^{-1}}}{b_w + j} \sum_{v \in V(S,z)} \prod_{x \in V} q^{v_x}.$$

Theorem now follows from Proposition 3.6. □

4. A motivation from the quantum knot theory: colored Jones function

sec.col

The motivation to the previous discussion comes from a study of the colored Jones function J_n which is a joint work with Stavros Garoufalidis in [GL]. Colored Jones function is the quantum group invariant of knots that corresponds to the $(n+1)$-dimensional irreducible representation of $\mathfrak{sl}_2$. In [GL] a non-commutative formula for the colored Jones function is presented.

Fix a generic planar projection $\mathcal{K}$ of an oriented knot with r crossings. Let c_i for $i = 1, \dots, r$ denote an ordering of the crossings of $\mathcal{K}$. Then $\mathcal{K}$ consists of r arcs a_i, which we label so that each arc a_i ends at the crossing i. We will single out a specific arc of $\mathcal{K}$ which we decorate by $\star$. Without loss of generality, we may assume that the crossings of a knot appear in increasing order, when we walk in the direction of the knot, and that the last arc is decorated by $\star$. We denote by K the long knot obtained by removing $\star$ from $\mathcal{K}$.

Given $\mathcal{K}$, we define a weighted directed graph $G_{\mathcal{K}}$. In a directed graph, each edge e is directed, i.e. has its starting vertex $s(e)$ and its terminal vertex $t(e)$.

def.arcgraph

Definition 4.1. The *arc-graph* $G_{\mathcal{K}}$ has r vertices $1, \dots, r$, r *blue directed edges* $(v, v+1)$ (v taken modulo r) and r *red directed edges* (u, v), where at the crossing u the arc that crosses over is labeled by a_v.

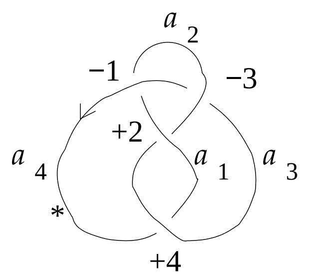
The vertices of $G_{\mathcal{K}}$ are equipped with a sign, where $\text{sign}(v)$ is the sign of the corresponding crossing v of $\mathcal{K}$, and the edges of $G_{\mathcal{K}}$ are equipped with a weight β, where the weight of the blue edge $(v, v+1)$ is $t^{-\text{sign}(v)}$, and the weight of the red edge (u, v) is $1 - t^{-\text{sign}(u)}$. Here t is a variable.

Finally, G_K denotes the digraph obtained by deleting vertex r from $G_{\mathcal{K}}$.

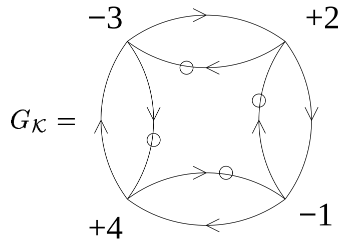
It is clear from the definition that from every vertex of $G_{\mathcal{K}}$, the blue outdegree is 1, the red outdegree is 1, and the blue indegree is 1. It is also clear that $G_{\mathcal{K}}$ has a Hamiltonian cycle that consists of all the blue edges. We denote by e_i^b (e_i^r) the blue (red) edge *leaving vertex i*.

ex.1

Example 4.2. For the figure 8 knot we have:



Its arc-graph $G_{\mathcal{K}}$ with the ordering and signs of its vertices is given by



where the blue edges are the ones with circles on them.

def.flow

Definition 4.3. A *flow* f on a digraph G is a function $f : \text{Edges}(G) \longrightarrow \mathbb{N}$ of the edges of G that satisfies the (Kirkhoff) *conservation law*

$$\sum_{e \text{ begins at } v} f(e) = \sum_{e \text{ ends at } v} f(e)$$

at all vertices v of G. Let $f(v)$ denote this quantity and let $\mathcal{F}(G)$ denote the *set of flows* of a digraph G.

If β is a weight function on the set of edges of G and f is a flow on G, then the *weight* $\beta(f)$ of f is given by $\beta(f) = \prod_e \beta(e)^{f(e)}$, where $\beta(e)$ is the weight of edge e.

In order to express the Jones polynomial as a function of the reduced arc graph G_K, we need to add the following two structures, which may be read off from the knot diagram.

- We associate in a standard way a *rotation* $\mathrm{rot}(e)$ to each edge e of G_K; the exact definition is not relevant here; it may be found in [GL].
- We linearly order the set of edges of G_K terminating at vertex $v \in \{1, \dots, r-1\}$ as follows: if we travel along the arc of K corresponding to vertex v, we 'see' one by one the arcs corresponding to the starting vertices of red edges entering v: this gives the linear order of the red edges entering v. Finally there is at most one blue edge entering v, and we make it smaller than all the red edges entering v. The linear order will be denoted by $<_v$ and let $P(e)$ denote the set of predecessors of an edge e in the corresponding linear order.

With these decorations we define

$$\mathrm{rot}(f) = \sum_{e \in E} f(e)\mathrm{rot}(e), \qquad \mathrm{exc}(f) = \sum_v \mathrm{sign}(v) f(e_v^b)(\sum_{e \in P(e_v^r)} f(e)), \qquad \delta(f) = \mathrm{exc}(f) - \mathrm{rot}(f).$$

Let $\mathcal{S}(G)$ denote the set of all *admissible* subgraphs C of G such that each component of C is a directed cycle. Note that $\mathcal{S}(G)$ may be identified with a finite subset of $\mathcal{F}(G)$ since the characteristic function of C is a flow.

Let $\mathcal{K}$ be a knot projection. The *writhe* of $\mathcal{K}$, $\omega(\mathcal{K})$, is the sum of the signs of the crossings of $\mathcal{K}$, and $\mathrm{rot}(K)$ is the *rotation number* of K, defined as follows: *smoothen* all crossings of $\mathcal{K}$, and consider the oriented circles that appear; one of them is special, marked by $\star$. The number of circles different from the special one whose orientation agrees with the special one, minus the number of circles whose orientation is opposite to the special one is defined to be $\mathrm{rot}(K)$. We further let $\delta(K, n) = 1/2(n^2\omega(\mathcal{K}) + n\mathrm{rot}(K))$, and $\delta(K) = \delta(K, 1)$.

The following theorem appears in [LW] (see also [GL]).

thm.arcjones

Theorem 6.

$$J(\mathcal{K})(t) = t^{\delta(K)} \sum_{c \in \mathcal{S}(G_K)} t^{\delta(c)} \beta(c).$$

The colored Jones function equals to the Jones polynomial of a proper 'cabling' of the knot diagram. Using graph theory, this may be described as follows.

def.cabled2

Definition 4.4. Fix a red-blue digraph G. Let $G^{(n)}$ denote the digraph with vertices a_j^k for $k = 1, \dots, r$ and $j = 1, \dots, n$. $G^{(n)}$ contains blue directed edges (a_j^l, a_j^{l+1}) with weight $t^{-\epsilon n}$ (where $\epsilon \in \{-1, +1\}$ is the sign of the crossing l) for each $l = 1, \dots, r$ ($l+1$ considered modulo r) and $j = 1, \dots, n$. Moreover, if (a_k, a_l) is a red directed edge of G, then $G^{(n)}$ contains red edges (a_i^k, a_j^l) for all $i, j = 1, \dots n$ with weight $t^{(j-1)}(1-t)$ resp. $t^{-(n-j)}(1-t^{-1})$, if the sign of the i crossing is -1 resp. $+1$.

We will denote the set of admissible even subgraphs of $G^{(n)}$ by $\mathcal{S}_n(G)$. The following theorem appears in [GL] as Theorem 10.

thm.frst

Theorem 7. *For every knot diagram $\mathcal{K}$ and every $n \in \mathbb{N}$, we have*

$$J_n(\mathcal{K})(t) = t^{\delta(K,n)} \sum_{c \in \mathcal{S}_n(G_K)} t^{\delta(c)} \beta(c).$$

Recall that for an integer m, we denote by

$$(m)_q = \frac{q^m - 1}{q - 1}$$

the *quantum integer* m. This defines the *quantum factorial* and the *quantum binomial coefficients* by

$$(m)_q! = (1)_q(2)_q \dots (m)_q \qquad \binom{m}{n}_q = \frac{(m)_q!}{(n)_q!(m-n)_q!}$$

for natural numbers m, n with $n \leq m$. We also define

$$\mathrm{mult}_q(f) = \prod_v \binom{f(v)}{f(e_v^b)}_{q^{\mathrm{sign}(v)}}.$$

One of the key propositions of [GL] (Theorem 7) is the following expression of the colored Jones function (as a deformed zeta function of the reduced arc graph).

thm.main

Theorem 8. *For oriented knot diagram $\mathcal{K}$ we have:*

$$J_n(\mathcal{K})(t) = t^{\delta(\mathcal{K},n)} \sum_{f \in \mathcal{F}(G_K)} \mathrm{mult}_t(f) t^{\delta(f)} \prod_{v \in V_K} t^{-\mathrm{sign}(v) n f(e_v^b)} \prod_{e red; t(e)=v} \prod_{j=0}^{f(e)-1} (1 - t^{-\mathrm{sign}(s(e))(n-j-\sum_{e' <_v e} f(e))}).$$

The proof of Theorem 8 is based on a rather complicated combinatorial construction. In this paper we present a curious interpretation of this construction as a chromatic operator applied to a power series whose coefficients are linear combinations of chord diagrams.

def.cd

Definition 4.5. Given reduced knot diagram K and a flow f on G_K, we define:

- A collection $I_1, \dots, I_p$ of intervals on $1, \dots, r-1$ is *relevant* (for K, f) if for each $1 \leq v \leq r-1$, the number of intervals starting at v equals $\sum f(e); e$ red edge entering v, and the number of intervals terminating at v equals $f(e_v^r)$. We fix an order on the intervals of the relevant collection starting in the same vertex v, according to $<_v$.
- Each relevant collection of intervals defines a set of *long chord diagrams*:
  For each $1 \leq v \leq r-1$ we introduce vertices $v_1, \dots, v_{o(v)}, v^1, \dots, v^{i(v)}$. We assume that all the new vertices appear in the introduced order along a line. For each interval we introduce a *chord* on this line. The chords corresponding to intervals starting at v will start at $v_1, \dots, v_{o(v)}$, in agreement with the fixed ordering of the intervals. The chords corresponding to intervals terminating at v will terminate at $v^1, \dots, v^{i(v)}$, in *an arbitrary order*. If D is a resulting long chord diagram, then we denote by $\deg(D)$ the number of long chord diagrams obtained from the same relevant collection of intervals as D. We assume that the chords (intervals) in a long chord diagram are ordered by their starting vertices.
- We denote by $\Delta(K, f)$ the set of the long chord diagrams obtained in this way from a relevant collection of intervals for K, f.

def.intersection

Definition 4.6. Let D be a long chord diagram. We define intersection graph of its chords $G(D) = (D, E(D))$ so that the chords of D form the set of vertices of $G(D)$, and there is an edge between two vertices if the corresponding chords intersect or one contains the other.

def.chromdef

Definition 4.7. Let D be a long chord diagram, $c \in D$ and $v \in V(G(D), n)$.

- We denote by $P(D, c)$ the set of chords $c' \in D$ which encircle the starting vertex of c,
- $\mathrm{def}_1(D, v, c)$ equals the number of chords c' of $P(D, c)$ satisfying $v_{c'} < v_c$,
- We denote by $Q(D, c)$ the set of chords $c' \in D$ which encircle the terminal vertex z of c and at least one starting vertex of a chord after z,
- $\mathrm{def}_2(D, v, c)$ equals the number of chords c of $Q(D, c)$ satisfying $v_{c'} < v_c$,
- $$M_t^{\mathrm{def}}(G(D), n) = \sum_{v \in V(G(D),n)} \prod_{c \in D} t^{v_c - \mathrm{def}_1(D,v,c) - \mathrm{def}_2(D,v,c)}.$$

re.qdet

Remark 4.8. We remark that def_1 here is equivalent to the def of a tree structure, see Definition 3.4. It is very close to the definition of sign for quantum determinants, first introduced by L. Fadeev, N. Reshetikhin and L. Takhtadjian in [FRT]. Indeed, Theorem 8 is used in [GL] to give a non-commutative formula for the colored Jones function.

The following two theorems belong to the main esults of the paper.

thm.ma2

Theorem 9.
$$J_n(K)(t) = t^{\delta(K,n)} \sum_{f\in\mathcal{F}(G_K)} Z_{n,f}(t) \sum_{D\in\Delta(K,f)} \deg(D)^{-1} M_t^{\mathrm{def}}(G(D),n),$$
where $Z_{n,f}(t)$ *is a Laurent polynomial in* t *with integer coefficients and parameters* f *and* n*:*
$$Z_{n,f}(t) = t^{\delta(f)} t^{n(f_b^- - f_b^+)}(1-t)^{f_r^-}(1-t^{-1})^{f_r^+} \prod_{e\in F_r^+} t^{-(n-1-|P(f,e)|)} \prod_{v;\mathrm{sign}(v)=+} t^{f(e_v^r)f(e_v^b)}.$$

Moreover for each n *only flows bounded by* n *may contribute a non-zero to the RHS.*

The terms $f_b^-, f_b^+, f_r^-, f_r^+, F_r^+$ *are defined in 4.1 and the term* $P(f,e)$ *is defined in Definition 4.11.*

thm.ma3

Theorem 10.
$$J_n(K)(t) = t^{\delta(K,n)} \sum_{f\in\mathcal{F}(G_K)} W_{n,f}(t) \sum_{D\in\Delta(K,f)} \deg(D)^{-1} M_t(G(D),n),$$
where $W_{n,f}(t)$ *is a Laurent polynomial in* t *with integer coefficients and parameters* f *and* n*:*
$$W_{n,f}(t) = t^{\delta(f)} t^{n(f_b^- - f_b^+)}(1-t)^{f_r^-}(1-t^{-1})^{f_r^+} t^{-(n-1)f_r^+} \prod_{e\in F_r^+} t^{|P(f,e)|)} \prod_{v;\mathrm{sign}(v)=+} t^{f(e_v^r)f(e_v^b)}$$
$$\prod_{v=1}^{r} [\binom{f(v)}{f(e_v^b)}_{t^{-1}} \binom{f(v)}{f(e_v^b)}^{-1} \prod_{j=f(e_v^b)}^{f(v+1)} \frac{(f(e_v^b)+j)_{t^{-1}}}{f(e_v^b)+j}].$$

sub.prbla

4.1. **Categorification of flows: proof of Theorem 9.** Recall Theorem 7. Each $c \in \mathcal{S}_n(G_K)$ projects to a flow on G_K. An analysis of the contribution of each flow is obtained via categorification of the flows and their multiplicities in [GL]. Next we briefly describe this.

Let f be a flow on G_K. Let F (resp. F_r) denote the multiset that contains each edge (resp. red edge) e of G_K with multiplicity $f(e)$.

Let F_r^+ denote the set of all red edges of F which leave a vertex with + sign. Let $f_r^+ = |F_r^+|$. Analogously we define $F_r^-, \ldots$.

If e is an edge of G_K then we let $F(e) \subset F$ be the set of all copies of e in F, and fix an *arbitrary* total order on each $F(e)$.

def.conf

Definition 4.9. Fix a flow f on G_K. A *flow configuration* of f is a sequence $C = (C_1, \ldots, C_{r-2})$ so that C_1 is a subset of $\{e \in F_r; e \text{ terminates in vertex } 1\}$ of $f(e_1^b)$ elements and for each $2 \le i < r-1$, C_i is a subset of $C_{i-1} \cup \{e \in F_r; e \text{ terminates in vertex } i\}$ of $f(e_i^b)$ elements.

Let us denote by $\mathcal{C}(f)$ the set of all flow configurations of f.

def.AConf

Definition 4.10. Let f be a flow on G_K, $n > 0$ a natural number, $C \in \mathcal{C}(f)$, and $v \in \{0, \ldots, n-1\}^{F_r}$. We say that a pair $P = (C, v)$, is *admissible* if, for every two edges $e, e' \in F_r$ such that $v_e = v_{e'}$ and e ends in vertex i and e' ends in vertex j and $j \ge i$, there exists an l, $i \le l < j$ such that $e \notin C_l$. We denote the set of admissible flow configurations by $\mathcal{AC}(f)$.

def.prd

Definition 4.11. Let $e' \in F(e)$. We define set $P(f,e')$ as follows: if $e_1' \in F(e_1)$ then $e_1' \in P(f,e')$ if $e_1 \in P(e)$ in G_K or $e = e_1$ and $e_1' < e'$ in our fixed total order of $F(e)$.

def.okk

Definition 4.12. Let $e \in F_r$. We define

- $\mathrm{def}_1(C,v,e) = |\{e' \in P(f,e) : v_{e'} < v_e\}|$,
- $\mathrm{def}_2(C,v,e) = |\{e' \in C_{d(e)} : v_{e'} < v_e\}|$, where $d(e)$ is the biggest index such that $d(e) \ge t(e)$ and $e \notin C_{d(e)}$.

def.Ifn

Definition 4.13. If $v \in \{0,\dots,n-1\}^{F_r}$ then we define $f_r^-(v) = \sum_{e\in F_r^-} v_e$ and we define $f_r^+(v)$ analogously.

The following formulas appear in paragraph 7.3 of [GL].

thm.catmm

Theorem 11.

- $$J_n(K)(t) = t^{\delta(K,n)} \sum_{f\in\mathcal{F}(G_K)} t^{\delta(f)} t^{n(f_b^- - f_b^+)} (1-t)^{f_r^-} (1-t^{-1})^{f_r^+}$$
$$\prod_{e\in F_r^+} t^{-(n-1-|P(f,e)|)} \prod_{i:\mathrm{sign}(i)=+} t^{f(e_v^r)f(e_v^b)} \sum_{(C,v)\in\mathcal{AC}(f,n)} \prod_{e\in F_r} t^{v_e - \mathrm{def}_1(C,v,e) - \mathrm{def}_2(C,v,e)}.$$
- $$J_n(K)(t) = t^{\delta(K,n)} \sum_{f\in\mathcal{F}(G_K)} t^{\delta(f)} t^{n(f_b^- - f_b^+)} (1-t)^{f_r^-} (1-t^{-1})^{f_r^+} t^{-(n-1)f_r^+}$$
$$\prod_{e\in F_r^+} t^{|P(f,e)|} \prod_{i:\mathrm{sign}(i)=+} t^{f(e_v^r)f(e_v^b)} \prod_{v=1}^{r} \binom{f(v)}{f(e_v^b)}_{t^{-1}} \prod_{e\in F_r} (n-|P(f,e)|)_t.$$

From flow structures to relevant collections of intervals. The formula of Theorem 11 may be interpreted in terms of chordal graphs. The basic observation is that each flow structure is a $(T,V,(A_w,b_w : w\in W))-$ tree structure where T is a path with vertices $1,\dots,r-1$ rooted at 1, $B_{w+1} = C_w$ and $A_w = \{e \in F_r; e$ terminates in vertex $w\}$. Now we recall that each chordal graph is the intersection graph of subtrees of a tree, and this representation may be obtained from its tree structure (see section 3). However, if a tree structure of a graph is a path, then the graph is the intersection graph of subpaths (intervals) of the path. This directly leads to the relevant collection of intervals. Hence the first part of Theorem 11 implies Theorem 9 and the second part of Theorem 11 together with Theorem 5 implies Theorem 10.

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