Planar graphs of odd-girth at least 9 are homomorphic to Petersen graph[1]

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Abstract

Let G be a graph and let $c : V(G) \to \binom{\{1,\ldots,5\}}{2}$ be an assignment of 2-element subsets of the set $\{1, \ldots, 5\}$ to the vertices of G such that for every edge vw, the sets $c(v)$ and $c(w)$ are disjoint. We call such an assignment a $(5, 2)$-*coloring*. A graph is (5,2)-colorable if and only if it has a homomorphism to the Petersen graph. The *odd-girth* of a graph G is the length of the shortest odd cycle in G (∞ if G is bipartite). We prove that every planar graph of odd-girth at least 9 is $(5, 2)$-colorable, and thus it is homomorphic to the Petersen graph. Also, this implies that such graphs have fractional chromatic number at most $\frac{5}{2}$. As a special case, this result holds for planar graphs of girth at least 8.

1 Introduction

One of the most natural generalizations of the usual graph coloring is the k-*tuple coloring*. We assign to each vertex a set of k distinct colors, instead of just one color, and require that the adjacent vertices obtain disjoint sets of colors. Such an assignment is called a k-*tuple coloring*, or a k-*tuple* n-*coloring* if n colors are used (where $k \leq n$). To be short, we use the term (n, k)-*coloring* instead. Obviously, an $(n, 1)$-coloring is just the usual n-coloring.

It is easy to see that these multi-colorings are homomorphisms to Kneser graphs. The *Kneser graph* $K(n, k)$ (where $2k \leq n$) is a graph with vertex set consisting of all k-element subsets of $\{1, 2, \ldots, n\}$ and with two vertices adjacent if and only if the subsets are disjoint. An (n, k)-coloring of G also can be viewed as a homomorphism of G to $K(n, k)$. Let us note that $K(5, 2)$ is isomorphic to the Petersen graph.

The *fractional chromatic number* of G, denoted $\chi_f(G)$, is the infimum of the fractions n/k such that G admits an (n, k)-coloring. The notion can also be alternatively

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defined using the linear relaxation of the integer programming formulation of the graph coloring problem. For a fixed k, the minimum n such that G admits an (n, k)-coloring is called the *k-tuple chromatic number* of G and it is denoted by $\chi^k(G)$. Therefore, $\chi_f(G) = \inf \chi^k(G)/k$. More details about this coloring can be found in the monograph of Scheinerman and Ullman [5].

Let C be a circle of (Euclidean) length r. An *r-circular coloring* of a graph G is a mapping c which assigns to each vertex v of G an open unit length arc $c(v)$ of C, such that for every edge $xy \in E(G)$, $c(x) \cap c(y) = \emptyset$. We say a graph G is *r-circular colorable* if there is an r-circular coloring of G. The *circular chromatic number* of a graph, denoted by $\chi_c(G)$, is defined as

$$\chi_c(G) = \inf\{r \mid G \text{ is } r\text{-circular colorable}\}.$$

The following well-known inequalities hold for each graph G,

$$\chi(G) - 1 < \chi_c(G) \leq \chi(G) \quad \text{and} \quad \chi_f(G) \leq \chi_c(G) \leq \chi(G).$$

It is also well-known that the circular chromatic number is always a rational number. For more details on the circular chromatic number see [9].

The concept of (n/k)-circular colorings is closely related to the concept of (n, k)-flows. Let G be a graph and D an orientation of G. For positive integers k and $n \geq 2k$, an *(n, k)-flow* f of D is a mapping that assigns to each edge e of D an integer $f(e) \in \{k, k+1, \ldots, n-k\}$ such that $\sum_{e \in E^+(v)} f(e) = \sum_{e \in E^-(v)} f(e)$ for every vertex v; Here $E^+(v)$ is the set of edges incident to v that are directed away from v, and $E^-(v)$ is the set of edges incident to v that are directed towards v in D. A graph G is said to admit an (n, k)-flow if G has an orientation D that admits an (n, k)-flow.

Notice that $(n, 1)$-flows are precisely nowhere-zero integer n-flows introduced by Tutte [6, 7]. One can find more about flows in the monograph of Zhang [8] and the survey of Jaeger [2]. The following well-known conjecture was proposed by Jaeger [3]:

Conjecture 1 (Circular-Flow Conjecture) *For any integer $k \geq 1$, every $4k$-edge-connected graph admits a $(2k+1, k)$-flow.*

Note that the Circular-Flow Conjecture for $k = 1$ and $k = 2$ implies the famous 3-Flow Conjecture and 5-Flow Conjecture of W. Tutte, respectively. For planar graphs, the flow problem can be dualized to the circular coloring problem. More precisely, an (n/k)-circular coloring of a planar graph G corresponds to an (n, k)-flow of the dual graph of G. Therefore, the restriction of Jaeger's conjecture to planar graphs is equivalent to the following:

Conjecture 2 *Every planar graph G of girth at least $4k$ has circular chromatic number at most $2 + \frac{1}{k}$.*

The *odd-edge connectivity* of a graph G is the size of the smallest odd edge-cut of G. The dual of odd-edge connectivity is an *odd-girth*. The *odd-girth* of a graph G is the length of the shortest odd cycle in G (∞ if G is bipartite). Klostermeyer and Zhang [4] proposed a strengthening of Conjecture 2, where the edge-connectivity condition is replaced by an odd edge-connectivity condition:

Conjecture 3 *Every planar graph G of odd-girth at least $4k+1$ has circular chromatic number at most $2 + \frac{1}{k}$.*

Regarding Conjecture 3, Zhu [10] proved that for every $k \geq 2$, planar graphs of odd-girth at least $8k-3$ have circular chromatic number at most $2+\frac{1}{k}$. Later Borodin, Kim, Kostochka and West [1] proved that a planar graph with girth at least $\frac{20t-2}{3}$ has circular chromatic number at most $2+\frac{1}{t}$. Both results imply that planar graphs of girth at least 12 are $(5,2)$-colorable.

Conjecture 3 for $k=2$ implies that the fractional chromatic of each planar graph of odd-girth at least 9 is at most $\frac{5}{2}$. We prove that this indeed is the case, i.e., that every planar graph of odd-girth at least 9 is $(5,2)$-colorable. This also implies that planar graphs of girth 8 have fractional chromatic number at most $\frac{5}{2}$. Other interpretation of this result is that such graphs have a homomorphism to Petersen graph.

2 Main Theorem

The following theorem is the main result presented in this paper.

Theorem 1 *Every planar graph of odd-girth at least* 9 *is* $(5,2)$*-colorable.*

To prove this theorem, we first show that we may restrict our attention to graphs without any short cycles. To deal with short faces, we use the following lemma proved by Klostermeyer and Zhang [4]:

Lemma 2 (Folding Lemma) *Let G be a planar graph with odd-girth $g > 3$. If $C = v_0v_1 \ldots v_{r-1}$ is a facial circuit of G with $r \neq g$, then there is an integer $i \in \{0,\ldots,r-1\}$ such that the graph G' obtained from G by identifying v_{i-1} and v_{i+1} (where indices are taken modulo r) is also of odd-girth g.*

To handle short cycles that are not faces, we show that under some assumptions, a coloring of the exterior of such a cycle can be extended to its interior:

Theorem 3 *Let H be a 2-connected planar graph such that H does not contain a cycle of length smaller than 9, except possibly for its outer face O_H whose length is at least 4. The graph H is $(5,2)$-colorable. Furthermore, if the length of O_H is at most 8, then any proper $(5,2)$-coloring of O_H can be extended to a proper coloring of H.*

Assuming that Theorem 3 holds, we can prove Theorem 1 easily. Given a plane graph G and a cycle C, let $\mathrm{Int}_G(C)$ be the subgraph of G consisting of vertices and edges that lie on C or inside C, and let $\mathrm{Out}_G(C)$ be the subgraph of G consisting of vertices and edges that lie on C or outside C. Thus, $\mathrm{Int}_G(C)$ and $\mathrm{Out}_G(C)$ share only the vertices and edges of C.

Proof of Theorem 1. For the sake of contradiction, let us assume that Theorem 1 is false and let G be a counterexample with minimum number of vertices. Obviously, G is connected. We may also assume that G is 2-connected, otherwise we can color each block separately and permute the colors so that the colorings agree on the cut-vertices.

Let us show that each face of G has size at least 9. Let $g \geq 9$ be the odd-girth of G. Assume that G contains a face of length different from g. By Lemma 2, there exist vertices v_1 and v_2 on that face such that the graph G' obtained from G by identifying v_1 and v_2 is a planar graph of odd-girth g. Note that the vertices v_1 and v_2 are not adjacent, because of the restriction on the odd-girth of G. By the minimality of G, the graph G'

is $(5,2)$-colorable. However, since v_1 and v_2 are not adjacent, this also gives a proper $(5,2)$-coloring of G, which is a contradiction.

If G does not contain any cycle of length less than 9, then G is $(5,2)$-colorable by Theorem 3. Therefore, assume that G contains a short cycle. Let C be a cycle in G of length at most 8 such that the number of vertices of $\mathrm{Int}_G(C)$ is minimal. Since G does not contain any face of length less than 9, the graph $\mathrm{Out}_G(C)$ has less vertices than G. By the minimality of G, there exists a proper $(5,2)$-coloring c of $\mathrm{Out}_G(C)$. The graph $\mathrm{Int}_G(C)$ does not contain any cycle shorter than 9, thus the coloring c can be extended to the vertices of $\mathrm{Int}_G(C)$ by Theorem 3. This shows that G can be $(5,2)$-colored, which is a contradiction. □

It remains to prove Theorem 3. We proceed by the discharging method. Let us assume that H is a minimal counterexample to the theorem throughout the rest of the paper, and let O_H be its outer face. We first assign charge to vertices and faces of H in such a way that the total amount of charge is negative. The bound on the length of faces ensures that the charge of each face is non-negative, and we never change the charge of faces, hence we know that at each moment the total charge of vertices is negative. The initial distribution of the charge is described in Section 4. We then redistribute the charge, and assuming that H is a minimal counterexample to Theorem 3, we show that the final charge of each vertex is non-negative. Since the amount of charge does not change during its redistribution, we obtain a contradiction. This will establish Theorem 3.

In Section 3, we describe some subgraphs that cannot appear in H, we call such subgraphs (configurations) *reducible.* The proof of reducibility of a configuration C usually proceeds in the following way. By the minimality of H, we know that $H - C$ can be properly colored. We then show that an arbitrary coloring of $H - C$ can be extended to C, thus showing that H is (5,2)-colorable and obtaining a contradiction.

The discharging proceeds in two phases. Let us call vertices of degree at least 3 *important.* In the first phase (Section 5), we consider vertices of degree 2 in H, and make their charge non-negative by moving charge from adjacent important vertices. We then analyze the charge of important vertices, and show that the subgraph H' induced by the important vertices with negative or small positive charge has a very special structure—in particular, it has maximum degree at most three, and only vertices of degree at most two may have negative charge.

In Section 6, we study the reducible configurations in H'. This enables us to restrict the structure of H'. Finally, we run the second phase of discharging over H', moving the charge from vertices of degree three to vertices of smaller degree in H', and show that one of 3-vertices is contained in a reducible configuration (Section 7).

Throughout the proof, we need to handle the precolored outer face O_H. We would run into problems if a vertex v of a reducible configuration C were precolored, since we would be unable to prove that the coloring of $H - C$ can be extended in a way that is consistent with the color of v. To avoid this problem, we add an extra charge to the vertices of O_H while still preserving the fact that the total amount of charge is negative. This extra charge ensures that the charge of the vertices of O_H is large enough, thus they do not appear in any configuration we consider—only configurations whose total charge is negative are important in the discharging method.

3 Reducible configurations

We say that a configuration (a subgraph) is *reducible* if it cannot appear in the minimal counterexample H, except possibly if some of its vertices are precolored. To use the discharging method we first need to prove that some configurations are reducible. We start with the following lemma that enables us to handle vertices of degree two in the minimal counterexample H of Theorem 3.

Lemma 4 *The following claims hold:*

(1) Let $P = v_1v_2v_3$ be a path of H such that $d(v_2) = 2$ and v_2 is not a precolored vertex of the outer face. If c is a coloring of $H - v_2$ such that $c(v_1) \cap c(v_3) \neq \emptyset$, then c can be extended to a proper coloring of H.

(2) Let $P = v_1v_2v_3v_4$ be a path of H such that $d(v_2) = d(v_3) = 2$ and neither v_2 nor v_3 is precolored. If c is a coloring of $H - \{v_2, v_3\}$ such that $c(v_1) \neq c(v_4)$, then the coloring c can be extended to a proper coloring of H.

(3) If P is a path $v_1v_2v_3v_4v_5$ in H such that $d(v_2) = d(v_3) = d(v_4) = 2$ and neither of v_2, v_3 and v_4 is precolored, then P is reducible.

Proof. Let us show each claim separately:

(1) Let $X = \{1, \ldots, 5\} \setminus [c(v_1) \cup c(v_2)]$. Size of X is at least two, thus we can choose $c(v_2) \subseteq X$.

(2) If $c(v_1) \cap c(v_4) = \emptyset$, then set $c(v_2) = c(v_4)$ and $c(v_3) = c(v_1)$. Otherwise $c(v_1) = \{a, b_1\}$ and $c(v_4) = \{a, b_2\}$ have exactly one common element a. Let $\{b_3, b_4\} = \{1, \ldots, 5\} \setminus \{a, b_1, b_2\}$. We set $c(v_2) = \{b_2, b_3\}$ and $c(v_3) = \{b_1, b_4\}$.

(3) Suppose that this configuration appears in H. There exists a coloring c of $H - \{v_2, v_3, v_4\}$ by the minimality of H. Let $c(v_2)$ be a color-pair disjoint with $c(v_1)$ and distinct from $c(v_5)$ and use the second claim of this lemma to color v_3 and v_4. In this way, we extend the coloring to H, thus obtaining a contradiction. □

We need to show that many subgraphs are reducible. For each such subgraph C, we consider the coloring of $H - C$ and extend it to C. We usually use counting arguments of the following type: if a vertex v of C is adjacent to exactly one vertex u of $H - C$, the vertex v can be colored by one of three colors that are not forbidden by the color of u. If we determine the numbers of available colors for each vertex of C in a similar way, and show that C can be colored from all possible lists of colors of the given length, then we prove that C is reducible. To make the statements and the proofs of the theorems simpler, we introduce a notion closely related to the list coloring.

Let $\mathcal{P} = \binom{\{1,\ldots,5\}}{2}$ be the set of all unordered pairs of the elements of $\{1, \ldots, 5\}$ and let us call the elements of $\mathcal{P}$ *color-pairs*. A *list* is a subset of $\mathcal{P}$, and if its size is k, then we call it a *k-list*. Lists are used to specify which color-pairs can be used at vertices in the considered configurations. For a given $X \subseteq \{1, \ldots, 5\}$, let $\mathrm{ss}(X) = \{p \in \mathcal{P}; p \subseteq X\}$ be the list of all 2-element subsets of X. Let $\mathrm{int}(X) = \{p \in \mathcal{P}; p \cap X \neq \emptyset\}$ be the list of all color-pairs that intersect X.

A *type* is a set of lists. Types are used to express properties of the possible colorings of the considered configurations. For example, if we know that a vertex v has one neighbor precolored with a color-pair p, then the color-pairs that can be used to color v must be disjoint with p, and the list of such color-pairs belongs to the type $\mathcal{N}_3$ defined below. Let us point out the following special types:

- $\mathcal{P}_{all} = \wp(\mathcal{P})$ is the set of all lists, and $\mathcal{P}_k$ is the set of all k-lists. Thus, $\mathcal{P}_{10} = \{\mathcal{P}\}$.
- $\mathcal{N}_4 \subset \mathcal{P}_4$ is the set of all 4-lists such that no color belongs to all color-pairs in the list. For example, the list $L = \{\{1,2\},\{1,3\},\{1,4\},\{1,5\}\}$ does not belong to $\mathcal{N}_4$.
- $\mathcal{N}_3 \subset \mathcal{P}_3$ is the type of all 3-lists that can be expressed as $\mathrm{ss}(X)$ for some 3-element subset X of $\{1,\dots,5\}$.
- $\mathcal{N}_7 \subset \mathcal{P}_7$ is the set of all 7-lists, whose complements belong to $\mathcal{N}_3$. In other words, $\mathcal{N}_7$ is the type of all lists that can be expressed as $\mathrm{int}(X)$ for some 2-element subset X of $\{1,\dots,5\}$.

If $\mathcal{A}$ and $\mathcal{B}$ are two types, we write $\mathcal{B} \geq \mathcal{A}$ if every list in $\mathcal{B}$ has a (not necessarily proper) subset that belongs to $\mathcal{A}$. That is, the lists in $\mathcal{B}$ are "greater" than the lists in $\mathcal{A}$. In particular, if $\mathcal{B} \geq \mathcal{A}$ and a coloring can be extended to a vertex whose list of available colors belongs to $\mathcal{A}$, then the coloring can also be extended to the same vertex with list from $\mathcal{B}$. Obviously, if $\mathcal{B} \subseteq \mathcal{A}$, then $\mathcal{B} \geq \mathcal{A}$. Also, $\mathcal{P}_m \geq \mathcal{P}_n$ for $m \geq n$.

Lemma 5 *The following relations hold:*

(1) $\mathcal{P}_7 \geq \mathcal{N}_3$*, and*

(2) $\mathcal{P}_9 \geq \mathcal{N}_7$*.*

Proof. Consider the first claim. We may assume that there exists a set X of seven color-pairs that does not contain a subset in $\mathcal{N}_3$. Each of the color-pairs has two elements; therefore, there must be a color that belongs to at least three color-pairs in X. We may assume that these color-pairs are $\{1,2\}$, $\{1,3\}$ and $\{1,4\}$. Hence, the color-pairs $\{2,3\}$, $\{2,4\}$ and $\{3,4\}$ cannot belong to X. By the size of X, all remaining color-pairs must belong to X. In particular, $\{1,5\}$ and $\{2,5\}$ belong to X and together with $\{1,2\}$ they form a list from $\mathcal{N}_3$, which is a contradiction.

Let us now show the second claim. Let X be an arbitrary list in $\mathcal{P}_9$, i.e, $X = \mathcal{P} \setminus \{p\}$ for a color-pair p. Let Y be any 2-element subset of $\{1,\dots,5\} \setminus p$. The set $\mathrm{int}(Y) \in \mathcal{N}_7$ is a subset of X, witnessing that $\mathcal{P}_9 \geq \mathcal{N}_7$. □

We say that a color-pair p_1 *forbids* a color-pair p_2 if p_1 and p_2 are not disjoint, and that a list L *forbids* a color-pair p if every color-pair in L forbids p. This means that if vertices u and v are adjacent and the color of u belongs to L, then the color of v cannot be p, or vice-versa, if the color of v is p, then we cannot color u from the list L. We often use a counting argument—if we determine that less than 10 color-pairs can be forbidden at a vertex v by a coloring of some of its neighbors, such a coloring can be extended to the vertex v. A color-pair that is not forbidden is called *free*.

So far, we were using the terms "configuration" and "reducible" intuitively. Let us now define them formally using the concept of types. A *configuration* C is a graph $G_C = (V_C, E_C)$ together with a function $\mathrm{type}_C : V_C \to \wp(\mathcal{P}_{all})$ that assigns a type to each vertex of C. Let us remark that we only consider configurations that do not contain precolored vertices of the face O_H. The configuration represents an induced subgraph of

H whose reducibility we study. We want to show that any precoloring of the vertices of $V(H) \setminus V_C$ can be extended to a $(5,2)$-coloring of whole H. The types of the vertices of the configuration reflect the constraints imposed by colors of vertices of $V(H) \setminus V_C$—each proper $(5,2)$-coloring of $V(H) \setminus V_C$ gives us a list of free colors at each vertex v of C, and the type of v in the configuration consists of all such lists.

An *instance* Q of the configuration C is a function from V_C to $\mathcal{P}_{all}$ such that $Q(v) \in \mathrm{type}_C(v)$ for each vertex v. The instance Q is *colorable* if there exists a proper coloring c of G_C such that $c(v) \in Q(v)$ for each vertex of C. The configuration C is *reducible* if each instance of C is colorable.

Note that if $\mathcal{X}$ and $\mathcal{Y}$ are types with $\mathcal{X} \geq \mathcal{Y}$ and we change the type of a vertex in a reducible configuration from $\mathcal{Y}$ to $\mathcal{X}$, then the new configuration is reducible as well. We use the following notation to simplify the descriptions of configurations. If $\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_k$ are types, then: $\mathrm{path}(\mathcal{X}_1\, \mathcal{X}_2\, \ldots\, \mathcal{X}_n)$ is a path $v_1 v_2 \ldots v_n$ on n vertices with $\mathrm{type}(v_i) = \mathcal{X}_i$ for each i. Similarly, $\mathrm{cycle}(\mathcal{X}_1\, \mathcal{X}_2\, \ldots\, \mathcal{X}_n)$ is a cycle $v_1 v_2 \ldots v_n$ on n vertices with types $\mathrm{type}(v_i) = \mathcal{X}_i$.

Lemma 6 *The following configurations are reducible:*

(1) $\mathrm{path}(\mathcal{P}_1\, \mathcal{P}_8)$,

(2) $\mathrm{path}(\mathcal{P}_2\, \mathcal{P}_6)$,

(3) $\mathrm{path}(\mathcal{P}_3\, \mathcal{P}_5)$, *and*

(4) $\mathrm{path}(\mathcal{N}_4\, \mathcal{P}_4)$.

Proof. Observe that a configuration $\mathrm{path}(\mathcal{X}\, \mathcal{P}_n)$ is reducible if and only if every list in $\mathcal{X}$ forbids at most $n-1$ color-pairs. In the rest of this proof, let a, b, c, d and e be mutually distinct elements of $\{1,\ldots,5\}$. Let us consider the configurations separately.

(1) A single color-pair $\{a, b\}$ forbids exactly the seven color-pairs of $\mathrm{int}(\{a, b\})$; therefore, $\mathrm{path}(\mathcal{P}_1\, \mathcal{P}_8)$ is reducible.

(2) If the two color-pairs are disjoint, say $\{a, b\}$ and $\{c, d\}$, then they forbid four color-pairs $\{a, c\}$, $\{a, d\}$, $\{b, c\}$ and $\{b, d\}$. Otherwise they share a common element, say the pairs are $\{a, b\}$ and $\{a, c\}$, then they forbid five color-pairs—the four color-pairs that contain a and the color-pair $\{b, c\}$. Therefore, $\mathrm{path}(\mathcal{P}_2\, \mathcal{P}_6)$ is reducible.

(3) Without loss of generality, there are the following four possibilities for three color-pairs:

- $\{a, b\}$, $\{a, c\}$ and $\{a, d\}$, forbidding the four color-pairs that contain a.
- $\{a, b\}$, $\{a, c\}$ and $\{b, c\}$, forbidding three color-pairs $\{a, b\}$, $\{a, c\}$ and $\{b, c\}$.
- $\{a, b\}$, $\{a, c\}$ and $\{b, d\}$, forbidding three color-pairs $\{a, b\}$, $\{a, d\}$ and $\{b, c\}$.
- $\{a, b\}$, $\{a, c\}$ and $\{d, e\}$, forbidding two color-pairs $\{a, d\}$ and $\{a, e\}$.

Therefore, $\mathrm{path}(\mathcal{P}_3\, \mathcal{P}_5)$ is reducible.

(4) Note that if a list X is a subset of a list Y and X forbids n color-pairs, then Y forbids at most n color-pairs. If L is any list from $\mathcal{N}_4$, then there is a list $L' \subset L$ of size three such that the intersection of elements of L' is empty. But such a list L' forbids at most three color-pairs, as follows from the proof of the previous claim of this lemma. Therefore, L also forbids at most three color-pairs, and hence path($\mathcal{N}_4\,\mathcal{P}_4$) is reducible. $\square$

Before we proceed with the next lemma, let us introduce the following convention: we use $\mathcal{X}^*$ inside a path or a cycle description of a configuration for an arbitrarily long (possibly empty) chain of vertices of type $\mathcal{X}$. For example, path($\mathcal{P}_2\,\mathcal{P}_7^*\,\mathcal{P}_9$) may be any of path($\mathcal{P}_2\,\mathcal{P}_9$), path($\mathcal{P}_2\,\mathcal{P}_7\,\mathcal{P}_9$), path($\mathcal{P}_2\,\mathcal{P}_7\,\mathcal{P}_7\,\mathcal{P}_9$), etc.

Note that a chain of vertices of type $\mathcal{P}_7$ "propagates" the type of the vertex adjacent to it. For example, let $uv_1v_2\dots v_kw$ be a path, and let the type of u be $\mathcal{P}_6$ (or any other type $\mathcal{X} \geq \mathcal{P}_6$) and let the type of each v_i ($1 \leq i \leq k$) be $\mathcal{P}_7$. Let Q be an instance of this configuration. The list $Q(u)$ forbids at most one color-pair q, because path($\mathcal{P}_6\,\mathcal{P}_2$) is reducible by Lemma 6(2). Let $Q_1 = Q(v_1) \setminus \{q\}$ be the list of v_1 without the forbidden color-pair. The size of Q_1 is at least 6, thus it also forbids at most one color-pair q_1 in the list of v_2. Similarly the list $Q_2 = Q(v_2) \setminus \{q_1\}$ forbids at most one color-pair in the list of v_3, etc. Finally, the non-forbidden part of $Q(v_k)$ forbids at most one color-pair in $Q(w)$. This in particular means that if $|Q(w)| > 1$, we may color the vertex w by a remaining color, and then choose a free color for v_k, v_{k-1}, $\dots$, v_1 and u. Thus, we can say that path($\mathcal{P}_6\,\mathcal{P}_7^*$) behaves as path($\mathcal{P}_6$). Similarly, since a list in $\mathcal{P}_5$ forbids at most two color-pairs by Lemma 6(3), path($\mathcal{P}_5\,\mathcal{P}_7^*$) behaves as path($\mathcal{P}_5$), and since a list in $\mathcal{P}_2$ forbids five color-pairs by Lemma 6(2), path($\mathcal{P}_2\,\mathcal{P}_7^*$) behaves as path($\mathcal{P}_2$). The lists in types $\mathcal{P}_4$ and $\mathcal{P}_3$ may forbid four color-pairs by Lemma 6(3), so path($\mathcal{P}_4\,\mathcal{P}_7^*$) and path($\mathcal{P}_3\,\mathcal{P}_7^*$) behave as path($\mathcal{P}_3$).

Lemma 7 *The following configurations are reducible:*

(1) path($\mathcal{P}_2\,\mathcal{P}_8\,\mathcal{P}_5$)*,*

(2) path($\mathcal{P}_2\,\mathcal{P}_7^*\,\mathcal{P}_{10}\,\mathcal{P}_7^*\,\mathcal{P}_2$)*, and*

(3) path($\mathcal{P}_2\,\mathcal{P}_7^*\,\mathcal{P}_9\,\mathcal{P}_7^*\,\mathcal{P}_3$)*.*

Proof. Let us show the reducibility of each of the configurations separately.

(1) The reducibility of the configuration path($\mathcal{P}_2\,\mathcal{P}_8\,\mathcal{P}_5$) is a simple consequence of Lemma 6(2) and (3). A vertex with list from type $\mathcal{P}_2$ forbids at most five color-pairs. A vertex with list from type $\mathcal{P}_5$ forbids at most two color-pairs. Therefore, there is at least one free color-pair in the list of the vertex of type $\mathcal{P}_8$.

(2) Since the type of vertices propagates through the chain of vertices of type $\mathcal{P}_7$, it is sufficient to show that path($\mathcal{P}_2\,\mathcal{P}_{10}\,\mathcal{P}_2$) is reducible. Let $v_1v_2v_3$ be the configuration and let Q be an instance of the configuration. A vertex of type $\mathcal{P}_2$ forbids at most five color-pairs. If either $Q(v_1)$ or $Q(v_3)$ forbids less than 5 color-pairs at v_2, then the instance is colorable. By the proof of Lemma 6(2), a list $L \in \mathcal{P}_2$ can forbid five color-pairs only if the color-pairs in L intersect each other. Suppose that both $Q(v_1)$ and $Q(v_3)$ forbid five color-pairs. Let a be the intersection of color-pairs in $Q(v_1)$ and b the intersection of color-pairs in $Q(v_3)$. If $a \neq b$, then let $p = \{a, b\}$,

otherwise let p be any color-pair that contains a. Observe that p is forbidden by both $Q(v_1)$ and $Q(v_3)$. This means that the total number of forbidden color-pairs in $Q(v_2)$ is at most 9, hence the instance is colorable.

(3) We argue similarly as in the previous claim that suffices to show that $\mathrm{path}(\mathcal{P}_2\,\mathcal{P}_9\,\mathcal{P}_3)$ is reducible. We use a similar argument as in the previous case. Let $v_1v_2v_3$ be the configuration and let Q be an instance of the configuration. A vertex of type $\mathcal{P}_2$ forbids at most five color-pairs and a vertex of type $\mathcal{P}_3$ forbids at most four color-pairs. Therefore, if the instance is not colorable, $Q(v_1)$ must forbid precisely five color-pairs and $Q(v_3)$ must forbid precisely four color-pairs. The proofs of Lemma 6(2) and (3) show that this may happen only if both $\bigcap_{p\in Q(v_1)} p \neq \emptyset$ and $\bigcap_{p\in Q(v_3)} p \neq \emptyset$. Let a be the element of intersection of $Q(v_1)$ and b the element of intersection of $Q(v_3)$. If $a \neq b$, then let $p = \{a, b\}$, otherwise let p be any color-pair that contains a. Observe that p is forbidden by both $Q(v_1)$ and $Q(v_3)$. This means that the total number of forbidden color-pairs in $Q(v_2)$ is at most 8, which shows that the instance is colorable. □

Lemma 8 *The configurations* $\mathrm{path}(\mathcal{P}_2\,\mathcal{P}_7^*\,\mathcal{P}_9\,\mathcal{P}_7^*\,\mathcal{P}_8\,\mathcal{P}_7^*\,\mathcal{P}_2)$ *are reducible.*

Proof. Let $v_1v_2\ldots v_n$ be one of the configurations described by the statement of this lemma, and let Q be its instance. Since $\mathcal{P}_2$ propagates over chain of vertices of type $\mathcal{P}_7^*$, it suffices to consider the case when v_2 has type $\mathcal{P}_9$ and v_{n-1} has type $\mathcal{P}_8$.

By Lemma 7(3), the configuration $\mathrm{path}(\mathcal{P}_2\,\mathcal{P}_9)$ on v_1v_2 forbids at most two color-pairs at v_3. If $n = 4$, then v_3 is of type $\mathcal{P}_8$, and hence $Q(v_3)$ contains at least six free color-pairs. This instance of the configuration $\mathrm{path}(\mathcal{P}_6\,\mathcal{P}_2)$ on v_3v_4 is colorable by Lemma 6(2). If $n > 4$, then the type of v_3 is $\mathcal{P}_7$ and $Q(v_3)$ contains at least five free color-pairs. Since the type $\mathcal{P}_5$ propagates over chain of vertices of type $\mathcal{P}_7$, it suffices to show that the configuration $\mathrm{path}(\mathcal{P}_5\,\mathcal{P}_8\,\mathcal{P}_2)$ is reducible. This follows by Lemma 7(1). □

Lemma 9 *The cyclic configurations* $\mathrm{cycle}(\mathcal{N}_3\,X\,\mathcal{N}_3\,\mathcal{P}_{10}\,\mathcal{P}_{10})$*, where X is an arbitrary path consisting of one vertex of type $\mathcal{P}_9$ and at least three vertices of type $\mathcal{N}_7$, are reducible.*

Proof. Let $v_1v_2\ldots v_k$ be one of these configurations, where v_1 and v_{k-2} are the vertices of type $\mathcal{N}_3$ and v_{k-1} and v_k have type $\mathcal{P}_{10}$. By symmetry, we may assume that the types of v_2 and v_3 are $\mathcal{N}_7$. Let Q be an instance of this configuration. We may assume that $Q(v_1) = \mathrm{ss}(\{1, 2, 3\})$. By Lemma 4(2), any coloring of the vertices v_1 and v_{k-2} by distinct color-pairs can be extended to v_{k-1} and v_k.

Suppose first that $Q(v_{k-2}) \neq Q(v_1)$, and thus $|Q(v_{k-2}) \cap Q(v_1)| \leq 1$. Now restrict the configuration to the vertices $v_1, \ldots, v_{k-2}$ and create an instance Q' such that $Q'(v_1) = Q(v_1) \setminus Q(v_{k-2})$ (i.e., a list that consists of at least two color-pairs) and $Q'(v_i) = Q(v_i)$ for $i > 1$. Every coloring of Q' can be extended to a coloring of Q, and the instance Q' is colorable by Lemma 7(3). Hence, Q is colorable as well.

In the rest of the lemma, we assume that $Q(v_{k-2}) = Q(v_1)$. We know that $Q(v_2) = \mathrm{int}(p)$, where p is some color-pair. Let us now consider the case that $p \notin Q(v_1)$. Without loss of generality, we may assume that either $p = \{1, 4\}$ or $p = \{4, 5\}$. Restrict the configuration to vertices $v_2, \ldots, v_{k-2}$ and create an instance Q' such that $Q'(v_2) = \{\{1, 4\}, \{1, 5\}, \{4, 5\}\} \subset Q(v_2)$, $Q'(v_{k-2}) = \{\{1, 2\}, \{1, 3\}\}$ and $Q'(v_i) = Q(v_i)$ for $2 < i < k-2$. It is easy to check that any coloring of Q' can be extended to a coloring of Q, since for each color-pair in $Q'(v_2)$ we may color v_1 with $\{2, 3\}$. The instance Q' is colorable by Lemma 7(3). Thus, Q is colorable as well.

Now we consider the case $p \in Q(v_1)$. Without loss of generality, we may assume $p = \{1,2\}$ and $Q(v_3) = \mathrm{int}(q)$, where q is one of the following color-pairs: $\{1,2\}$, $\{1,3\}$, $\{1,5\}$, $\{3,5\}$ or $\{4,5\}$. Let us restrict the configuration to vertices $v_3, \ldots, v_{k-2}$. Let Q' be the instance such that $Q'(v_{k-2}) = \{\{1,2\},\{2,3\}\}$ and $Q'(v_i) = Q(v_i)$ for $3 < i < k-2$. The list of v_3 is defined as follows:

- If $1 \in q$, then $Q'(v_3) = \{\{1,3\},\{1,4\},\{1,5\}\}$.
- If $q = \{3,5\}$, then $Q'(v_3) = \{\{2,5\},\{3,5\},\{2,3\}\}$.
- If $q = \{4,5\}$, then $Q'(v_3) = \{\{2,5\},\{3,5\},\{2,4\}\}$.

Any coloring c of Q' can be extended to a coloring of Q: if $1 \in q$ or $c(v_3) = \{3,5\}$, then we set $c(v_1) = \{1,3\}$, otherwise we choose $c(v_1) \in \{\{1,2\},\{2,3\}\}$ distinct from $c(v_{k-2})$. In both cases, there is a free color-pair in $Q(v_2)$. The instance Q' is colorable by Lemma 7(3), and hence Q is also colorable. □

We define the following operations on types:

- $\mathrm{cut}_1(\mathcal{X}) = \{X'; (\exists X \in \mathcal{X})\ X' \subseteq X, |X \setminus X'| \leq 1\}$, i.e., the lists in this set can be obtained from those in $\mathcal{X}$ by removing at most one color-pair.
- $\mathrm{cut}_3(\mathcal{X}) = \{X'; (\exists X \in \mathcal{X})\ (\exists p \in \mathcal{P})\ X' = X \cap \mathrm{int}(p)\}$, i.e., the lists in this set can be obtained from those in $\mathcal{X}$ by removing color-pairs that are disjoint with a color-pair p.
- $\mathrm{cut}_7(\mathcal{X}) = \{X'; (\exists X \in \mathcal{X})\ (\exists p \in \mathcal{P})\ X' = X \setminus \mathrm{int}(p)\}$, i.e., the lists in this set can be obtained from those in $\mathcal{X}$ by removing color-pairs that intersect a color-pair p.

Let us write cut_a^k for applying k times the operation cut_a, in particular cut_a^0 is identity. These operations are used to reflect the effect of precoloring vertices to the lists of remaining vertices. Note that the operations commute with each other, i.e., $\mathrm{cut}_a(\mathrm{cut}_b(\mathcal{X})) = \mathrm{cut}_b(\mathrm{cut}_a(\mathcal{X}))$ for each $a, b \in \{1,3,7\}$.

Lemma 10 *The following relations hold:*

(1) $\mathrm{cut}_1(\mathcal{P}_n) \geq \mathcal{P}_{n-1}$,

(2) $\mathrm{cut}_3(\mathcal{P}_{10}) = \mathcal{N}_7$,

(3) $\mathrm{cut}_7(\mathcal{P}_{10}) = \mathcal{N}_3$, *and*

(4) $\mathrm{cut}_3^2(\mathcal{P}_{10}) \geq \mathcal{N}_4$.

Proof. The first three claims are trivial. Let us consider the fourth one. Let $L \in \mathrm{cut}_3^2(\mathcal{P}_{10})$. Note that $|L| \geq 4$. If $|L| \geq 5$, then L obviously contains four color-pairs whose intersection is empty. Thus, we only need to consider the case when $|L| = 4$. This only happens when the color-pairs p_1 and p_2 used to obtain L from $\mathcal{P}_{10}$ by cut_3 operations are disjoint. We may assume that $p_1 = \{1,2\}$ and $p_2 = \{3,4\}$. The list $L = \{\{1,3\},\{1,4\},\{2,3\},\{2,4\}\}$ belongs to $\mathcal{N}_4$, thus proving the fourth relation. □

A vertex is called *important* if its degree is at least three, and an edge is called *important* if both of its vertices are important. Let us call a path with k inner vertices of degree two connecting two important vertices a *k-link*. In particular, an important edge is a 0-link. Links with $k > 0$ are called *proper*. By Lemma 4(3), only 0-, 1- and 2-links may appear in H.

Let G be a fixed subgraph and v an important vertex belonging to G. We say that G is *closed* with respect to v if all links incident to v are either completely contained in G, or share only the vertex v with G. For such a vertex v, let $U(v)$ be the set of important vertices that do not belong to G and that are joined with v by a link, and let $U^*(v)$ be the set consisting of v, vertices in $U(v)$ and the vertices of the links connecting v to vertices of $U(v)$. Suppose that v is joined by n_k k-links to vertices in $U(v)$, for $0 \leq k \leq 2$. Let $\mathcal{U}_v = \mathrm{cut}_7^{n_0}(\mathrm{cut}_3^{n_1}(\mathrm{cut}_1^{n_2}(\mathcal{P}_{10})))$. We say that G is closed if it is closed with respect to all important vertices in G.

The following lemma shows how we can determine the type of a vertex v in a configuration C from the links through that v is connected to vertices that do not belong to C. This lemma basically restates Lemma 4, and tells us that links to vertices of C can be eliminated and replaced by setting the types of vertices of the configuration using the appropriate cut operations.

Lemma 11 *Let v be an important vertex of a subgraph G such that G is closed with respect to v. Let c be a proper coloring of the graph induced by $U(v)$. If L is the list of color-pairs p such that $c(v) = p$ for some extension of c to the subgraph induced by $U^*(v)$, then $L \in \mathcal{U}_v$.*

Proof. If $u \in U(v)$ is adjacent to v, then the color-pair $c(u)$ forbids all color-pairs that intersect $c(u)$ at v. By Lemma 4, if u is joined to v by a 1-link, then the color-pair $c(u)$ forbids all color-pairs that are disjoint with $c(u)$. Finally, if u is joined to v by a 2-link, then the color-pair $c(u)$ forbids the color-pair $c(u)$ at v. The operations cut_7, cut_3 and cut_1 modify the lists in the type they are applied to in the same way, and since the lists for all possible choices of the color-pair $c(u)$ are included in $\mathcal{U}_v$, we infer that $L \in \mathcal{U}_v$. $\square$

If we determine the types of important vertices of an induced subgraph of H using Lemma 11 and the resulting configuration is reducible, then we obtain a contradiction with the fact that H is a counterexample to Theorem 3, as the following lemma shows. This provides us with a powerful tool for determining the subgraphs that cannot appear in H.

Lemma 12 *Let G be closed subgraph without precolored vertices. Let C be the configuration on G, where the type of each important vertex v of C is $\mathcal{U}_v$, and the type of each 2-vertex is $\mathcal{P}_{10}$. If the configuration C is reducible, then G cannot appear in H.*

Proof. Suppose for the sake of contradiction that G appears in H. Let $W = V_C \cup \bigcup_{v \in V_C}(U^*(v) \setminus U(v))$ be the set consisting of V_C and vertices of degree two contained in proper links that share only one vertex with C. Let $V' = V(H) \setminus W$. By the minimality of H, the subgraph induced by V' has a proper $(5,2)$-coloring c_1. Let v be a vertex of C. By Lemma 11, the list L_v of color-pairs that are not forbidden by c at v belongs to $\mathcal{U}_v$. Let Q be the instance of C in that the list L_v is selected at each vertex v of C. Since C is reducible, the instance Q is colorable. Thus, a proper coloring c_2 of C from the lists of free color-pairs exists. By Lemma 4, the union of colorings c_1 and c_2 can be extended to the remaining 2-vertices of the links that belong to W (note that these vertices cannot be precolored, since the vertices of G are not precolored, and the precolored vertices form a cycle in H). This yields a proper $(5, 2)$-coloring of H, which is a contradiction. □

4 Initial Charge

We assign charge to vertices and faces of H. For every $v \in V(H)$, we define the initial vertex charge $\mathrm{ch}_0(v)$ as

$$\mathrm{ch}_0(v) = \frac{7}{3}d(v) - 6,$$

where $d(v)$ denotes the degree of v in the graph H. Let $F(H)$ be the set of faces of the graph H. For every face $f \in F(H)$, distinct from O_H, we define the initial face charge $\mathrm{ch}_0(f)$ as

$$\mathrm{ch}_0(f) = \frac{2}{3}r(f) - 6,$$

where $r(f)$ denotes the length of f. The charge of the outer face O_H is set to $\frac{2}{3}r(O_H) + \frac{11}{2}$. By Euler's formula, the total amount of charge is

$$\begin{aligned}\sum_{v\in V(H)} \mathrm{ch}_0(v) + \sum_{f\in F(H)} \mathrm{ch}_0(f) &= \left(\frac{14}{3}|E(H)| - 6|V(H)|\right) \\ &\quad + \left(\frac{4}{3}|E(H)| - 6|F(H)|\right) + \frac{23}{2} \\ &= 6(|E(H)| - |V(H)| - |F(H)|) + \frac{23}{2} \\ &= -\frac{1}{2}.\end{aligned}$$

Here we see the charge assigned to the vertices of small degrees:

$d(v)$	2	3	4	5	6
$\mathrm{ch}_0(v)$	$-\frac{4}{3}$	1	$\frac{10}{3}$	$\frac{17}{3}$	8

Note that the charge of each face f distinct from O_H is at least $\frac{2}{3}r(f) - 6 \geq 0$, since $r(f) \geq 9$. We redistribute the charge of the vertices in two phases such that the total amount of charge does not change. However, we show that in the minimal counterexample H, the final charge of each vertex and face is non-negative, thus contradicting the existence of the minimal counterexample and proving Theorem 3.

5 First Phase of Discharging

In the first phase, we ensure that all 2-vertices have non-negative charge. We use the following redistribution rules to move charge from the important vertices to links:

(R1) If xyz is a 1-link in H such that x has degree 3, 4 or 5 and y is not precolored, then x sends $\frac{2}{3}$ to y.

(R2) If $xyzu$ is a 2-link in H such that x has degree 3, 4 or 5 and y is not precolored, then x sends $\frac{4}{3}$ to y.

(R3) If x is a vertex of degree at least 6 adjacent to a 2-vertex y, and y is not precolored, then x sends $\frac{4}{3}$ to y.

(R4) If the length r of O_H is at most 8, the face O_H sends $\frac{4}{3}$ to each vertex incident to it.

The charge after the first phase is denoted by ch_1. The charge of 2-vertices becomes non-negative after the first phase. Furthermore, note that charge of a vertex w of degree at least 6 is non-negative:

$$\mathrm{ch}_1(w) \geq \frac{7}{3}d(w) - 6 - \frac{4}{3}d(w) = d(w) - 6 \geq 0.$$

If the face O_H is precolored, then each vertex w of O_H has at least two neighbors to that it does not send any charge, hence its charge is at least

$$\mathrm{ch}_1(w) \geq \frac{7}{3}d(w) - 6 - \frac{4}{3}(d(w) - 2) + \frac{4}{3} = d(w) - 2 \geq 0.$$

The charge of inner faces does not change, thus it is still non-negative. The charge of O_H changes only if its length r is at most 8, and then the final charge of O_H is at least $(\frac{2}{3}r + \frac{11}{2}) - \frac{4}{3}r \geq \frac{1}{6}$. Therefore, the charge of each face after the first phase of discharging is non-negative, hence the sum of the charges of vertices is negative. In the rest of the paper, we only take into account the charge of vertices.

Let us now analyze the possible neighborhoods of important vertices of H. We first consider neighborhoods of 3-vertices.

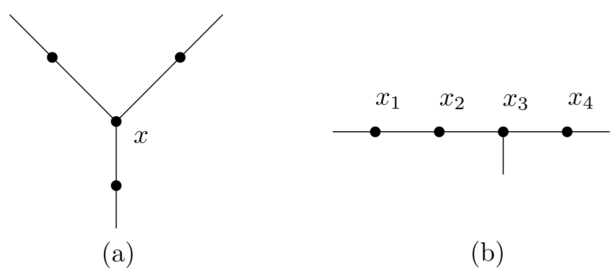

Figure 1: Reducible 3-vertex subgraphs

Lemma 13 *Subgraphs (a) and (b) in Figure 1 cannot appear in H, unless the 3-vertex is precolored.*

Proof. Consider first the subgraph (a). Using Lemma 12, the type of x after the links are eliminated is $\mathrm{cut}_3^3(\mathcal{P}_{10}) \geq \mathcal{P}_1$. Since a configuration consisting of a single vertex of type $\mathcal{P}_1$ is reducible, this subgraph cannot appear in H.

Now consider the subgraph (b). Let v_1, v_3 and v_4 be the vertices adjacent to x_1, x_3 and x_4 outside of the subgraph, respectively. Let c be a coloring of $H - \{x_1, x_2\}$. Note that $c(v_3) \neq c(v_4)$ because of the vertices x_3 and x_4. The set of colors p such that the coloring c' defined by $c'(v_3) = c(v_3)$, $c'(v_4) = c(v_4)$ and $c'(x_3) = p$ is proper and can be extended to x_4 is $\mathrm{int}(c(v_4)) \setminus \mathrm{int}(c(v_3))$ and since $c(v_3) \neq c(v_4)$, this set has size at least two. Hence, we can change colors of x_3 and x_4 so that $c(x_3) \neq c(v_1)$. We extend the coloring c to vertices x_1 and x_2 using Lemma 4. Since H is not $(5,2)$-colorable, the subgraph (b) cannot appear in H. □

The complete list of possible subgraphs in the neighborhood of a non-precolored 3-vertex that do not contain one of these two reducible subgraphs is given in Figure 2. The final charge and the type of the central vertex when all incident proper links are eliminated (as in Lemma 11) are also shown in the figure. The edges of 1-links and 2-links are drawn thin and the important edges are drawn thick.

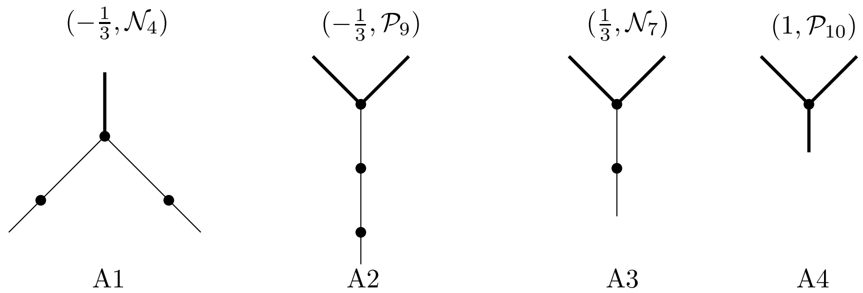

Figure 2: 3-vertex neighborhoods

Let us now consider neighborhoods of 4-vertices.

Lemma 14 *If A_1, A_2 and A_3 are three subsets of $\{1, \ldots, 5\}$ of size 3, then there exists a color-pair that is a subset of two of the sets A_1, A_2 and A_3. Consequently,* $\mathrm{cut}_3^3(\mathcal{P}_{10}) \geq \mathcal{P}_2$.

Proof. We may assume that $A_1 = \{1, 2, 3\}$. No two of the sets A_1, A_2 and A_3 may be disjoint, thus assume that $|A_1 \cap A_2| = |A_1 \cap A_3| = 1$. But then $\{4, 5\} \subset A_2, A_3$. $\square$

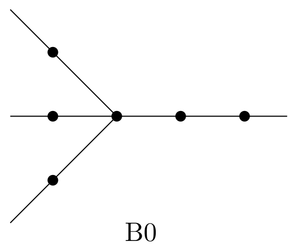

Figure 3: A reducible 4-vertex subgraph

Lemma 15 *The subgraph B0 in Figure 3 cannot appear in H, unless the 4-vertex is precolored.*

Proof. By Lemma 14, $\mathcal{X} = \mathrm{cut}_3^3(\mathcal{P}_{10}) \geq \mathcal{P}_2$. The type of the important vertex of the configuration is $\mathrm{cut}_1(\mathcal{X}) \geq \mathcal{P}_1$, and the subgraph cannot appear in H by Lemma 12. $\square$

The complete list of possible subgraphs in the neighborhood of a non-precolored 4-vertex that do not contain $B0$ is given in Figure 4. Next, we study neighborhoods of non-precolored 5-vertices.

Lemma 16 *The subgraph C0 in Figure 5 cannot appear in H, unless the 5-vertex is precolored.*

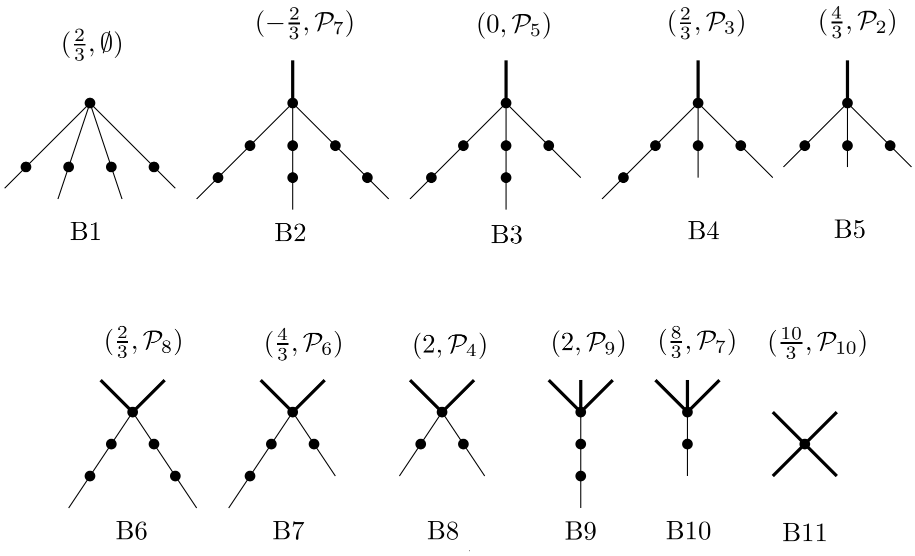

Figure 4: 4-vertex neighborhoods

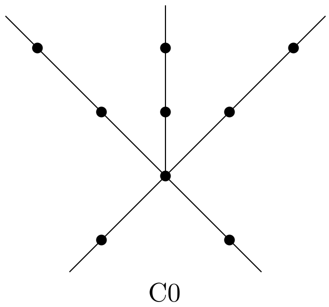

Figure 5: A reducible 5-vertex subgraph

Proof. The type of the important vertex of the configuration is $\mathrm{cut}_1^3(\mathrm{cut}_3^2(\mathcal{P}_{10})) \geq \mathcal{P}_1$, and the subgraph cannot appear in H by Lemma 12. $\square$

Let $d_{\mathrm{imp}}(v)$ be the number of important neighbors of a vertex v of H. We show that vertices of big degree have charge at least $\frac{2d_{\mathrm{imp}}(v)}{3}$, which will provide enough charge to distribute in the second phase of discharging.

Lemma 17 *If v is a 5-vertex in H such that $\mathrm{ch}_1(v) < \frac{2d_{\mathrm{imp}}(v)}{3}$, then v belongs to the subgraph $C1$ depicted in Figure 6.*

Proof. If $d_{\mathrm{imp}}(v) = 0$, then v would be incident to at most two 2-links by Lemma 16, hence $\mathrm{ch}_1(v) \geq \frac{17}{3} - 2 \cdot \frac{4}{3} - 3 \cdot \frac{2}{3} = 1$. If $d_{\mathrm{imp}}(v) \geq 2$, then $\mathrm{ch}_1(v) \geq \frac{17}{3} - (5 - d_{\mathrm{imp}}(v))\frac{4}{3} = \frac{4d_{\mathrm{imp}}(v)}{3} - 1 \geq \frac{2d_{\mathrm{imp}}(v)}{3} + \frac{1}{3}$.

Now we may assume that $d_{\mathrm{imp}}(v) = 1$. If v is incident to a 1-link, then $\mathrm{ch}_1(v) \geq \frac{17}{3} - 3 \cdot \frac{4}{3} - \frac{2}{3} = 1 > \frac{2}{3}$. Therefore, v is incident to four 2-links and belongs to the configuration $C1$. $\square$

Finally, we consider vertices of degree at least 6.

Lemma 18 *Every vertex v of H of degree $d \geq 6$ has charge $\mathrm{ch}_1(v) \geq \frac{2d_{\mathrm{imp}}(v)}{3}$.*

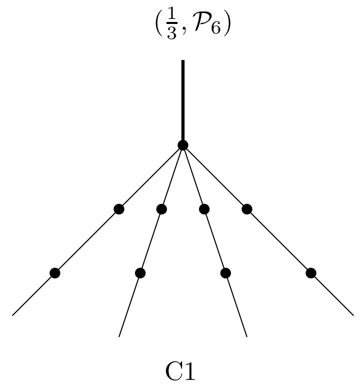

Figure 6: Another 5-vertex subgraph

Proof. The charge of v is at least

$$\frac{7}{3}d - 6 - (d - d_{\text{imp}}(v))\frac{4}{3} = d - 6 + \frac{4}{3}d_{\text{imp}}(v) \geq \frac{2d_{\text{imp}}(v)}{3}.$$

□

6 Further reducible configurations

In this section we focus on the reducibility of the configurations that have negative charge after the first phase of discharging. In the second phase of discharging, we want to make the charge of all vertices non-negative by moving charge over the important edges. It turns out that it is always sufficient to move charge at most $\frac{2}{3}$ over each edge, and thus we only need to consider configurations consisting of vertices whose charge is smaller than $\frac{2}{3}d_{\text{imp}}$. We may classify the important vertices by types and charge after the first phase:

$\mathcal{A}$. *A 4-vertex v incident to one important edge and three 2-links* (B2 in Figure 4). Its charge after the first stage is $-\frac{2}{3}$. By Lemma 11, when we consider reducibility of a configuration C containing v and none of the three important vertices to that v is connected through the 2-links belongs to C, we may assume that the type of v is $\mathcal{P}_7$. For sake of briefness we omit the references to Lemma 11 in the rest of the enumeration.

$\mathcal{B}$. *A 3-vertex incident to one important edge and two 1-links* (A1 in Figure 2). Its charge is $-\frac{1}{3}$, and when the 1-links are omitted, its type is $\mathcal{N}_4$.

$\mathcal{C}$. *A 4-vertex incident to one important edge, one 1-link and two 2-links* (B3 in Figure 4). Or *a 5-vertex incident to one important edge and four 2-links* (C1 in Figure 6). Charge of such a vertex is at least 0 and when the proper links are omitted, its type is $\geq \mathcal{P}_5$.

$\mathcal{D}$. *A 3-vertex incident to two important edges and one 2-link* (A2 in Figure 2). Its charge is $-\frac{1}{3}$ and when the 2-link is omitted, its type is $\mathcal{P}_9$.

$\mathcal{E}$. *A 3-vertex incident to two important edges and one 1-link* (A3 in Figure 2). Its charge is $\frac{1}{3}$ and when the 1-link is omitted, its type is $\mathcal{N}_7$.

$\mathcal{F}$. *A 4-vertex incident to two important edges and two 2-links* (B6 in Figure 4). Its charge is $\frac{2}{3}$ and when the 2-links are omitted, its type is $\mathcal{P}_8$.

$\mathcal{G}$. *A 3-vertex incident to three important edges* (A4 in Figure 2). Its charge is 1.

$\mathcal{H}$. *A vertex incident to d important edges and with charge at least $\frac{2d}{3}$.*

To distinguish the items in this classification from "types" introduced earlier, we say that the vertex described in the item $\mathcal{X}$ of this classification has *class* $\mathcal{X}$. If $\mathcal{X}$ and $\mathcal{Y}$ are classes, let us say that a vertex v is of class $(\mathcal{XY})$ if v is of class $\mathcal{X}$ or $\mathcal{Y}$.

Let us create a graph H' from H by iterating the following operation as long as possible: If there is an induced closed subgraph G with at least two important vertices such that it is joined to the rest of the graph by d important edges and the total charge of vertices in G is $c \geq \frac{2d}{3}$, then replace the subgraph G by a single vertex v of class $\mathcal{H}$ with charge c. We let all links and important edges that connected G with its complement originate in v instead. This operation does not change the total charge of the vertices, thus it is enough to prove that the total charge of vertices of H' is non-negative in order to obtain a contradiction. Note that the construction does not necessarily determine H' uniquely. However, it guarantees that any closed subgraph of H' with at least two important vertices has charge at most $\frac{2}{3}$ per outgoing edge.

The configurations whose reducibility we consider do not contain vertices of class $\mathcal{H}$. We also never pose any assumptions on whether the vertices adjacent to the configurations we consider are distinct or not in the following proofs, in particular decontracting the vertices of class $\mathcal{H}$ does not affect affect the reducibility of the configurations. Therefore, if we find a reducible configuration in H', the corresponding configuration in H is reducible as well, thus obtaining a contradiction with the minimality of H.

If the outer face O_H is precolored, we contract it to a vertex of class $\mathcal{H}$ in this step, so that we do not need to care about it in the rest of the proof. This is possible, since the degree of the created vertex is $\sum_{v \in O_H}(d(v) - 2)$, and each vertex v of the face has charge at least $d(v) - 2$. This also implies that any cycle in H' of length less than 9 contains a vertex of class $\mathcal{H}$.

A *string* is a path or a cycle (if it matters, we use terms a *path string* and a *cycle string* to distinguish between these two possibilities) in H' whose edges are all important and whose vertices do not have class $(\mathcal{GH})$. Vertices of class $(\mathcal{ABC})$ may of course only be the end vertices of a path string, while the inner vertices of a string must be of class $(\mathcal{DEF})$. A path string that contains a vertex of class $(\mathcal{ABC})$ is called a *leaf string.* Note that we do not impose any restrictions on where the proper links from the vertices of the string lead to (i.e., they may connect vertices inside the string). We call a string *spanned* in case there exists a proper link such that both of its important vertices belong to the string. The *charge* of a string is the sum of the charges of its vertices.

We use the notation path$(T_1 \ldots T_k)$ and cycle$(T_1 \ldots T_k)$ for the strings whose vertices have classes $T_1, \ldots, T_k$, in the order of their occurrence on the path or the cycle. We let $\mathcal{X}^k$ stand for a path of k vertices of class $\mathcal{X}$. We also use the following notation for describing the inner vertices of strings: we let $(\mathcal{D}, \mathcal{EF})^{m,n}$ mean a string that consists of m vertices of class $\mathcal{D}$ and n vertices of class $(\mathcal{EF})$, in any order. We write $(\mathcal{D}, \mathcal{EF})^{\leq m,n}$ for a string that consists of at most m vertices of class $\mathcal{D}$ and n vertices of class $(\mathcal{EF})$, in any order, and similarly for other possible combinations of numbers and $\geq$ or $\leq$. We sometimes use this notation inside path$(\ldots)$ or cycle$(\ldots)$ to denote a substring in this form. For example, path$((\mathcal{D}, \mathcal{EF})^{2,1})$ is one of the strings path$(\mathcal{D}^2\,\mathcal{E})$, path$(\mathcal{D}^2\,\mathcal{F})$, path$(\mathcal{D}\,\mathcal{E}\,\mathcal{D})$, path$(\mathcal{D}\,\mathcal{F}\,\mathcal{D})$, path($\mathcal{E}\,\mathcal{D}^2$), or path($\mathcal{F}\,\mathcal{D}^2$). For consistency, we also sometimes use $(\mathcal{D},\mathcal{EF})^{0,n}$ for strings that consist entirely from vertices of class $(\mathcal{EF})$, and $(\mathcal{D},\mathcal{EF})^{n,0}$ for strings that consist only of vertices of class $\mathcal{D}$.

In the following lemmas, we describe the strings in H' that are not reducible. Lemmas 19—21 deal with the leaf strings that contain a vertex of class $\mathcal{A}$, $\mathcal{B}$ or $\mathcal{C}$. Lemma 22 considers the path strings. Finally, Lemma 23 shows that H' does not contain any cycle strings with negative charge (and thus no cycle strings at all, by construction of H').

Lemma 19 *The strings in H' that are not reducible and that contain a vertex of class $\mathcal{A}$ are of the form* path($\mathcal{A}\,(\mathcal{D},\mathcal{EF})^{0,\leq 3}$)*. Moreover, these strings are not spanned.*

Proof. Let s be a string in H' that contains a vertex of class $\mathcal{A}$. Note that if s is one of the strings from the statement of the lemma, s cannot be spanned because of the restrictions on the lengths of cycles in H'.

Suppose now that s is not one of the strings described in the statement. The string s cannot contain more than three vertices of class $(\mathcal{EF})$ in a row, otherwise the charge of the substring formed by the vertices of class $(\mathcal{EF})$ would be at least $\frac{4}{3}$ and this substring would become a vertex of class $\mathcal{H}$ during the construction of the graph H'. Therefore, s must contain a substring s' of form path($\mathcal{A}\,(\mathcal{D},\mathcal{EF})^{0,\leq 3}\mathcal{X}$), where the class $\mathcal{X}$ is different from $(\mathcal{EF})$ (i.e., $\mathcal{X}$ is one of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ or $\mathcal{D}$).

Since H' does not contain a cycle of length less than 9 without a vertex of class $\mathcal{H}$, the string s' cannot be spanned. Therefore, we may use Lemma 11 to determine the types of the vertices of s'. Let us show that regardless of the choice of $\mathcal{X}$, the string s' is reducible.

By Lemma 6(2) and the fact that the type $\mathcal{P}_2$ propagates over vertices of type $\mathcal{P}_7$, the configuration path($\mathcal{P}_7^*\,\mathcal{P}_2$) is reducible. Let v be the vertex of s' with class $\mathcal{X}$. After taking the links into account, the type of each vertex of s', except for v, is $\geq \mathcal{P}_7$. For any possible choice of $\mathcal{X}$, the type of v, after eliminating the links and possibly one important edge from v using Lemma 11, is $\geq \mathcal{P}_2$. Therefore, s' is reducible, and it cannot appear in H'. □

Lemma 20 *The strings in H' that are not reducible and that contain a vertex of class $\mathcal{B}$ are of the form* path($\mathcal{B}\,(\mathcal{D},\mathcal{EF})^{0,\leq 2}$) *or* path($\mathcal{B}\,(\mathcal{D},\mathcal{EF})^{1,\leq 3}$)*. Moreover, these strings are not spanned.*

Proof. The string cannot contain a vertex of class $\mathcal{A}$ by Lemma 19. Let us consider several cases regarding the string of vertices of class $(\mathcal{DEF})$ adjacent to the vertex of class $\mathcal{B}$. Note that the string cannot contain more vertices of class $(\mathcal{EF})$ than specified in each of the cases, otherwise it would contain a subgraph with charge at least $\frac{2}{3}$ per outgoing important edge. None of the strings may be spanned, otherwise H' would contain a cycle of length at most 8 not containing a vertex of class $\mathcal{H}$.

path($\mathcal{B}\,(\mathcal{D},\mathcal{EF})^{0,\leq 1}\,\mathcal{B}$)**:** By Lemma 11, it suffices to show that configurations path($\mathcal{N}_4\,\mathcal{N}_4$) and path($\mathcal{N}_4\,\mathcal{P}_7\,\mathcal{N}_4$) are reducible. For the former one, this follows from Lemma 6(4). For the later one, by Lemma 6(4), a list from $\mathcal{N}_4$ forbids at most three color-pairs, hence we may always choose a color-pair that is not forbidden for the middle vertex.

path($\mathcal{B}\,(\mathcal{D},\mathcal{EF})^{\leq 1,\leq 1}\,\mathcal{C}$)**:** The corresponding configuration path($\mathcal{N}_4\,\mathcal{P}_7^*\,\mathcal{P}_5$) is reducible, as $\mathcal{P}_5$ is propagated over chain of $\mathcal{P}_7$'s and the final configuration path($\mathcal{N}_4\,\mathcal{P}_5$) is reducible.

path($\mathcal{B}\,(\mathcal{D},\mathcal{EF})^{1,\leq 2}\,\mathcal{B}$)**:** The string is not spanned, and the corresponding configurations path($\mathcal{N}_4\,\mathcal{P}_7^*\,\mathcal{P}_9\,\mathcal{P}_7^*\,\mathcal{N}_4$) are reducible by Lemma 7(3), since $\mathcal{N}_4 \geq \mathcal{P}_3$.

path($\mathcal{B}\,(\mathcal{D},\mathcal{EF})^{1,\leq 3}$)**:** This is one of the configurations described in the statement of the lemma.

A string with prefix path($\mathcal{B}\,(\mathcal{D},\mathcal{EF})^{1,\leq 3}\,\mathcal{D}$)**:** Let s be this prefix. Note that s cannot be spanned: in order to avoid a cycle of length at most 8, the link would have to be a 2-link between the end-vertices of the path. However, a vertex of class $\mathcal{B}$ is incident only to 1-links. Consequently, the string path($\mathcal{B}\,(\mathcal{D},\mathcal{EF})^{1,\leq 3}\,\mathcal{D}$) corresponds to a configuration path($\mathcal{N}_4\,\mathcal{P}_7^*\,\mathcal{P}_9\,\mathcal{P}_7^*\,\mathcal{P}_2$)—the type $\mathcal{P}_2$ of the last vertex of the path is obtained from $\mathcal{P}_9$ (type of vertex of class $\mathcal{D}$) by applying operation cut_7, to take into account the important vertex adjacent to it outside of the configuration. This configuration is reducible by Lemma 7(3).

This shows that indeed the only such strings in H' that are not reducible are the non-spanned strings described in the claim of this lemma. □

Lemma 21 *The strings in H' that are not reducible and that contain a vertex of class $\mathcal{C}$ are of the form* path($\mathcal{C}\,(\mathcal{D},\mathcal{EF})^{0,\leq 1}$) *or* path($\mathcal{C}\,(\mathcal{D},\mathcal{EF})^{1,\leq 2}$)*. Moreover, these strings are not spanned.*

Proof. Let s be a string in H' that contains a vertex of class $\mathcal{C}$. By Lemmas 19 and 20, the string s cannot contain a vertex of class ($\mathcal{AB}$). We distinguish several cases. Note that in each of the cases the considered string cannot contain more vertices of class ($\mathcal{EF}$) than specified, otherwise s would contain a subgraph with charge at least $\frac{2}{3}$ per outgoing important edge and this subgraph would be eliminated during the construction of H'. The considered strings also cannot be spanned, since the length of the cycle would be less than 9. If s contains at least two vertices of class $\mathcal{D}$, then s has a prefix path($\mathcal{C}\,(\mathcal{D},\mathcal{EF})^{1,\leq 2}\,\mathcal{D}$). However, the corresponding configuration path($\mathcal{P}_5\,\mathcal{P}_7^*\,\mathcal{P}_9\,\mathcal{P}_7^*\,\mathcal{P}_2$) is reducible by Lemma 7(3). Otherwise, if s contains two vertices of class C, then s is path($\mathcal{C}\,(\mathcal{D},\mathcal{EF})^{\leq 1,0}\,\mathcal{C}$), but the corresponding configurations path($\mathcal{P}_5\,\mathcal{P}_5$) and path($\mathcal{P}_5\,\mathcal{P}_9\,\mathcal{P}_5$) are reducible. Therefore, the only possible strings are indeed those described in the statement of the lemma. □

Finally there is a classification of path strings that are composed from vertices of class ($\mathcal{DEF}$) only. To simplify the arguments we characterize only the strings whose charge is negative.

Lemma 22 *The path strings in H' that are not reducible, contain only vertices of class ($\mathcal{DEF}$) and have negative charge are non-spanned strings of one of the following forms:* path($\mathcal{D}$)*,* path($(\mathcal{D},\mathcal{EF})^{2,\leq 1}$)*, or* path($(\mathcal{D},\mathcal{EF})^{3,\leq 2}$)*.*

Proof. A string with m vertices of class $\mathcal{D}$ may contain at most $m-1$ vertices of class ($\mathcal{EF}$), in order for it to have negative charge. The strings described in the statement of this lemma are non-spanned, because the length of the cycle that a spanning link would create is at most 7.

Let us assume that that $s = v_1 \dots v_k$ is a string in H' with negative charge that consists only of vertices of class ($\mathcal{DEF}$) and that contains at least four vertices of class $\mathcal{D}$. We want to show that s is reducible. Additionally, we may assume that s is minimal, i.e., it does not contain a substring with more than three vertices of class $\mathcal{D}$ and negative charge.

By the minimality of s, its substring $s' = v_2 \dots v_k$ either has non-negative charge, or contains at most three vertices of class $\mathcal{D}$. Since the charge of vertices of classes $(\mathcal{EF})$ is positive, this means that v_1 is of class $\mathcal{D}$. Similarly, the class of v_k is $\mathcal{D}$. Furthermore, if s contains more than four vertices of class $\mathcal{D}$, then v_2 and v_{k-1} must also be of class $\mathcal{D}$, otherwise we can remove the first (or the last) two vertices from s.

Since the charge of a vertex of class $\mathcal{D}$ is $-\frac{1}{3}$, if the charge of s is less than $-\frac{1}{3}$, the string s must contain exactly four vertices of class $\mathcal{D}$, otherwise we can remove the first vertex from s. In case the charge of s is exactly $-\frac{1}{3}$, the string s contains at most six vertices of class $\mathcal{D}$—if there are at least 7 vertices of class $\mathcal{D}$, let s_1 be the shortest prefix of s that contains four vertices of class $\mathcal{D}$, and let s_2 be the shortest suffix of s that contains the last vertex of class $\mathcal{D}$ that belongs to s_1. Obviously s_1 and s_2 both contain at least four vertices of class $\mathcal{D}$. Let c be the charge of s and c_i be the charge of s_i for $i = 1, 2$. The strings s_1 and s_2 share exactly one vertex of class $\mathcal{D}$, hence $c = c_1 + c_2 + \frac{1}{3}$. Therefore, at least one of c_1 and c_2 must be negative, contradicting the minimality of s.

The string s must be spanned, otherwise by Lemma 8, the corresponding configuration path$(\mathcal{P}_2\,\mathcal{P}_7^*\,\mathcal{P}_9\,\mathcal{P}_7^*\,\mathcal{P}_9\,\mathcal{P}_7^*\,\mathcal{P}_2)$ is reducible. It follows that there are no minimal strings with charge less than $-\frac{1}{3}$, since such strings consist of four vertices of class $\mathcal{D}$ and at most two vertices of class $(\mathcal{EF})$ and such strings cannot be spanned by girth constraints. Thus we may assume that the charge of s is $-\frac{1}{3}$.

Let us first consider the case that s contains exactly four vertices of class $\mathcal{D}$. Such a string must contain three vertices of class $\mathcal{E}$ in order to be spanned, and there must be a 2-link between v_1 and v_7. But the resulting configuration is reducible by Lemma 9.

If s contains five vertices of class $\mathcal{D}$, we consider several possibilities:

- *The string s contains four vertices of class $\mathcal{E}$.* The path $v_1 \dots v_8$ is a substring of s with four vertices of class $\mathcal{D}$ and four vertices of class $\mathcal{E}$, where v_1, v_2 and v_8 are vertices of class $\mathcal{D}$. The vertices of the configuration cannot be joined by a 1-link, because the vertices of class $\mathcal{D}$ are incident only to 2-links and the vertices of class $\mathcal{E}$ are too close to each other. Therefore, there must be a 2-link between two vertices of class $\mathcal{D}$. Let v_i and v_j $(i < j)$ be two vertices joined by a 2-link such that $j - i$ is the smallest possible. The difference $j - i$ must be at least 6, and at least one of v_{i+1}, $\dots$, v_{j-1} must be of class $\mathcal{D}$, since there are only 4 vertices of class $\mathcal{E}$. Therefore, the configuration $v_i \dots v_j$ together with the 2-link is reducible by Lemma 9.

- *The string s contains two vertices of class $\mathcal{E}$ and one vertex of class $\mathcal{F}$.* The vertices of class $(\mathcal{EF})$ cannot be all consecutive, since they would form a substring with charge $\frac{4}{3}$. There is a substring $v_i \dots v_j$ $(i < j \leq i + 4)$ such that v_i and v_j are of class $\mathcal{D}$, and there is one more vertex of class $\mathcal{D}$ and a vertex of class $\mathcal{F}$ inside the substring. This substring cannot be spanned and is reducible by Lemma 8.

- *The string s contains two vertices of class $\mathcal{F}$.* We consider the substring on vertices $v_1 \dots v_6$. This substring cannot be spanned and it contains four vertices of class $\mathcal{D}$, thus it is reducible by Lemma 8.

Finally, consider the case that s contains six vertices of class $\mathcal{D}$, and at most five vertices of class $(\mathcal{EF})$. We know that classes of v_1 and v_2 are $\mathcal{D}$. If classes of v_3 and v_4 are both $(\mathcal{EF})$, the substring on vertices $v_4 \dots v_k$ has negative charge, contradicting the minimality of s. On the other hand, s cannot contain four consecutive vertices of class $\mathcal{D}$, thus exactly one of v_3 and v_4 has class $\mathcal{D}$. Similarly, exactly one of v_{k-3} and v_{k-2} is of class $\mathcal{D}$. Since s has charge $-\frac{1}{3}$, it contains either five vertices of class $\mathcal{E}$, three vertices of class $\mathcal{E}$ and one of class $\mathcal{F}$, or one vertex of class $\mathcal{E}$ and two of class $\mathcal{F}$.

If s contains five vertices of class $\mathcal{E}$, considering that s cannot contain four consecutive vertices of class $\mathcal{E}$ by the construction of H', it follows that s is equal to $s_1 = \text{path}(\mathcal{D}^2\,\mathcal{E}\,\mathcal{D}\,\mathcal{E}^3\,\mathcal{D}\,\mathcal{E}\,\mathcal{D}^2)$. If s contains three vertices of class $\mathcal{E}$ and one of class $\mathcal{F}$, it similarly follows that s is equal to $s_2 = \text{path}(\mathcal{D}^2\,\mathcal{E}\,\mathcal{D}\,\mathcal{E}\,\mathcal{F}\,\mathcal{D}\,\mathcal{E}\,\mathcal{D}^2)$. In case s contains one vertex of class $\mathcal{E}$ and two of class $\mathcal{F}$, the classes of v_3 and v_7 cannot be $\mathcal{F}$ by the minimality of s. However, the two vertices of class $\mathcal{F}$ would then have to be adjacent in s, which is not possible by the construction of H'.

The string s_1 contains a substring $s_1' = \text{path}(\mathcal{D}^2\,\mathcal{E}\,\mathcal{D}\,\mathcal{E}^3\,\mathcal{D})$. If s_1' is non-spanned, then it is reducible by Lemma 8. There may be a 2-link between two vertices of class $\mathcal{D}$, but in that case s_1 together with the link is reducible by Lemma 9. The string s_2 contains a substring $\text{path}(\mathcal{D}\,\mathcal{E}\,\mathcal{D}\,\mathcal{E}\,\mathcal{F}\,\mathcal{D})$ that cannot be spanned, and hence is reducible by Lemma 8.

Therefore, the only non-leaf path strings with negative charge in H' are those listed in the statement of this lemma. □

Let us now show that no cycle strings appear in H'.

Lemma 23 *H' does not contain a cycle string.*

Proof. A cycle string consists of a vertices of class $(\mathcal{D}\mathcal{E}\mathcal{F})$, and its length must be at least 9. If its charge is negative, then it contains a substring of length 6 with negative charge. Such a string cannot be spanned and it is reducible by Lemma 22. Cycle strings with non-negative charge are contracted during construction of H'. □

Let us note that strings in H' do not have large negative charge:

Lemma 24 *Each leaf string in H' has charge at least $-\frac{2}{3}$, and each non-leaf path string has charge at least -1.*

Proof. Let us inspect all possible strings. By Lemma 19, the leaf strings that contain a vertex of class $\mathcal{A}$ are of form $\text{path}(\mathcal{A}\,(\mathcal{D},\mathcal{E}\mathcal{F})^{0,\leq 3})$, and thus their charge is at least $-\frac{2}{3}$.

By Lemma 20, the leaf strings that contain a vertex of class $\mathcal{B}$ contain only one such vertex and at most one vertex of class $\mathcal{D}$, hence their charge is at least $-\frac{2}{3}$.

By Lemma 21, the leaf strings that contain a vertex of class $\mathcal{C}$ contain at most one vertex of class $\mathcal{D}$, hence their charge is at least $-\frac{1}{3}$.

By Lemma 22, a non-leaf path string with negative charge contains at most three vertices of class $\mathcal{D}$, thus its charge is at least -1. □

7 Second Phase of Discharging

Now we are ready to perform the second phase of discharging in the graph H'. Note that all vertices whose charge after the first phase is negative are inside strings. A path string $s = v_1 v_2 \dots v_k$ is *adjacent* to a vertex v of class $(\mathcal{G}\mathcal{H})$ if v and v_1 are adjacent. A string is *maximal* if it is not a proper substring of another string in H'. In this phase, the vertices of classes $(\mathcal{G}\mathcal{H})$ send charge to the adjacent maximal strings, thus making the charge of the vertices in the strings non-negative. We then argue that the vertices of class $\mathcal{G}$ with negative charge must belong to reducible configurations. The rules for discharging are the following:

(R5) A vertex of class $\mathcal{H}$ sends charge of $\frac{2}{3}$ to each adjacent maximal string.

(R6) A vertex of class $\mathcal{G}$ adjacent to a maximal leaf string of charge $c < 0$ sends charge of $-c$ to the string.

(R7) A vertex of class $\mathcal{G}$ adjacent to a maximal non-leaf string of charge $c < 0$ sends $-\frac{c}{2}$ to the string.

Each vertex of class $\mathcal{H}$ sends charge at most $\frac{2}{3}$ per edge by Rule R5, thus its final charge is non-negative. By Lemma 24, each leaf string in H' has charge at least $-\frac{2}{3}$ and each non-leaf path string has charge at least -1. Therefore, each maximal leaf or path string with charge $c < 0$ receives charge at least $-c$ from the adjacent vertices of class $(\mathcal{GH})$ and its final charge in non-negative. By Lemma 23, the cycle strings do not appear in H'. It remains to show that also the charge of vertices of class $\mathcal{G}$ is non-negative. Let us start with some definitions and lemmas.

By the classification of the strings presented in the previous section, the maximal path strings with negative charge in H' are non-spanned, and we can divide them into the following sets according to the charge they receive from the adjacent vertices of $\mathcal{G}$:

- $C_{\frac{2}{3}}$ is the set of strings to that vertices of class $\mathcal{G}$ send $\frac{2}{3}$, i.e., path($\mathcal{A}$) and path($\mathcal{D}\,\mathcal{B}$).
- $C_{\frac{1}{2}}$ is the set containing a single string path($\mathcal{D}^3$) to that vertices of class $\mathcal{G}$ send $\frac{1}{2}$.
- $C_{\frac{1}{3}}$ is the set of strings to that vertices of class $\mathcal{G}$ send $\frac{1}{3}$, consisting of

  path($\mathcal{E}\,\mathcal{A}$), path($\mathcal{B}$), path($\mathcal{E}\,\mathcal{D}\,\mathcal{B}$), path($\mathcal{D}\,\mathcal{E}\,\mathcal{B}$), path($\mathcal{D}\mathcal{C}$), path($\mathcal{D}^2$), and all path strings that consist of 3 vertices of class $\mathcal{D}$ and one of class $\mathcal{E}$.
- $C_{\frac{1}{6}}$ is the set of strings to that vertices of class $\mathcal{G}$ send $\frac{1}{6}$, consisting of all path strings with n_d vertices of class $\mathcal{D}$, n_e vertices of class $\mathcal{E}$ and n_f vertices of class $\mathcal{F}$, where the triple (n_d, n_e, n_f) is one of $(1, 0, 0)$, $(2, 1, 0)$, $(3, 0, 1)$ or $(3, 2, 0)$.

Let $C_{\text{all}} = C_{\frac{2}{3}} \cup C_{\frac{1}{2}} \cup C_{\frac{1}{3}} \cup C_{\frac{1}{6}}$. We are going to consider configurations formed by strings in C_{all} adjacent to a vertex of $\mathcal{G}$. The main problem is dealing with the spanned configurations. Since some of these strings are quite long, it is impossible to directly use the girth restrictions to show that the configurations cannot be spanned. To solve this problem, we restrict our attention to prefixes of these strings that are short enough so that the girth argument is applicable. The following lemma claims that that we can replace each string in C_{all} by a single vertex of type $\mathcal{P}_2$ when considering the reducibility of a subgraph of H':

Lemma 25 *Let G be a closed subgraph of H' and let s be a path string $v_1 v_2 \ldots v_k$ in G belonging to C_{all} such that the important vertex adjacent to v_1 distinct from v_2 also belongs to G, but (in case s is a non-leaf string) the important vertex adjacent to v_k distinct from v_{k-1} does not belong to G. Let K' be a configuration obtained from G by removing the vertices $v_2, \ldots, v_k$ and the 2-vertices of the proper links incident to them and determining the type of vertices of K' using Lemma 11. We create a configuration K from K' by setting the type of v_1 to $\mathcal{P}_2$. If K is reducible, then G is reducible as well.*

Proof. We proceed similarly as in proof of Lemma 12. Depending on s, we determine a suitable number m ($2 \leq m \leq k+1$) as described below. Then, we fix a coloring c' of H except for $V(K)$, vertices v_2, v_3, …, v_{m-1}, and the 2-vertices of the proper links incident to them. We extend the coloring c' to vertices v_2,…, v_{m-1} in such a way that there are at least two free colors at v_1, thus obtaining a partial coloring c. Since K is reducible, its instance Q obtained from the precoloring c is colorable, and thus c can be further extended to whole graph H. This will show the reducibility of G. It remains to show how to determine m and how to extend c' to c. We discuss several cases that cover all strings in C_{all}.

If the class of v_1 is $\mathcal{D}$, we set $m = 2$. By Lemma 11, the type of v_1 in K' is at least $\text{cut}_7(\mathcal{P}_9) \geq \mathcal{P}_2$, and thus if K is reducible, then K' is reducible as well and the statement follows from Lemma 12. Similarly, if $k = 1$, we note that the type of v_1 in K' is at least $\mathcal{P}_2$, and thus the reducibility of K implies reducibility of K'.

If the class of v_1 is $\mathcal{E}$ and the class of v_2 is $(\mathcal{AD})$, then we set $m = 3$. There are at least two free color-pairs a_1 and a_2 at v_2, with respect to partial coloring c'. The type of v_1 is $\mathcal{N}_7$. Suppose that $\text{int}(p) = Q(v_1)$. At least one of a_1 or a_2 (say a_1) is distinct from p. Thus, a_1 forbids at most five color-pairs in $Q(v_1)$ and we may color v_2 with a_1.

If the class of v_1 and v_2 is $\mathcal{E}$ and the class of v_3 is $\mathcal{D}$, then we set $m = 4$. Similarly as in the previous case, there are at least two free colors at v_3 and we may choose a color for v_3 such that it keeps two free colors at v_2. Repeating the same argument, we color v_2 in such a way that there are at least two free color-pairs at v_1.

The remaining case is that the class of v_1 is $\mathcal{F}$ and the class of v_2 is $\mathcal{D}$, then we set $m = 3$. There are at least two free colors a_1 and a_2 at v_2. Let L be a subset of the list of v_1 that contains exactly 8 color-pairs. It cannot be the case that both $\text{int}(a_1)$ and $\text{int}(a_2)$ are subsets of L, since $|\text{int}(a_1) \cup \text{int}(a_2)| \geq 9$. We may assume that $\text{int}(a_1) \nsubseteq L$, hence $|L \setminus \text{int}(a_1)| \geq 2$. We color v_2 with the color-pair a_1, which leaves at least two free color-pairs at v_1. □

A string in $C_{\frac{1}{3}} \cup C_{\frac{1}{2}}$ contains one of the following substrings adjacent to v: $\text{path}(\mathcal{E}\,\mathcal{D}^2)$, $\text{path}(\mathcal{E}\,\mathcal{D}\,\mathcal{B})$, $\text{path}(\mathcal{D}\,\mathcal{E}\,\mathcal{B})$, $\text{path}(\mathcal{D}\,\mathcal{E}\,\mathcal{D})$, $\text{path}(\mathcal{D}^2)$, $\text{path}(\mathcal{D}\,\mathcal{C})$, $\text{path}(\mathcal{E}\,\mathcal{A})$, or $\text{path}(\mathcal{B})$. Let $C_\#$ be the set consisting of these strings. We show that each of these strings forbids only a few color-pairs. Note that unlike Lemma 25, we must assume that there are no links from such string to the rest of the configuration.

Lemma 26 *Suppose that $s = v_1 \dots v_k \in C_\#$, and let us consider a closed configuration that includes s and a vertex v of class $\mathcal{G}$ adjacent to s, but contains neither any of the proper links incident to s, nor (in case s is not a leaf string) the important vertex adjacent to v_k distinct from v_{k-1}. The string s forbids at most two color-pairs at v, with the exception of* $\text{path}(\mathcal{B})$ *that forbids at most three color-pairs.*

Proof. Let us consider the elements of $C_\#$ separately.

If $s = \text{path}(\mathcal{E}\,\mathcal{D}^2)$ or $s = \text{path}(\mathcal{E}\,\mathcal{D}\,\mathcal{B})$, then the type of v_1 is $\mathcal{N}_7$, the type of v_2 is $\mathcal{P}_9$ and the type of v_3 is $\geq \mathcal{P}_2$, by Lemma 12. By Lemma 7(3), the configuration $\text{path}(\mathcal{P}_3\,\mathcal{P}_7\,\mathcal{P}_9\,\mathcal{P}_2)$ is reducible, thus the string s forbids at most two color-pairs at v.

Similarly, if $s = \text{path}(\mathcal{D}\,\mathcal{E}\,\mathcal{D})$ or $s = \text{path}(\mathcal{D}\,\mathcal{E}\,\mathcal{B})$, the type of v_1 is $\mathcal{P}_9$, the type of v_2 is $\mathcal{P}_7$ and the type of v_3 is $\geq \mathcal{P}_2$. Again by Lemma 7(3), the configuration $\text{path}(\mathcal{P}_3\,\mathcal{P}_9\,\mathcal{P}_7\,\mathcal{P}_2)$ is reducible and s forbids at most two color-pairs at v.

If $s = \text{path}(\mathcal{D}^2)$ or $s = \text{path}(\mathcal{D}\,\mathcal{C})$, then the type of v_1 is $\mathcal{P}_9$ and the type of v_2 is $\geq \mathcal{P}_2$. Since the configuration $\text{path}(\mathcal{P}_3\,\mathcal{P}_9, \mathcal{P}_2)$ is reducible by Lemma 7(3), s forbids at most two color-pairs at v.

If $s = \text{path}(\mathcal{E}\,\mathcal{A})$, then the type of v_1 is $\mathcal{N}_7$ and the type of v_2 is $\mathcal{P}_7$. The vertex v_2 forbids at most one color-pair at v_1, and v_1 then forbids at most one color-pair at v, by Lemma 6(2).

Finally, if $s = \text{path}(\mathcal{B})$, then the type of v_1 is $\mathcal{N}_4$, and it forbids at most three color-pairs at v by Lemma 6(4). $\square$

The strings in $C_{\frac{2}{3}} \cup C_{\frac{1}{2}}$ forbid even fewer color-pairs.

Lemma 27 *Suppose that $s = v_1 \dots v_k \in C_{\frac{2}{3}} \cup C_{\frac{1}{2}}$, and let K be a configuration that includes s and a vertex v of class $\mathcal{G}$ adjacent to s, but contains neither any of the proper links incident to s, nor (in case s is not a leaf string) the important vertex adjacent to v_k distinct from v_{k-1}. The string s forbids at most one color-pair at v.*

Proof. Let us consider each possible string separately.

If $s = \text{path}(\mathcal{A})$, then the type of v_1 is $\mathcal{P}_7$. By Lemma 6(2), the configuration $\text{path}(\mathcal{P}_2\,\mathcal{P}_7)$ is reducible, thus s forbids at most one color-pair at v.

If $s = \text{path}(\mathcal{D}\,\mathcal{B})$, then the type of v_1 is $\mathcal{P}_9$ and the type of v_2 is $\mathcal{N}_4$. By Lemma 7(3), the configuration $\text{path}(\mathcal{P}_2\,\mathcal{P}_9\,\mathcal{N}_4)$ is reducible, thus showing that s forbids at most one color-pair at v.

Finally, if $s = \text{path}(\mathcal{D}^3)$, then the type of v_1 and v_2 is $\mathcal{P}_9$ and the type of v_3 is $\mathcal{P}_2$. By Lemma 8, the configuration $\text{path}(\mathcal{P}_2\,\mathcal{P}_9\,\mathcal{P}_9\,\mathcal{P}_2)$ is reducible, thus also in this case, s forbids at most one color-pair at v.

We are now ready to show that the final charge of vertices of class $\mathcal{G}$ is non-negative.

Theorem 28 *The charge of each vertex of classes ($\mathcal{G}$) after the second phase of discharging is non-negative.*

Proof. Let v be a vertex of class $\mathcal{G}$. The charge of v before the second phase is 1. We prove that if the charge of the vertex v would become negative after applying the rules ($R5$)–($R7$), then the configuration in the neighborhood of v is reducible. If v is adjacent to at most one maximal string with negative charge, then v sends at most 1 unit of charge by Lemma 24, hence the final charge of v is nonnegative. The vertex v cannot be joined to a single string with negative charge through more than one edge, otherwise the string together with v forms a cycle of length less than 9. Let s_1, s_2 and s_3 be the strings adjacent to v. There are the following cases in that the charge of v after the second phase is negative (up to symmetry):

- *s_1 belongs to $C_{\frac{2}{3}}$ and and s_2 belong to $C_{\frac{2}{3}} \cup C_{\frac{1}{2}}$.* The configuration formed by s_1, v and s_2 cannot be spanned by the girth constraints. By Lemma 27, each of s_1 and s_2 forbids at most one color-pair at v. The precoloring of s_3 forbids at most seven color-pairs at v. Since there is at least one free color-pair at v, the configuration is reducible.

- *$s_1 \in C_{\frac{2}{3}}$, $s_2 \in C_{\frac{1}{3}}$ and $s_3 \in C_{\frac{1}{3}} \cup C_{\frac{1}{6}}$.* Consider the configuration formed by v, s_1, the prefix substring of s_2 contained in $C_{\#}$, and a single vertex of type $\mathcal{P}_2$ that replaces s_3 using Lemma 25. This configuration cannot be spanned. By Lemmas 27, 26 and 6(2), the strings forbid at most 1, 3 and 5 color-pairs at v, respectively. Therefore, there is a free color-pair at v and the configuration is reducible.

- $s_1, s_2 \in C_{\frac{1}{2}}$ *and* $s_3 \in C_{\frac{1}{6}} \cup C_{\frac{1}{3}} \cup C_{\frac{1}{2}}$. We consider the configuration formed by v, the substrings $\text{path}(\mathcal{D}^2) \in C_\#$ of s_1 and s_2, and a vertex of type $\mathcal{P}_2$ that replaces the string s_3 using Lemma 25. This configuration cannot be spanned by the girth constraints. By Lemmas 26 and 6(2), the strings forbid at most 2, 2 and 5 color-pairs at v, respectively, hence the configuration is reducible.

- $s_1 \in C_{\frac{1}{2}}$ *and* $s_2, s_3 \in C_{\frac{1}{3}}$. We consider the configuration formed by v, $s_1 = \text{path}(\mathcal{D}^3)$, the prefix substring s_2' of s_2 that belongs to $C_\#$ and a vertex of type $\mathcal{P}_2$ that replaces s_3. Consider first the case that the configuration is not spanned. By Lemmas 27, 26 and 6(2), the numbers of color-pairs forbidden by the strings at v are at most 1, 3 and 5, respectively, hence the configuration is reducible.

  Let us now consider the case that the configuration is spanned. By the girth restrictions, this can only happen if there is a 2-link from the last vertex of s_1 to s_2'. The girth restrictions also exclude the case $s_2' = \text{path}(\mathcal{B})$, and thus s_2' forbids at most two color-pairs, by Lemma 26. We shorten s_1 to its substring $\text{path}(\mathcal{D}^2) \in C_\#$ in the configuration. The new configuration cannot be spanned, and by Lemmas 26 and 6(2), the numbers of forbidden color-pairs at v are at most 2, 2 and 5, respectively. Therefore, this smaller configuration is reducible.

This shows that charge of v is non-negative after the second phase of discharging. □

This is the last piece needed in order to finish the proof of our main result.

Proof of Theorem 3. We proceed by contradiction. We assume that H is a minimal counterexample to this theorem. We assign charge to the vertices and faces of H as described in Section 4 such that the total amount of charge is negative. We run the first phase of discharging over H as described in Section 5, arguing that the charge of all faces and 2-vertices is non-negative after this phase. We construct the graph H' described in Section 6, such that the sum of the charges of vertices of H' is negative. Finally, we run the second phase of discharging as described in Section 7. The final charge of each vertex of class $(\mathcal{GH})$ and of each string is non-negative, which is a contradiction. □

8 Conclusion

An immediate consequence of Theorem 1 is:

Corollary 29 *Every planar graph of girth at least* 8 *is* $(5,2)$*-colorable.*

From R. Naseraser, we learned that no planar graph of odd-girth 7 is known that does not map to Petersen graph. This motivates the following question:

Problem 1 *Is there a planar graph of odd-girth 7 with fractional chromatic number greater than* $\frac{5}{2}$*? Or at least one that does not map to Petersen graph?*

References

[1] O. Borodin, S. Kim, A. Kostochka, D. West, *Homomorphisms from sparse graphs with large girth*, J. Comb. Theory Ser. B **90** (2004), 147–159.

[2] F. Jaeger, *Nowhere-zero flow problems*, Selected Topics in Graph Theory 3, (L. W. Beineke and R. J. Wilson eds.), Academic Press, London, (1988), 71–95.

[3] F. Jaeger, *On circular flows in graphs*, Finite and Infinite Sets, (Eger, 1981), Colloquia Mathematica Societatis János Bolyai 37, North Holland, (1984), 391–402.

[4] W. Klostermeyer, C. Q. Zhang, $(2+\epsilon)$*-coloring of planar graphs with large odd-girth*, J. Graph Theory **33** (2000), 109–119.

[5] E. Scheinerman, D. Ullman, *Fractional Graph Theory*, Wiley-Interscience, New York 1997, 200–211.

[6] W. T. Tutte, *A contribution to the theory of chromatic polynomials*, Canad. J. Math., **6** (1954), 80–91.

[7] W. T. Tutte, *On the imbeddings of linear graphs in surfaces*, Proc. London Math. Soc. **51** (1954), 474–483.

[8] C. Q. Zhang, *Integer Flows and Cycle Covers of Graphs*, Marcel Dekker, New York, 1997.

[9] X. Zhu, *Circular chromatic number: a survey*, Discrete Math. **229** (2001), 371–410.

[10] X. Zhu, *Circular chromatic number of planar graphs of large odd girth*, Electronic J. Comb. **8** (2001), #R25.