On tension-continuous mappings

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Abstract

Tension-continuous (shortly TT) mappings are mappings between the edge sets of graphs. They generalize graph homomorphisms. From another perspective, tension-continuous mappings are dual to the notion of flow-continuous mappings and the context of nowhere-zero flows motivates several questions considered in this paper.

Extending our earlier research we define new constructions and operations for graphs (such as graphs $\Delta_M(G)$) and give evidence for the complex relationship of homomorphisms and TT mappings. Particularly, solving an open problem, we display pairs of TT-comparable and homomorphism-incomparable graphs with arbitrarily high connectivity.

We give a new (and more direct) proof of density of TT order and study graphs such that TT mappings and homomorphisms from them coincide; we call such graphs homotens. We show that most graphs are homotens, on the other hand every vertex of a nontrivial homotens graph is contained in a triangle. This provides a justification for our construction of homotens graphs.

Key words graphs – homomorphisms – tension-continuous mappings
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Contents

1 Introduction

It is a traditional mathematical theme to study the question when a map between the sets of substructures is induced (as a lifting) by a mapping of underlying structures. In a combinatorial setting (and as one of the simplest instances of this general paradigm) this question takes the following form:

Question 1. Given undirected graphs G, H and a mapping $g : E(G) \to E(H)$ does there exist a mapping $f : V(G) \to V(H)$ such that $g(\{x, y\}) = \{f(x), f(y)\}$ for every edge $\{x, y\} \in E(G)$?

In the positive case we say that g *is induced by* f. It is easy to see that such mapping f is a homomorphism $G \xrightarrow{hom} H$ and that to each homomomorphism corresponds exactly one induced mapping g. Thus Question 1 asks which mappings g between edge sets are induced by a homomorphism. Various instances of this problem were considered for example by Whitney [21], the first author [15], Kelmans [10], and by Linial, Meshulam, and Tarsi [13]. More recently, DeVos, Nešetřil, and Raspaud [3] isolated the following necessary condition for a mapping $g : E(G) \to E(H)$ to be induced by a homomorphism.

$$\text{For every cut } C \subseteq E(H) \text{ the set } g^{-1}(C) \text{ is a cut of } G. \qquad (1)$$

Here, a *cut* means the edge set of a spanning bipartite induced subgraph. It is natural to call any mapping g satisfying condition (1) a *cut-continuous* mapping $G \xrightarrow{cc} H$. Cut-continuous mappings extend and generalize the notion of a homomorphism and the relationship of these two notions is the central theme of this paper. We provide evidence in both directions. We present various examples of cut-continuous mapping that are not induced, in particular in Proposition 4 we construct such mappings between highly connected graphs, thereby solving a problem from our previous paper [19]. On the other hand, as described in Section 4, for most of the graphs all cut-continuous mappings are induced.

Cut-continuous mappings were defined and investigated in [3, 19] in the more general context of nowhere-zero flows and circuit covers. As such, the tension-continuous mappings (being duals of flow-continuous mappings) have deep combinatorial meaning. For example, for a cubic graph G the number of cut-continuous mappings $G^* \xrightarrow{cc} K_3$ equals the number of 1-factorizations of G. (Consequently, there is a cut-continuous mapping $K_4 \xrightarrow{cc} K_3$, while there is clearly no homomorphism $K_4 \xrightarrow{hom} K_3$.) On a similar note, let T be a graph with two vertices, one edge connecting them and one loop. It is known that the number of homomorphisms $f : G \xrightarrow{hom} T$ equals to the number of independent sets of the graph G, a graph parameter that is important and hard to compute. The corresponding parameter, the number of cut-continuous mappings $g : G \xrightarrow{cc} T$ is simple to compute (but still interesting): it is equal to the number of cuts in G, that is to $2^{|V(G)|-k}$, where k is the number of components of G.

The analysis of flow problems by means of edge mappings between graphs was pioneered by Jaeger [9]; the basic definitions were stated and developed in [3]. In [19] we studied tension-continuous (mainly $\mathbb{Z}_2$-tension-continuous, that is cut-continuous) mappings more thoroughly. Here we extend and complement results of [19] by treating tension-continuous mappings in an arbitrary abelian group instead of $\mathbb{Z}_2$. We also solve several open problems from [19]. Particularly, we find examples of k-connected graphs that are equivalent with respect to tension-continuous mappings and not with respect to homomorphisms (Proposition 4 in Section 3). On the positive side we give a characterization of a large class of graphs where tension-continuous mappings coincide with homomorphisms. Such graphs (called here left and right homotens graphs) are studied in Sections 4 and 5. This also implies a shorter proof of some results of [19], particularly of universality (Theorem 4) and density (Theorem 6) of tension-continuous mappings. The proof of the latter uses construction $\Delta_M(G)$ (defined in Section 5), which is interesting in itself.

2 Definition & Basic Properties

2.1 Basic notions—flows and tensions

We refer to [4, 8] for basic notions on graphs and their homomorphisms.

By a graph we mean a finite directed graph with multiple edges and loops allowed. We write uv (or sometimes (u, v)) for an edge from u to v (one of them, if there are several parallel edges). A *circuit* in a graph is a connected subgraph in which each vertex is adjacent to two edges. For a circuit C, we let C^+ and C^- be the sets of edges oriented in either direction. We will say that (C^+, C^-) is a *splitting* of edges of C.

A *cycle* is an edge-disjoint union of circuits. Given a graph G and a set X of its vertices, we let $\delta(X)$ denote the set of all edges with one end in X and the other in $V(G) \setminus X$; we call each such edge set a *cut* in G. Let M be a ring (by this we mean an asociative ring with unity). We say that a function $\varphi : E(G) \to M$ is an *M-flow on G* if for every vertex $v \in V(G)$

$$\sum_{e \text{ enters } v} \varphi(e) = \sum_{e \text{ leaves } v} \varphi(e)\,.$$

A function $\tau : E(G) \to M$ is an *M-tension on G* if for every circuit C in G (with (C^+, C^-) being the splitting of its edges) we have

$$\sum_{e \in C^+} \tau(e) = \sum_{e \in C^-} \tau(e)\,.$$

We remark that for definition of flows and tensions we could use any abelian group. But as our emphasis is on finite graphs, we are interested in finitely generated abelian groups. Every such group is of form $\mathbb{Z}^k \times \prod \mathbb{Z}_{n_i}^{k_i}$, therefore we can introduce a ring structure on it. In proof of Lemma 14 we present a way how results about general abelian groups can be inferred from finitely generated ones.

Note that M-tensions on a graph G form a module over M (or even a vector space, if M is a field). Its dimension is $|V(G)| - k(G)$, where $k(G)$ denotes the number of components of G. This module will be called the *M-tension module* of G.

For a cut $\delta(X)$ we define

$$\varphi_X(uv) = \begin{cases} 1 & \text{if } u \in X \text{ and } v \notin X \\ -1 & \text{if } u \notin X \text{ and } v \in X \\ 0 & \text{otherwise}\,. \end{cases}$$

Any such φ_X is called *elementary M-tension*. It is easy to prove that elementary M-tensions generate the M-tension module.

Remark that every M-tension is of form δp, where $p : V(G) \to M$ is any mapping and $(\delta p)(uv) = p(v) - p(u)$ (in words, tension is a difference of a potential).

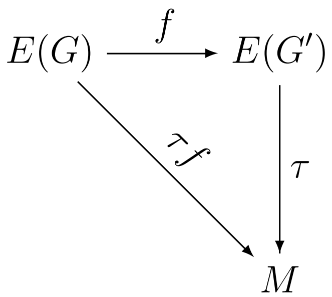
For M-flows the situation is similar to M-tensions: all M-flows on G form a module (the *M-flow module* of G) of dimension $|E(G)| - |V(G)| + k(G)$; it is generated by *elementary flows* (those with a circuit as a support) and it is orthogonal to the M-tension module.

The above are the basic notions of algebraic graph theory. For a more thorough introduction to the subject see [4]; we only mention two more basic observations:

A cycle can be characterized as a support of a $\mathbb{Z}_2$-flow and a cut as a support of a $\mathbb{Z}_2$-tension. If G is a plane graph then each cycle in G corresponds to a cut in its dual G^*; each flow on G corresponds to a tension on G^*.

2.2 Tension-continuous mappings

The following is the principal notion of this paper: Let M be a ring, let G, G' be graphs and let $f : E(G) \to E(G')$ be a mapping between their edge sets. We say f is an *M-tension-continuous mapping* (shortly TT_M mapping) if for every M-tension τ on G', the composed mapping τf is an M-tension on G. The scheme below illustrates this definition. It also shows that f "lifts tensions to tensions", thus suggesting the term TT mapping.



We write $f : G \xrightarrow{TT_M} H$ if f is a TT_M mapping from G to H (or, more precisely, from $E(G)$ to $E(H)$). In the important case $M = \mathbb{Z}_n$ we write TT_n instead of $TT_{\mathbb{Z}_n}$, when M is clear from the context we omit the subscript.

Of course if $M = \mathbb{Z}_2$ then the orientation of edges does not matter. Hence, if G, H are undirected graphs and $f : E(G) \to E(H)$ any mapping, we say that f is *$\mathbb{Z}_2$-tension-continuous* (TT_2) if for some (equivalently, for every) orientation $\overrightarrow{G}$ of G and $\overrightarrow{H}$ of H, f is TT_2 mapping from $\overrightarrow{G}$ to $\overrightarrow{H}$. As cuts correspond to $\mathbb{Z}_2$-tensions, with this provision TT_2 mappings of undirected graphs are exactly the *cut-continuous* mappings: mappings between edge sets of undirected graphs such that preimage of every cut is a cut.

For general ring M, the orientation is important. Still, we define that a mapping $f : E(G) \to E(H)$ between undirected graphs G, H is TT_M if for some orientation $\overrightarrow{G}$ of G and $\overrightarrow{H}$ of H, f is TT_M mapping from $\overrightarrow{G}$ to $\overrightarrow{H}$. This definition may seem a bit arbitrary, but in fact it is a natural one: clearly it is equivalent to ask that for each $\overrightarrow{H}$ there is an $\overrightarrow{G}$ such that f is a TT_2 mapping from $\overrightarrow{G}$ to $\overrightarrow{H}$ (we just change orientation of edges of $\overrightarrow{G}$ according to change of orientation of edges of $\overrightarrow{H}$). We will elaborate more on this in Proposition 1.

Convention. Unless specifically specified, our results hold for both the directed and undirected case.

Recall that $h : V(G) \to V(G')$ is called a *homomorphism* if for any $uv \in E(G)$ we have $f(u)f(v) \in E(G')$; we shortly write $h : G \xrightarrow{hom} G'$. We define a quasiorder $\preccurlyeq_h$ on the class of all graphs by

$$G \preccurlyeq_h G' \iff \text{there is a homomorphism } h : G \xrightarrow{hom} G'.$$

Homomorphisms generalize colorings: a k-coloring is exactly a homomorphism $G \xrightarrow{hom} K_k$, hence $\chi(G) \le k$ iff $G \preccurlyeq_h K_k$. For an introduction to the theory of homomorphisms see [8].

Motivated by the homomorphism order $\preccurlyeq_h$, we define for a ring M an order $\preccurlyeq_M$ by

$$G \preccurlyeq_M G' \iff \text{there is a mapping } f : G \xrightarrow{TT_M} G'.$$

This is indeed a quasiorder, see Lemma 1. We write $G \approx_M H$ iff $G \preccurlyeq_M H$ and $G \succcurlyeq_M H$, and similarly we define $G \approx_h H$; we say G and H are TT_M-equivalent, or hom-equivalent, respectively. Occasionally, we also write $G \xrightarrow{TT_M} H$ (instead of $G \preccurlyeq_M H$) to denote the existence of some TT_M mapping.

We define analogies of other notions used for study of homomorphisms: a graph G is called *TT_M-rigid* if there is no non-identical mapping $G \xrightarrow{TT_M} G$. Graphs G, H are called *TT_M-incomparable* if there is no mapping $G \xrightarrow{TT_M} H$, neither $H \xrightarrow{TT_M} G$.

If G is an undirected graph, its *symmetric orientation* $\overleftrightarrow{G}$ is a directed graph with the same set of vertices and with each edge replaced by an oriented 2-cycle, we will say these two edges are opposite. The following result clarifies the role of orientations.

Proposition 1. *Let G, H be undirected graphs, $E(H) \neq \emptyset$, let M be a ring. Then the following are equivalent.*

1. *For some orientation $\overrightarrow{G}$ of G and $\overrightarrow{H}$ of H it holds that $\overrightarrow{G} \xrightarrow{TT_M} \overrightarrow{H}$.*

2. *For each orientation* $\overrightarrow{H}$ *of* H *exists* $\overrightarrow{G}$ *of* G *such that* $\overrightarrow{G} \xrightarrow{TT_M} \overrightarrow{H}$.

3. *For symmetric orientations* $\overleftrightarrow{G}$ *of* G *and* $\overleftrightarrow{H}$ *of* H *it holds that* $\overleftrightarrow{G} \xrightarrow{TT_M} \overleftrightarrow{H}$.

Proof. If $M = \mathbb{Z}_2^k$ then all statements are easily equivalent, so suppose $M \neq \mathbb{Z}_2^k$. Take a mapping $f_1 : \overrightarrow{G} \xrightarrow{TT_M} \overrightarrow{H}$. We may suppose that $\overrightarrow{G} \subseteq \overleftrightarrow{G}$ and $\overrightarrow{H} \subseteq \overleftrightarrow{H}$. Thus if e', e'' are opposite edges end $e' \in E(\overrightarrow{G})$, then we let $f_3(e')$ be $f_1(e')$ and $f_3(e'')$ be the edge opposite to $f_1(e')$. As cycles of $\overrightarrow{G}$ together with the 2-cycles consisting of opposite edges generate the cycle space of $\overleftrightarrow{G}$, mapping f_3 is TT_M, hence 1 implies 3. Next take any $\overrightarrow{H}$, suppose again $\overrightarrow{H} \subseteq \overleftrightarrow{H}$, and let opposite edges e', e'' of $\overleftrightarrow{G}$ correspond to $e \in G$. At least one of the edges $f_3(e')$, $f_3(e'')$ connects the same vertices (in the same direction) as some edge $\bar{e}$ of $\overrightarrow{H}$; we let this one of e', e'' to be an edge of $\overrightarrow{G}$ and let f_2 map it to $\bar{e}$. Clearly, f_2 is a TT_M mapping; therefore 3 implies 2. Finally 2 implies 1 is trivial. □

2.3 Basic properties

In this section we summarize some properties of TT mappings which will be needed in the sequel.

Lemma 1. *Let* $f : G \xrightarrow{TT_M} H$ *and* $g : H \xrightarrow{TT_M} K$ *be* TT_M *mappings. Then the composition* $g \circ f$ *is a* TT_M *mapping.*

Lemma 2. *Let* $f : G \xrightarrow{TT_M} H$*, let* H' *be a subgraph of* H *that contains all edges* $f(e)$ *for* $e \in E(G)$*. Then* $f : G \to H'$ *is* TT_M *as well.*

Proof. Take any M-tension τ' on H'. Let $\tau' = \delta p'$ for $p' : V(H') \to M$. If $V(H) = V(H')$ let $p = p'$, otherwise extend p' arbitrarily to get p. Now $\tau = \delta p$ is an M-tension on H that agrees with τ' on $V(H')$. Hence $\tau' f = \tau f$, and as τf is an M-tension, $\tau' f$ is an M-tension, too. □

An easy corollary of these observations is the monomorphism-epimorpism factorization of TT_M mappings.

Corollary 1. *Let* $f : G \xrightarrow{TT_M} H$*. Then there is a graph* H' *and* TT_M *mappings* $f_1 : G \xrightarrow{TT_M} H'$*,* $f_2 : H' \xrightarrow{TT_M} G$ *such that* f_1 *is surjective and* f_2 *injective.*

Another easy (but useful) way to modify TT_M mapping is by adding parallel edges. The next result shows, that we may in many respects restrict ourselves to bijective TT_M mappings (this approach was taken by [13, 10]). A bijection $G \xrightarrow{TT_M} H$ may be viewed as an identification $E(G) = E(H)$, therefore we in fact study when the tension module of H is a submodule of tension module of G (this language was used in [20]).

Lemma 3. *Let $f : G \xrightarrow{TT_M} H$ be a TT_M mapping of (directed or undirected) graphs. Then there is a graph H' and a mapping $f' : E(G) \to E(H')$ such that*

- *f' is TT_M,*
- *f' is bijective,*
- *we can get H' by adding parallel edges and deleting edges from H.*
- *for each edge $a \in E(G)$ the edge $f'(a)$ connects the same vertices as $f(a)$.*

Proof. For an edge $e \in E(H)$ we let $c(e) = |f^{-1}(e)|$ be the number of edges that map to e. We replace each edge of H by $c(e)$ parallel edges in the same direction (in case of directed graphs) as e and keep all vertices; we let H' denote the resulting graph. We define $f'(a)$ to be any one of the parallel edges that replaced $f(a)$, making sure that f' is injective (therefore bijective). Clearly, for any $p : V(H) = V(H') \to M$, if we consider the M-tensions $\tau = \delta p$ of H and $\tau' = \delta p$ of H', then $f \circ \tau = f' \circ \tau'$. Thus if f was a TT_M mapping, f' is TT_M as well. □

If C is a circuit with a splitting (C^+, C^-), we say that C is *M-balanced* if $(|C^+| - |C^-|) \cdot 1 = 0$ (with 0, 1, and operations in M). Otherwise, we say C is *M-unbalanced.* Let $g_M(G)$ denote the length of the shortest M-unbalanced circuit in G, if there is none we put $g_M(G) = \infty$. For the particular case $M = \mathbb{Z}_2$, a circuit is M-balanced if it is even, hence $g_{\mathbb{Z}_2}(G)$ is the odd-girth of G. We also have $G \xrightarrow{TT_M} \overrightarrow{K}_2$ iff any constant mapping $E(G) \to M$ is an M-tension. This clearly happens precisely when all circuits in G are M-balanced, equivalently, if $g_M(G) = \infty$. As a consequence of this, the function g_M provides us with an invariant for the existence of TT_M mappings, as shown in the next two lemmas.

Lemma 4. *Let M be a ring, let G, H be directed graphs, let $f : G \xrightarrow{TT_M} H$. If C is an M-unbalanced circuit in G then $f(C)$ contains an M-unbalanced circuit.*

Proof. The inclusion homomorphism $C \to G$ induces a TT_M mapping, composition with f yields $C \xrightarrow{TT_M} H$. By Lemma 2 we get a mapping $C \xrightarrow{TT_M} f(C)$. If all circuits in $f(C)$ are M-balanced, then $f(C) \xrightarrow{TT_M} \overrightarrow{K}_2$ and, by composition we have $C \xrightarrow{TT_M} \overrightarrow{K}_2$. This contradicts the fact that C is M-unbalanced. □

Lemma 5. *Let $G \preccurlyeq_M H$ be directed graphs. Then $g_M(G) \geq g_M(H)$.*

Proof. If $g_M(G) = \infty$, the conclusion holds. Otherwise, let C be an M-unbalanced circuit of length $g_M(G)$ in G. By Lemma 4, $f(C)$ contains an M-unbalanced circuit. It is of size at least $g_M(H)$ and at most $g_M(G)$. □

An alternative definition of tension-continuous mappings (proved in [3]) is often useful. For mappings $f : E(G) \to E(H)$ and $\varphi : E(G) \to M$ we let φ_f denote the *algebraical image of* φ: that is we define a mapping $\varphi_f : E(H) \to M$ by

$$\varphi_f(e') = \sum_{e \in f^{-1}(e')} \varphi(e)\,.$$

Lemma 6. *Let $f : E(G) \to E(H)$ be a mapping. Then f is M-tension-continuous if and only if for every M-flow φ on G, its algebraical image φ_f is an M-flow. Moreover, it is enough to verify this property for the basis of the flow module (elementary flows supported by an elementary cycle).*

We formulate this explicitly for $M = \mathbb{Z}_2$. Mapping f is cut-continuous if and only if for every cycle C in G, the set of edges of H, to which an odd number of edges of C maps, is a cycle.

For a homomorphism (of directed or undirected graphs) $h : V(G) \to V(G')$ we let $h^\sharp$ denote the *induced mapping on edges*, that is $h^\sharp((u,v)) = (h(u), h(v))$, or $h^\sharp(\{u,v\}) = \{h(u), h(v)\}$. If h is an *antihomomorphism*, that is for every edge $(u,v) \in E(G)$ we have $(h(v), h(u)) \in E(G')$ (h reverses every edge), we define $h^\flat((u,v)) = (h(v), h(u))$ and call it a mapping induced by antihomomorphism. If G' has parallel edges, then $h^\sharp$ is not unique: we just ask that $h^\sharp$ maps each of the edges (u,v) to some of the edges $(h(u), h(v))$; similarly for homomorphisms of undirected graphs and for antihomomorphisms. The following easy lemma is the starting point of our investigation.

Lemma 7. *Let G, H be (directed or undirected) graphs, M a ring. For every (anti)homomorphism f from G to H the induced mapping $f^\sharp$ ($f^\flat$, respectively) from G to H is M-tension-continuous. Consequently, from $G \preccurlyeq_h H$ follows $G \preccurlyeq_M H$.*

Proof. It is enough to prove Lemma 7 for homomorphisms of directed graphs. So let $f : G \to H$ be such homomorphism, $\varphi : V(H) \to M$ a tension. We may assume that φ is an elementary tension corresponding to the cut $\delta(X)$. Then the cut $\delta(f^{-1}(X))$ determines precisely the tension $\varphi \circ f$. $\square$

The main theme of this paper is to find similarities and differences between orders $\preccurlyeq_h$ and $\preccurlyeq_M$. In particular we are interested in when the converse to Lemma 7 holds. Now, we present a more precise version of Question 1 stated in the introduction.

Problem 1. Let $f : E(G) \to E(H)$. Find suitable conditions for f, G, H that will guarantee that whenever f is TT_M, then it is induced by a homomorphism (or an antihomomorphism); i.e. that there is a homomorphism (or an antihomomorphism) $g : V(G) \to V(H)$ such that $f = g^\sharp$ (or $f = g^\flat$).

Shortly, we say a mapping is *induced* if it is induced by a homomorphism or an antihomomorphism. Problem 1 leads us to the following definitions.

Definition 1. *We say a graph G is* left M-homotens *if for every loopless graph H every TT_M mapping from G to H is* induced *(that is induced by a homomorphism or an antihomomorphism). For brevity we will often call left M-homotens graphs just M-homotens graphs (following [19]).*

On the other hand, H is a right M-homotens *graph if for every graph G statements $G \xrightarrow{hom} H$ and $G \xrightarrow{TT_M} H$ are equivalent.*

We should note here, that the precise analogy of left M-homotens graphs—every TT_M mapping is induced—is not interesting, as this is much too strong requirement. For simplicity, suppose $M = \mathbb{Z}_2$. Let H be such graph, let $\Delta(H)$ be as defined before Lemma 8. The mapping $f : \Delta(H) \xrightarrow{TT_2} H$ given by $f(\{A, B\}) = A \,\Delta\, B$ is induced by an (anti)homomorphism, say g. Now this can happen only if for every $A \in V(\Delta(H))$ vertex $g(A)$ is adjacent to every edge e of H. (To see this, note that $f(\{A, A \,\Delta\, e\}) = e$, therefore $g(A)$ is one of the end vertices of e.) And this in turn can happen only if H is edgeless, or if $H = K_2$.

Definition of left M-homotens makes sense for both directed and undirected graphs. If $M = \mathbb{Z}_2^k$ then there are only trivial directed M-homotens graphs (namely an orientation of a matching). Thus, we restrict to study of undirected homotens graphs in this case [19]. For other rings, Proposition 2 states that the orientation does not play any role; this will be useful in Section 4 in our study of directed M-homotens graphs.

For $M \neq \mathbb{Z}_2^k$ we might study undirected M-homotens graphs, too. The relationship between these two notions (undirected graph is homotens versus some its orientation is homotens) is not clear. For every M, the latter notion implies the former one; however, somewhat surprisingly, both notions are equivalent for many rings M (at least for such, in which the equation $x + x = 0$ has no nonzero solution). (For right homotens graphs, the above discussion applies, too.)

Proposition 2. *Let G_1, G_2 be two directed graphs, such that we can get G_2 from G_1 by changing directions of edges, deleting and adding multiple edges. Let M be a ring. Then G_1 is left M-homotens if and only if G_2 is left M-homotens.*

Proof. Suppose G_1 is not homotens, that is there is a graph H_1 and a mapping $f_1 : G_1 \xrightarrow{TT_M} H_1$ that is not induced. By Lemma 3 we may suppose that f_1 is injective. We modify f_1 and H_1, to get a non-induced mapping $f_2 : G_2 \xrightarrow{TT_M} H_2$. If we change an orientation of an edge, we change an orientation of the corresponding edge in H_1. If we add an edge parallel to some edge e of G_1 then we map it to a new edge of H_1, parallel to $f_1(e)$. It is clear, that we get a TT_M mapping that is not induced. □

3 Examples

We illustrate the complex relationship of homomorphisms and TT mappings by several examples presenting the similarities and (mainly) the differences in concrete independent settings. Towards the former, we provide an infinite chain and antichain of $\preccurlyeq_{\mathbb{Z}_2}$, thereby exhibiting a similar behaviour of homomorphisms and TT mappings. On the other hand, we show that arbitrarily high connectivity of the source and target graphs does not force $TT_{\mathbb{Z}}$ mappings (much the less TT_M mappings) and homomorphisms to coincide. Finally, we show that an equivalence class of $\approx_{\mathbb{Z}_2}$ can contain exponentially many equivalence classes of $\approx_h$.

Proposition 3 appears already in [3], we include a proof for the convenience of the reader. Note that this result will be strongly generalized by Theorems 3, 4, and 6.

Proposition 3. *Graphs K_{2^t} form a strictly increasing chain in $\preccurlyeq_{\mathbb{Z}_2}$ order, that is $K_4 \prec_{\mathbb{Z}_2} K_8 \prec_{\mathbb{Z}_2} K_{16} \prec_{\mathbb{Z}_2} \cdots$. There are graphs $G_1, G_2, \dots$ that form an infinite antichain: there is no mapping $G_i \xrightarrow{TT_2} G_j$ for $i \neq j$.*

Proof. By Proposition 6 of [3] (compare also Corollary 8 of this paper), for any graph G

$$G \xrightarrow{hom} K_{2^k} \Longleftrightarrow G \xrightarrow{TT_2} K_{2^k} . \qquad (2)$$

This implies the first part. For the second part, let G_t be the Kneser graph $K(n,k)$ with $k = t(2^t - 2)$ and $n = 2k + 2^t - 2$. It is known that $\chi(G_t) = n - 2k + 2 = 2^t$. This by equivalence (2) implies that $G_i \overset{TT_2}{\not\longrightarrow} G_j$ for $i > j$. The remaining part follows from Lemma 5: It is known that the shortest odd cycle in $K(n,k)$ is the smallest odd number greater or equal to $n/(n-2k)$, which means that $g_{\mathbb{Z}_2}(G_t) = 2t + 1$. □

The differences of TT mappings and homomorphisms are easy to find. For example let $\{e_1, e_2, e_3\}$ be the edges of K_3, and color the edges of K_4 properly by three colors. We send both edges of color i to e_i. This mapping is easily checked to be TT_2, so we have $K_4 \xrightarrow{TT_2} K_3$ but obviously there is no homomorphism $K_4 \to K_3$. On the contrary, $TT_{\mathbb{Z}}$ mappings are more restricted and, indeed, there is no $TT_{\mathbb{Z}}$ mapping from an orientation of K_4 to an orientation of K_3. A simple example of $TT_{\mathbb{Z}}$ mapping that is not induced by a homomorphism is a noncyclic permutation of edges of an oriented circuit. E.g., let $E(\overrightarrow{C}_5) = \{e_0, e_1, \dots, e_4\}$ in this order, and define $f(e_i) = e_{2i \bmod 5}$. Then f is $TT_{\mathbb{Z}}$, on the other hand, f maps adjacent edges to nonadjacent edges, hence is not induced by a homomorphism. By applying the arrow construction—that is by replacing each oriented edge by a suitable graph (see [8] and also proof of Proposition 5 for more details) it is easy to produce graphs G, H such that $G \xrightarrow{TT_{\mathbb{Z}}} H$ but $G \overset{hom}{\not\longrightarrow} H$. No graphs G, H obtained in this manner are 3-connected; Whitney's theorem (two 3-regular graphs with the same cycle matroid are isomorphic) seems to suggest, that this situation may not repeat for graphs with higher connectivity. Therefore, the following lemma may be a bit surprising.

Proposition 4. *For every k there are k-connected graphs G, H such that $G \xrightarrow{TT_{\mathbb{Z}}} H$ but $G \overset{hom}{\not\longrightarrow} H$. Therefore, for each k exists a k-connected graph that is not $\mathbb{Z}$-homotens.*

Proof. Fix a k, let G, H be graphs illustrated for $k = 4$ in Figure 1. [1] (The construction is due to Shih [20].)

[1] If we wish to construct directed graphs, consider any orientation of them, such that corresponding edges of G and of H are oriented in the same way.

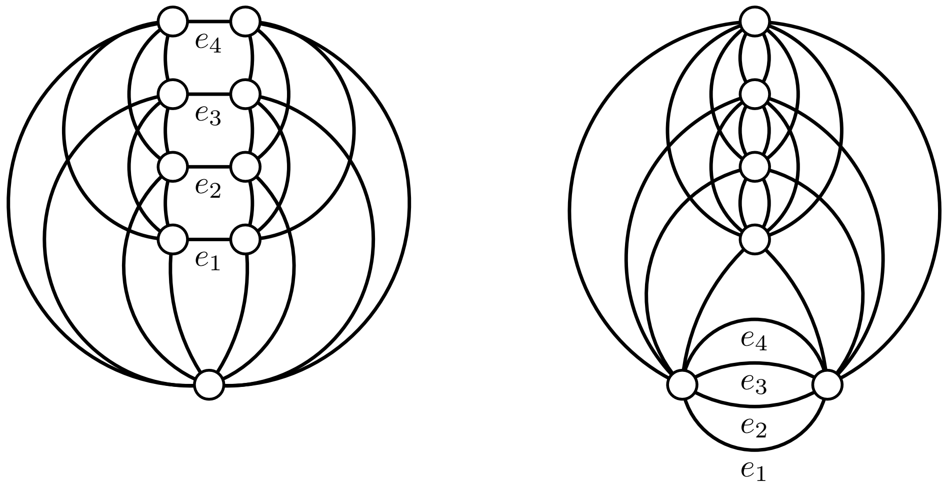


Figure 1: The left graph is an example of highly connected graph that is not $\mathbb{Z}$-homotens; the right one is a witness for the former not being $\mathbb{Z}$-homotens.

Clearly both G and H are k-connected and there there is no homomorphism between them. The natural bijection between G and H—we identify the left K_k's in G and H, the right K_k's in G and H, and the edges e_i as depicted in the Figure—is easily checked to be $TT_{\mathbb{Z}}$. $\square$

Further examples of graphs with negative answer to Problem 1 are listed in [19], here we only mention the perhaps most spectacular example: Petersen graph admits a TT_2 mapping to C_5. This mapping (and many others) may be obtained using the following construction: Given an (undirected) graph $G = (V, E)$ write $\Delta(G)$ for the graph $(\mathcal{P}(V), E')$, where $AB \in E'$ iff $A\Delta B \in E$ (here $\mathcal{P}(V)$ denotes the set of all subsets of V and $A\Delta B$ the symmetric difference of sets A and B).

Lemma 8. *Let G, H be undirected graphs. Then $G \xrightarrow{TT_2} H$ iff $G \xrightarrow{hom} \Delta(H)$.*

We can formulate analogous construction and result for rings $M \neq \mathbb{Z}_2$; this is done in Section 5.1. We conclude this section by a more quantitative example.

Proposition 5. *There are 2^{cn} undirected graphs with n vertices that form an antichain in the homomorphism order, yet all of them are TT_2-equivalent.*

Proof. To simplify notation, we will construct $\binom{n}{\lfloor n/2 \rfloor}$ graphs with $sn+1$ vertices, this clearly proves the proposition. We use the *replacement operation* of [8]. Let H be a graph (we explain later how do we choose it), let a, b, x_1, $\ldots$, x_5 be pairwise distinct vertices of H. Next, we take an oriented path with n edges and

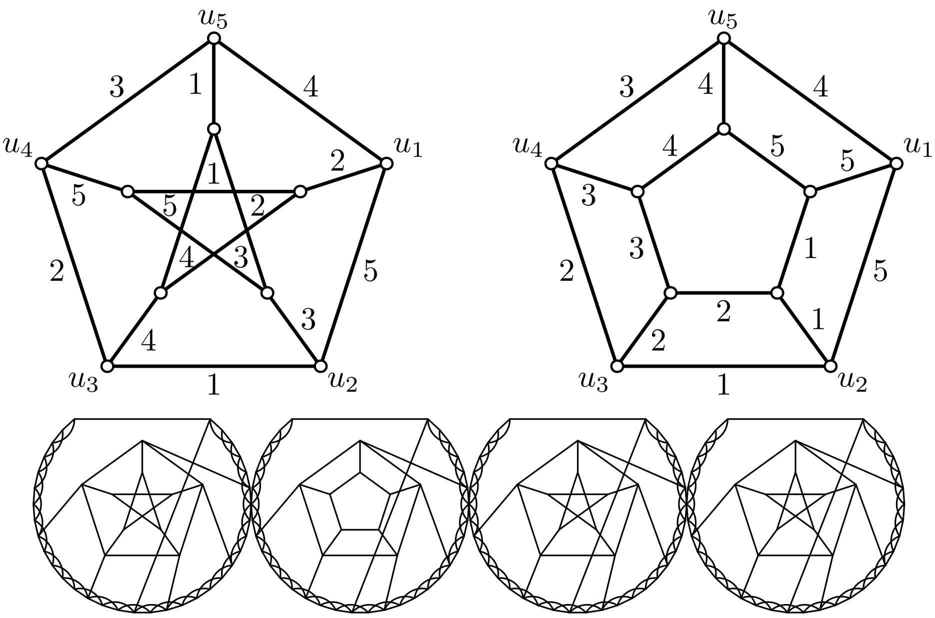


Figure 2: Petersen graph and the prism of C_5—two TT_2-equivalent graphs used in the proof of Proposition 5. Below is an example of the construction for $n = 4$, $t = (1, 0, 1, 1)$.

replace each of them by a copy of H. That is, we take $H_1, \ldots, H_n$—isomorphic copies of H—and identify vertex b of H_i with a of H_{i+1} (for every $i = 1, \ldots, n-1$). Let G be the resulting graph.

Finally, for each $t \in \{0, 1\}^n$ we present a graph G_t. We let F_i be a copy of the Petersen graph P if $t_i = 1$, and a copy of the prism of C_5—graph R in Figure 3—if $t_i = 0$. We construct the graph G_t as a vertex-disjoint union of G, $F_1, \ldots, F_t$ plus some 'connecting edges': for every $i = 1, \ldots, n$ and $j = 1, \ldots, 5$ we let x_i^j denote the copy of x_j in $H_i \subset G$ and u_i^j the copy of u_j in F_i; we let $x_i^j u_i^j$ be an edge of G_t. Note that each G_t has $(|V(P)| + |V(H)| - 1)n + 1$ vertices.

Claim 1. H can be chosen so that the only homomorphism $G \to G$ is the identity. Moreover the vertices x_i can be chosen so that the distance between any two of them is at least 4.

This follows immediately from techniques of [8], e.g. we can take H_9 from the Figure 4.9 of [8] as our graph H.

Claim 2. If $G_t \xrightarrow{hom} G_{t'}$ then $t_i \le t'_i$ holds for each i.

Take any homomorphism $f : G_t \xrightarrow{hom} G_{t'}$, fix an i, and let F_i (F'_i) be the copy of P or R that constitute the i-th part of graph G_t ($G_{t'}$ respectively). By Claim 1, f maps the vertices of G identically, in particular $f(x_i^j) = x_i^j$. As the only path of length 3 connecting vertices x_i^j and $x_i^{j \bmod 5+1}$ is the one containing vertices u_i^j

and $u_i^{j \bmod 5+1}$, mapping f satisfies $f(u_i^j) = u_i^j$ as well. Consequently, f maps vertices of F_i to vertices of F_i'. To show $t_i \le t_i'$ it remains to observe that there is no homomorphism $P \xrightarrow{hom} R$.

Claim 3. For every t, t' we have $G_t \xrightarrow{TT_2} G_{t'}$.

We map every edge of G and every edge $x_i^j u_i^j$ and $u_i^j u_i^{j \bmod 5+1}$ identically (we call such edges *easy edges*). We map edges of F_i in G_t to edges of the outer pentagon of F_i in $G_{t'}$ by sending an edge to the outer edge with the same number in Figure 3. To check that this is indeed a TT_2 mapping we use Lemma 6: if C is a cycle contained in some F_i then we easily check that algebraical image of C is a cycle. If C contains only easy edges that it is mapped identically, so its algebraical image is again a cycle. As every cycle can be written as a symmetric difference of these two types, we conclude that we have constructed a TT_2 mapping.

Now we are ready to finish the proof. Consider a set A containing all vertices of $\{0,1\}^n$ with $\lfloor n/2 \rfloor$ coordinates equal to 1. By Claim 2, graphs G_t, $G_{t'}$ are homomorphically incomparable for distinct $t, t' \in A$. On the other hand, by Claim 3, all of the graphs are TT_2-equivalent. $\square$

In this proof we can use other building blocks instead of Petersen graph and the pentagonal prism. To be concrete, we can take graphs G, H from Proposition 4 and use graphs $G \dot\cup H$ and $H \dot\cup H$. If we slightly modify the construction, we can prove version of Proposition 5 for $TT_{\mathbb{Z}}$ mappings, and therefore for TT_M mappings for arbitrary M. Moreover, by another small change of the construction, we can guarantee that all of the constructed graphs are k-connected (for any given k).

It would be interesting to know if 2^{cn} from Proposition 5 can be improved. Note that in the homomorphism order $\preccurlyeq_h$ the maximal antichain has full cardinality [12], that is there are

$$\frac{1}{n!}\binom{\binom{n}{2}}{\lfloor \frac{1}{2}\binom{n}{2} \rfloor}(1 - o(1))$$

homomorphically incomparable graphs with n-vertices. Proposition 5 claims that at least 2^{cn} of these graphs are contained in one equivalence class of $\approx_M$.

4 Left homotens graphs

In this section we point out similarities between homomorphisms and TT_M mappings by defining a class of graphs that force any TT_M mapping from them to be induced. We prove a surprising result that most graphs have this property. In Section 4.2 we use these graphs to find an embedding of category of graphs and homomorphism to the category of graphs and TT_M mappings, simplifying and generalizing a result of [19].

4.1 A sufficient condition

Recall (Definition 1) that a graph G is left M-homotens if every TT_M mapping from G (to any graph) is induced. The characterization of left M-homotens graphs seems to be a difficult problem; in this section we obtain a general sufficient condition in terms of *nice* graphs. This notion was introduced and proved to be a sufficient condition in [19] but only for $M = \mathbb{Z}_2^k$. Here, we prove it to be sufficient for all rings *different* from $\mathbb{Z}_2^k$. (Restricting to $M \neq \mathbb{Z}_2^k$ enables us to slightly weaken the sufficient condition.)

In Proposition 4 we saw that high connectivity does not imply homotens. In Corollary 4 we will see that every vertex of a homotens graph is incident with a triangle. In view of this, a sufficient condition for homotens has to be somewhat restrictive.

Definition 2. *We say that an undirected graph G is* nice *if the following holds*

1. *every edge of G is contained in some triangle*
2. *every triangle in G is contained in some copy of K_4*
3. *every copy of K_4 in G is contained in some copy of K_5*
4. *for every K, K' that are copies of K_4 in G there is a sequence of vertices v_1, v_2, ..., v_t such that*
    - $V(K) = \{v_1, v_2, v_3, v_4\}$,
    - $V(K') = \{v_t, v_{t-1}, v_{t-2}, v_{t-3}\}$,
    - $v_i v_j$ *or* $v_j v_i$ *is an edge of G whenever* $1 \leq i < j \leq t$ *and* $j \leq i + 3$.

We say that a graph is weakly nice *if conditions 1, 2, and 4 in the list above are satisfied. Finally, we say that a* directed *graph is (weakly) nice, if the underlying undirected graph is (weakly) nice.*

Before we prove Theorem 2, which we are aiming to, we restate here analogous result that appears as Theorem 13 in [19].

Theorem 1. *Let G, H be undirected graphs, let G be nice, and let $f : G \xrightarrow{TT_2} H$. Then f is induced by a homomorphism of the underlying undirected graphs. Shortly, every undirected nice graph is $\mathbb{Z}_2$-homotens.*

Theorem 2. *Let G, H be (directed or undirected) graphs, let G be weakly nice, let $M \neq (\mathbb{Z}_2)^r$ any ring. Suppose $f : G \xrightarrow{TT_M} H$. Then f is induced by a homomorphism or an antihomomorphism. Shortly, every weakly nice graph is M-homotens.*

We take time out for a technical lemma.

Lemma 9. *Let M be a ring that is not isomorphic to a power of $\mathbb{Z}_2$. Let $f : \overrightarrow{K}_4 \xrightarrow{TT_M} H$, where H is any loopless graph and $\overrightarrow{K}_4$ any orientation of K_4. Then f is induced by an injective homomorphism or antihomomorphism. Moreover, this (anti)homomorphism is uniquely determined.*

Proof. Suppose first that $f(\overrightarrow{K}_4)$ is a three-colorable graph, i.e., that there is a homomorphism $h : f(\overrightarrow{K}_4) \to \overleftrightarrow{K}_3$, where $\overleftrightarrow{K}_3$ is the directed graph with three vertices and all six oriented edges among them. A composition of TT_M mapping $f : \overrightarrow{K}_4 \xrightarrow{TT_M} f(\overrightarrow{K}_4)$ with $h^\sharp$ gives $g : \overrightarrow{K}_4 \xrightarrow{TT_M} \overleftrightarrow{K}_3$. Consider the three cuts of size 4 in $\overleftrightarrow{K}_3$: X_1, X_2, X_3. As M is not a power of $\mathbb{Z}_2$, $1 + 1 \neq 0$; let φ_i be M-tension that attains value ± 1 on X_i and 0 elsewhere. We can choose φ_i so, that for every $e \in E(\overleftrightarrow{K}_3)$ we have $\{\varphi_1(e), \varphi_2(e), \varphi_3(e)\} = \{0, \pm 1\}$. As g is TT_M, mappings $\psi_i = \varphi_i g$ are M-tensions and for every $e \in E(\overrightarrow{K}_4)$ we have $\{\psi_1(e), \psi_2(e), \psi_3(e)\} = \{0, \pm 1\}$. (*)

Call an M-tension *simple* if it attains only values 0 and ± 1. We will show that three simple M-tensions ψ_1, ψ_2, ψ_3 on $\overrightarrow{K}_4$ with property (*) do not exist.

To this end, we will characterize sets $\operatorname{Ker} \psi = \{e \in E(\overrightarrow{K}_4), \psi(e) = 0\}$ for simple M-tensions ψ. Let ψ be such tension. Pick $v \in V(\overrightarrow{K}_4)$ and let e_1, e_2, e_3 be adjacent to v. Note that ψ is determined by its values on e_1, e_2, e_3. We may suppose that each e_i is going out of v; otherwise we change orientation of some edges and the sign of ψ on them. Further, we may suppose that $|\{i, \psi(e_i) = 1\}| \geq |\{i, \psi(e_i) = -1\}|$; otherwise we consider $-\psi$. Thus, we distinguish the following cases (see Figure 3).

- $\psi(e_i) \in \{0, 1\}$ for each i.
  Let z be the number of e_i such that $\psi(e_i) = 0$. Then ψ is generated by a cut with $z + 1$ vertices on one side of the cut. Therefore, the set $\operatorname{Ker} \psi$ is either the edge set of a $\overrightarrow{K}_4$, of a triangle, or it is a pair of disjoint edges.

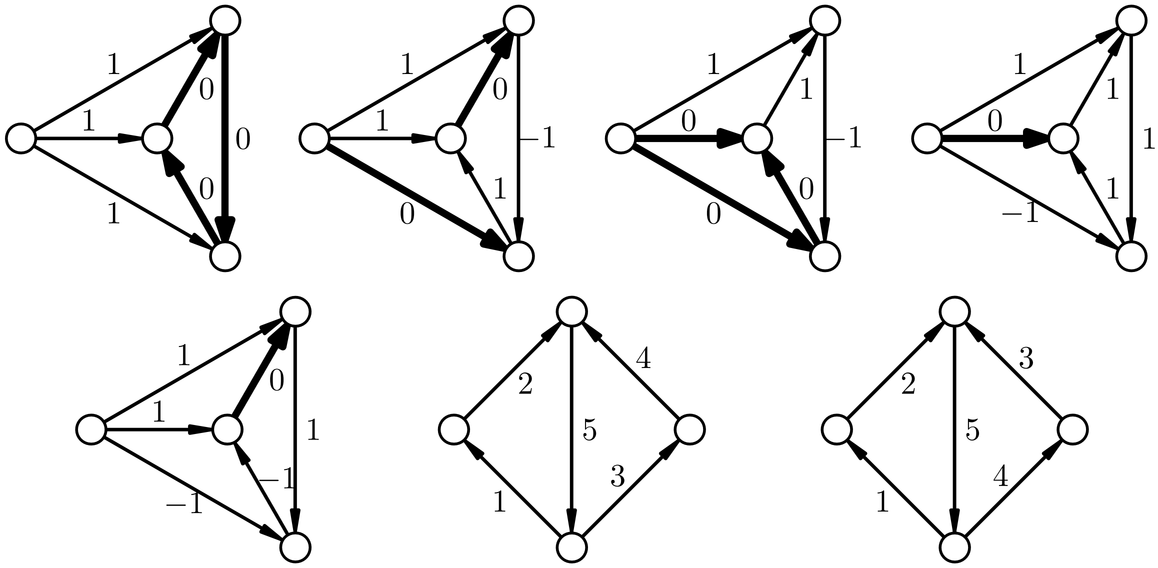


Figure 3: Illustration of proof of Lemma 9.

- $\psi(e_1) = 1$, $\psi(e_2) = 0$, $\psi(e_3) = -1$.
  In this case $\operatorname{Ker}\psi$ is a single edge. Note, that this case (and the next one) may happen only if $1 + 1 + 1 = 0$.

- $\psi(e_1) = \psi(e_2) = 1$, $\psi(e_3) = -1$.
  In this case too, $\operatorname{Ker}\psi$ is a single edge.

Hence, $E(\overrightarrow{K}_4)$ is partitioned into three sets, whose sizes are in $\{1, 2, 3, 6\}$. Therefore, there are two possibilities:

- $6 = 3 + 2 + 1$: The complement of a triangle is a star of three edges, there are no two disjoint edges in it.

- $6 = 2+2+2$: In this case, all three ψ_i's are generated by a cut. Suppose $\overrightarrow{K}_4$ is oriented as in Figure 3, the values of ψ_1 are indicated. It is not possible to fulfill the condition (*) on both edges from $\operatorname{Ker}\psi_1$.

So far we have proved, that the chromatic number of $f(\overrightarrow{K}_4)$ is at least four. As $f(\overrightarrow{K}_4)$ has at most 6 edges, its chromatic number is exactly four. Let $V_1, \ldots, V_4$ be the color classes. There is exactly one edge between two distinct color classes (otherwise the graph is three-colorable). Thus, f is a bijection. Next, $|V_i| = 1$ for every i (as otherwise, we can split one color-class to several pieces and join these to the other classes; again, the graph would be three-colorable). Consequently, $f(\overrightarrow{K}_4)$ is some orientation of K_4.

We call *star* a set of edges sharing a vertex. If we let φ be a simple M-tension on $f(\overrightarrow{K}_4)$ corresponding to a cut which is a star, then φf is a simple tension that is nonzero exactly on three edges (f is a bijection). By the characterization of zero sets of simple tensions we see that preimage of each star is a star. As f is a bijection and preimage of every star is a star, also image of every star is a star. This allows us to define a vertex bijection $g : V(\overrightarrow{K}_4) \to V(f(\overrightarrow{K}_4))$ by letting $g(u) = u'$ iff the f-image of the star with u as the central vertex is the star centered at u'. Stars sharing an edge map to stars sharing an edge, hence f is induced by g, which is either a homomorphism or an antihomomorphism. $\square$

Proof (Theorem 2). It is convenient to suppose that G contains no parallel edges (Proposition 2). Let K be a copy of K_4 in G (by this we mean here that K is some orientation of K_4). By Lemma 9 the restriction of f to K is induced by an (anti)homomorphism, let it be denoted by h_K. That is, we assume $f|_{E(K)} = h_K^\sharp$ (or $f|_{E(K)} = h_K^\flat$).

As every edge is contained in some copy of K_4, it is enough to prove that there is a common extension of all mappings $\{h_K \mid K \subseteq G,\ K \simeq K_4\}$ (we may define it arbitrarily on isolated vertices of G).

We say that h_K and $h_{K'}$ *agree* if for any $v \in V(K) \cap V(K')$ we have $h_K(v) = h_{K'}(v)$ and either both h_K, $h_{K'}$ are homomorphisms or both are antihomomorphisms. Thus, we need to show that any two mappings h_K, $h_{K'}$ agree.

First, let K, K' be copies of K_4 that intersect in a triangle. Then h_K and $h_{K'}$ agree (note that this does not necessarily hold if the intersection is just an edge, see Figure 3).

Now suppose K, K' are copies of K_4 that have a common vertex v. Since the graph G is weakly nice, we find $v_1, v_2, \ldots, v_t$ as in Definition 2. Let $G_i = G[\{v_i, v_{i+1}, v_{i+2}, v_{i+3}\}]$: every G_i is a copy of K_4, $G_1 = K$ and $G_{t-3} = K'$. Suppose $v = v_l = v_r$, where $l \in \{1, 2, 3, 4\}$, $r \in \{t-3, t-2, t-1, t\}$. Consider a closed walk $W = v_l, v_{l+1}, \ldots, v_{r-1}, v_r$. Let $v'_i = h_{G_i}(v_i)$ for $l \le i \le r-3$ and $v'_i = h_{G_{r-3}}(v_i)$ for $r-3 \le i \le r$. Mappings h_{G_i} and $h_{G_{i+1}}$ agree, hence $v'_i v'_{i+1} = f(v_i v_{i+1})$ is an edge of H. So $W' = v'_l, v'_{l+1}, \ldots, v'_{r-1}, v'_r$ is a walk in H.

Let φ be 'a ± 1-flow around W', formally

$$\varphi(e) = \sum_{\substack{l \le i \le r-1 \\ e=(v_i, v_{i+1})}} 1 - \sum_{\substack{l \le i \le r-1 \\ e=(v_{i+1}, v_i)}} 1 \,.$$

Clearly φ is an M-flow. Similarly, define $\varphi'(e)$ from W'. We have $\varphi' = \varphi_f$, hence φ' is a flow (Lemma 6). This can happen only if W' is a closed walk, that is $v'_l = v'_r$.

By definition, $v'_r = h_{K'}(v)$. As mappings h_{G_i} and $h_{G_{i+1}}$ agree, we have that $h_{G_i}(v_{i+j}) = h_{G_{i+j}}(v_{i+j})$ for $j \leq 3$. Consequently, $v'_l = h_K(v)$, which finishes the proof. □

Combining Theorems 1 and 2 we obtain a corollary.

Corollary 2. *An undirected nice graph is left M-homotens for every ring M. A (directed or undirected) weakly nice graph is left M-homotens for every ring $M \neq \mathbb{Z}_2^k$.*

Extending our conditions that guarantee that a graph is M-homotens, we present the following lemma, which will be used in Section 4.2. Note that the assumption about spanning subgraphs is needed.

Lemma 10. *Suppose H contains a connected spanning M-homotens graph. Then H is M-homotens.*

Proof. Let $f : H \xrightarrow{TT_M} K$, let G be the connected spanning M-homotens subgraph of H. Restriction of f to $E(G)$ is TT_M, hence $f(e) = g^\sharp(e)$ for each $e \in E(G)$ and some (anti)homomorphism g. Let $e = uv \in E(H) \setminus E(G)$. We have to prove $f(e) = (g(u), g(v))$. Let P be a path from u to v in G. By treating the closed walk $P \cup \{uv\}$ as W in the end of the proof of Theorem 2, we conclude the proof. □

4.2 Applications

In this section we provide several applications of nice graphs (that is of Theorem 2 and Corollary 2). Particularly, we prove that 'almost all' graphs are left M-homotens for every ring M and construct an embedding of category $\mathcal{G}_{hom}$ into $\mathcal{G}_{TT_M}$. This result was proved (for $M = \mathbb{Z}_2$) in [19] by an ad-hoc construction. Here we follow a more systematic approach—we employ a modification of an edge-based replacement operation (see [8]). As a warm-up we prove an easy, but perhaps surprising result.

Corollary 3. *For every graph G there is a graph G' containing G as an induced subgraph such that for every ring M every TT_M mapping from G' to arbitrary graph is induced by a homomorphism (i.e., G' is M-homotens).*

Proof. We take as G' the (complete) join of G and K_5; that is, we let $V(G') = V(G) \cup \{v_1, v_2, \dots, v_5\}$, and $E(G') = E(G) \cup \{\text{all edges containing some } v_i\}$. By Theorem 2 it is enough to show that G' is nice. Every copy of K_t ($t < 5$) in G' can be extended to K_5 by adding some vertices v_i. One can also show routinely that any two copies of K_4 in G' are 'K_4-connected'—condition 4 in Definition 2. □

The following theorem was our main motivation for introducing (weakly) nice graphs. Note that 'a.a.s.' means, as usual, 'asymptotically almost surely', that is 'with probability tending to 1'.

Theorem 3. *Let M be a ring.*

1. *Complete graph K_k is M-homotens for $k \geq 5$ (and for $k \geq 4$ if $M \neq (\mathbb{Z}_2)^t$).*
2. *The random graph $G(n, 1/2)$ is M-homotens a.a.s.*
3. *The random k-partite graph is M-homotens a.a.s. for $k \geq 5$ (and for $k \geq 4$ if $M \neq (\mathbb{Z}_2)^t$). Explicitly,*

$$\lim_{n\to\infty} \Pr[G = G(n, 1/2) \textit{ is } M\textit{-homotens} \mid G \textit{ is } k\textit{-partite}] = 1\,.$$

4. *The random K_k-free graph is M-homotens a.a.s. for $k \geq 6$ (and for $k \geq 5$ if $M \neq (\mathbb{Z}_2)^t$).*

If $M \neq \mathbb{Z}_2$, then in each of the statement, any orientation of the considered graph is M-homotens, too.

Proof. As K_t is nice (weakly nice for $t = 4$), 1 follows by Corollary 2. In [19] we proved that the random graph is a.a.s. nice, so again, Corollary 2 implies 2. By [11], a random K_k-free graph is a.a.s. $(k-1)$-partite, hence 3 implies 4. The proof of 3 is similar to the proof that the random graph is a.a.s. nice, we sketch it for convenience.

Let A_1, …, A_k be the parts of the random k-partite graph. By standard arguments, all A_i's are a.a.s. approximately of the same size, in particular all are non-empty. It is a routine to verify parts 1, 2, and (in case $k \geq 5$) 3 of Definition 2. For part 4, let $V(K) = \{v_1, \dots, v_4\}$, $V(K') = \{v_9, \dots, v_{12}\}$. We pick i_1, …, i_4 so that $v_t \notin A_{i_k}$, except possibly if $t = k$ or $t = k + 8$. We attempt to pick $v_5 \in A_{i_1}, \dots, v_8 \in A_{i_4}$ to satisfy the condition 4. The probability that a particular 4-tuple fails is at most $\left(1 - 2^{-18}\right)^{n/2k}$. Hence, the probability that some copies K, K' of K_4 are 'bad' is at most $n^8 c^n$ (for some $c < 1$). □

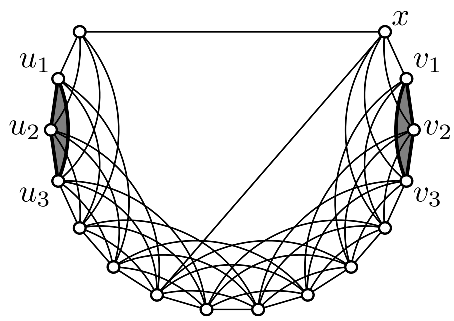


Figure 4: The graph I used in triangle-based replacement (proof of Theorem 4).

We proceed by another application of Corollary 2 — we show that the structure of TT_M mappings is at least as rich as that of homomorphisms.

Theorem 4. *There is a mapping F that assigns graphs to graphs, such that for any ring M and for any graphs G, H (we stress that we consider loopless graphs only) holds*

$$G \preccurlyeq_h H \iff F(G) \preccurlyeq_M F(H) .$$

Moreover F can be extended to a 1-1 correspondence for mappings between graphs: if $f : G \to H$ is a homomorphism, then $F(f) : F(G) \to F(H)$ is a TT_M mapping and any TT_M mapping between $F(G)$ and $F(H)$ is equal to $F(f)$ for some homomorphism $f : G \xrightarrow{hom} H$. (In category-theory terms, F is an embedding of the category of all graphs and their homomorphisms into the category of all graphs and all TT_M-mappings between them.)

Proof. We will use a modification of edge-based replacement (see [8]). Let I be the graph in Figure 4 with arbitrary (but fixed) orientation. To construct $F(G)$, we will replace each of the vertices of G by a triangle and each of the edges of G by a copy of I, gluing different copies on triangles. More precisely, let $U = V(G) \times \{0, 1, 2\}$, for every edge $e \in E(G)$ let I_e be a separate copy of I. If $e = (u, v)$ then we identify vertex u_i ($i \in \{0, 1, 2\}$) with (u, i) in U, and vertex v_i with (v, i) in U. Let $F(G)$ be the resulting graph; we write shortly $F(G) = G * I$. If $f : V(G) \to V(H)$ is a homomorphism then we define $F(f) : E(F(G)) \to E(F(H))$ as follows: let $e = (u, v)$ be an edge of G and a an edge of $E(I_e)$. Let e' be the image of e under f. In the isomorphism between I_e and I'_e the edge a gets mapped to some a'. We put $F(f)(a) = a'$. It is easily seen that $F(f)$ is a $TT_{\mathbb{Z}}$ (thus TT_M) mapping that is induced by a homomorphism, we let $\varphi(f)$ denote this homomorphism. Now, we turn to the more difficult step of proving that every TT_M mapping from G to H is $F(f)$ for some $f : G \xrightarrow{hom} H$. We will need several auxiliary claims.

Claim 1. I is critically 6-chromatic.

Take any K_5 in I, color in by 5 colors. There is a unique way how to extend it, which fails, so $\chi(I) \geq 6$. Clearly 6 colors suffice. Moreover, if we delete any vertex of I then it is possible to color the remaining vertices consecutively $1, 2, 3, 4, 5, 1, 2, \ldots, 5$.

Claim 2. I is rigid.

That is, the only homomorphism $f : I \to I$ is the identity. By Claim 1, f cannot map I to its subgraph, hence f is an automorphism. There is a unique vertex x of degree 9, so f fixes it. There is a unique hamiltonian cycle $x = x_1, \ldots, x_?$ such that $x_i x_j$ is an edge whenever $|i - j| \leq 4$, therefore this cycle has to be fixed by f too. This leaves two possibilities, but only one of them maps the edge $x_1 x_?$ properly.

Claim 3. I is K_5-connected.

That is, for every two vertices a, b of I there is a path $a = a_1, a_2, \ldots, a_k = b$ such that $a_i a_j$ is an edge whenever $|i - j| \leq 4$.

Claim 4. Whenever H is a graph and $g : I \xrightarrow{hom} H * I$ a homomorphism, there is an edge $e \in E(H)$ such that g is an isomorphism between I and I_e.

If g maps all vertices of I to one of the I_e's, then we are done by Claim 2. If not, let a, b be vertices of I such that $g(a)$ is a vertex of I_e (for some edge $e = uv \in E(H)$) and $g(b)$ is not. Choose a path $a = a_1, a_2, \ldots, a_k = b$ as in Claim 3. Let a_i be the last vertex on this path that is a vertex of I_e. Not all three vertices a_{i-1}, a_{i-2}, a_{i-3} can be in the 'connecting triangle' $\{v\} \times \{0, 1, 2\}$, on the other hand each of them is connected to a_{i+1}, a contradiction.

Claim 5. For every graph H the graph $H * I$ is nice.

This is an easy consequence of Lemma 10.

To finish the proof, let $h : F(G) \xrightarrow{TT} F(H)$ be a TT_M mapping. As graph $G * I$ is nice, it is M-homotens by Corollary 2. Therefore h is induced by a homomorphism, say $g : F(G) \xrightarrow{hom} F(H)$. By Claim 4, g maps an I_e to an $I_{e'}$, therefore there is a homomorphism $f : V(G) \to V(H)$ such that $g = \varphi(f)$ and $h = F(f)$, as claimed. □

4.3 A necessary condition

In this section we present a necessary condition for a graph to be $\mathbb{Z}$-homotens.[2] As mentioned earlier, odd circuits are the simplest examples of graphs that are not $\mathbb{Z}$-homotens. Similarly, no graph with a vertex of degree 2 is $\mathbb{Z}$-homotens, except of a triangle. This way of thinking can be further strengthened and generalized, yielding Theorem 5. To state our result in a compact way, we introduce a definition from [5]. We say that a graph G is *chromatically k-connected* if for every $U \subseteq V(G)$ such that $G - U$ is disconnected the induced graph $G[U]$ has chromatic number at least k. Equivalently [5], G is chromatically k-connected, iff every homomorphic image of G is k-connected.

Theorem 5. *Let M be a ring. If a graph is connected and M-homotens then it is chromatically 3-connected.*

Proof. Suppose G is a counterexample to the theorem. Hence, vertices of G can be partitioned into sets A, B, U, L, such that $A \cup B$ separates U from L; that is there is no edge from U to L, moreover A, B are independent sets. We may suppose $A \cup B$ is a minimal set that separates U from L. We are going to prove that G is not $\mathbb{Z}$-homotens, therefore by Lemma 16 not M-homotens as well.

We identify all vertices of A to a single vertex a, and all vertices of B to a vertex b. Let F be the resulting graph, and $f : G \to F$ be the identifying homomorphism. We define a $TT_{\mathbb{Z}}$ mapping g from F as follows. For $u \in U$ we map edge (u, a) (if it exists) to (b, u), (a, u) to (u, b), (u, b) to (a, u), and (b, u) to (u, a). For $u, v \in U$ we map edge (u, v) (if it exists) to (v, u). Every other edge is mapped to itself. We let F' denote the resulting graph (it has the same set of vertices as F). It is straightforward to use Lemma 6 to verify that g is indeed $TT_{\mathbb{Z}}$.

Hence $gf^\sharp$ is a $TT_{\mathbb{Z}}$ mapping; we need to show that it is not induced. At least one of A, B is non-empty. Suppose it is A and pick $x \in A$. As $A \cup B \setminus \{x\}$ is not a separating set ($A \cup B$ is a minimal one), there are vertices $u \in U$ and $l \in L$ that are adjacent to x, without loss of generality (x, u), (x, l) are edges of G. By definition of g we have $gf^\sharp((x, l)) = (x, l)$ and $gf^\sharp((x, u)) = (u, y)$. Therefore $gf^\sharp$ maps two adjacent edges to two nonadjacent edges, hence it is not induced. □

The following corollary deduces a simpler necessary condition, though a weaker one: We can prove that the graph of icosahedron is not $\mathbb{Z}$-homotens by using

[2]As any $TT_{\mathbb{Z}}$ mapping is TT_M for every ring M (Lemma 16), each M-homotens graph is also $\mathbb{Z}$-homotens. Therefore, the presented condition is necessary for a graph to be M-homotens, too. To illustrate that $\mathbb{Z}_2$-homotens is indeed stronger condition than $\mathbb{Z}$-homotens, we note that no 4-chromatic graph is $\mathbb{Z}_2$-homotens—it admits a TT_2 mapping to K_3.

Theorem 5 (the neighborhood of an edge is a C_6), but not using Corollary 4.

Corollary 4. *Let G be a connected graph with at least four vertices. Suppose the neighbourhood of some $v \in V(G)$ induces a bipartite graph. Then G is not M-homotens for any ring M.*

Consequently, every vertex of a homotens graph is incident with an odd wheel (in particular with a triangle), except if it is contained in a component of size at most three.

Proof. Let A, B be the color-classes of neighborhood of v. If there is a vertex nonadjacent to v, then we can use Theorem 5. So suppose v is connected to every vertex of G. Then every other vertex has a bipartite neighborhood. The only case that stops us from using Theorem 5 is when $|A|, |B| \le 1$, that is when G has at most three vertices. □

A somewhat surprising consequence of Corollary 4 is that no triangle-free graph is homotens. This immediately answers a question of [19]. It also implies, that a connected cubic graph is M-homotens only if it is a K_4 and M is not a power of $\mathbb{Z}_2$. More generally, we have the following result (compare Theorem 3).

Corollary 5. *Let $r \ge 3$ be integer, M ring. The probability that a random r-regular graph is M-homotens is bounded by a constant less than 1, if size of the graph is large enough.*

Proof. It is known [22] that the probability that random r-regular graph is triangle-free tends to a nonzero limit, hence we can apply Theorem 4. □

Corollary 4 also indicates that complete graphs involved in the definition of nice graphs are necessary, at least to some extent. However, the condition of Corollary 4 (or Theorem 5) is far from being sufficient: for example the graph from Proposition 4 is chromatically k-connected and not $\mathbb{Z}$-homotens. In particular, we do not know whether there are K_4-free homotens graphs. By [11], a random K_4-free graph is a.a.s. 3-partite, hence not chromatically 3-connected, hence by Theorem 5 not $\mathbb{Z}$-homotens. Still, it is possible that K_4-free $\mathbb{Z}$-homotens graphs exist, promising candidates are Kneser graphs $K(4n-1, n)$, which are chromatically 3-connected for large n [5].

Question 2. Is the Kneser graph $K(4n-1, n)$ $\mathbb{Z}$-homotens, if n is large enough?

5 Right homotens graphs

In this section we complement Section 4 by study of graphs which, when used as target graphs, make existence of TT mappings and of homomorphisms coincide. Recall that a graph H is called right M-homotens if the existence of a TT_M mapping from an arbitrary graph to H implies the existence of a homomorphism. Right homotens graphs (in comparison with left homotens ones) provide more structure; in this section we characterize them by means of special Cayley graphs and state a question aiming to find a better characterization.

5.1 Free Cayley graphs

Free Cayley graphs were introduced by Naserasr and Tardif [14] (see also thesis of Lei Chu [1]) in order to study chromatic number of Cayley graphs. They will serve us as a tool to study TT mappings, in particular we will use them to study right homotens graphs and to prove density in Section 6.

Let M be a ring, let H be a graph. For a vertex $v \in V(H)$ we let $e_v : V(H) \to M$ be the indicator function, that is $e_v(u) = 1$ if $v = u$ and $e_v(u) = 0$ otherwise. We define graph[3] $\Delta_M(H)$ with vertices $M^{V(H)}$, where (f, g) is an edge iff $g - f = e_v - e_u$ for some edge $(u, v) \in E(H)$. We can see that $\Delta_M(H)$ is a Cayley graph, it is called the *free Cayley graph* of H. We begin our study of free Cayley graphs with a simple observation and with a useful lemma, which is due to Naserasr and Tardif (for a proof, see [1]).

Proposition 6. *Graph $\Delta_M(H)$ contains H as an induced subgraph.*

Proof. Take functions $\{e_v \mid v \in V(H)\} \subseteq V(\Delta_M(H))$. $\square$

Lemma 11. *Let M be a ring, H a Cayley graph on M^k (for some integer k) and G an arbitrary graph. Then any homomorphism $G \xrightarrow{hom} H$ can be (uniquely) extended to a mapping $\Delta_M(G) \to H$ that is both graph and ring homomorphism.*

The following easy lemma appears in [3] (although without explicit mention of graphs Δ_M).

[3]More precisely, we define $\Delta_M(H)$ to be a directed graph. However, if $\overleftrightarrow{H}$ is a symmetric orientation of an undirected graph H, then $\Delta_M(\overleftrightarrow{H})$ is a symmetric orientation of some undirected graph H', we may let $\Delta_M(H) = H'$. The whole Section 5.1 may be modified for undirected graphs by similar changes.

Lemma 12. *$G \xrightarrow{TT_M} H$ is equivalent with $G \xrightarrow{hom} \Delta_M(H)$.*

Note that Lemma 8 is a special case of Lemma 12, as graphs $\Delta(G)$ defined in Section 3 are isomorphic to $\Delta_{\mathbb{Z}_2}(G)$. Lemmas 11 and 12 have as immediate corollary an embedding result that nicely complements Theorem 4. In contrary with Theorem 4 though, our embedding is not functorial, it is just embedding of quasiorder $(\mathcal{G}, \preccurlyeq_M)$ in $(\mathcal{G}, \preccurlyeq_h)$.

Corollary 6. *$G \xrightarrow{TT_M} H$ is equivalent with $\Delta_M(G) \xrightarrow{hom} \Delta_M(H)$.*

Proof. If $G \xrightarrow{TT_M} H$ then by Lemma 12 we have $G \xrightarrow{hom} \Delta_M(H)$ and by Lemma 11 the result follows. For the other implication, by Proposition 6 graph G maps homomorphically to $\Delta_M(H)$, and application of Lemma 12 yields $G \xrightarrow{TT_M} H$. □

We remark that Corollary 6 provides an embedding of category of TT_M mappings to category of Cayley graphs with mappings that are both ring and graphs homomorphisms.

5.2 Right homotens graphs

We start with two simple observations concerning right homotens graphs. The first one is a characterization of right homotens graphs by means of Δ_M. It does not, however, give an efficient method (polynomial algorithm) to verify if a given graph is right homotens, neither a good understanding of right homotens graphs. Hence, we will seek better characterizations (compare with Corollary 7 and Question 3).

Proposition 7. *A graph H is right M-homotens if and only if $\Delta_M(H) \xrightarrow{hom} H$.*

Proof. For the 'only if' part it is enough to observe that $\Delta_M(H) \xrightarrow{TT_M} H$ for every graph H: clearly $\Delta_M(H) \xrightarrow{hom} \Delta_M(H)$ and we use Lemma 12. For the other direction, if $G \xrightarrow{TT_M} H$ then by Lemma 12 we have $G \xrightarrow{hom} \Delta_M(H)$ and by composition (Lemma 1) we have $G \xrightarrow{hom} H$. □

Lemma 13. *Assume $H \xrightarrow{hom} H'$ and $H' \xrightarrow{TT_M} H$. If H is right M-homotens then H' is right M-homotens as well.*

Proof. If H is right M-homotens, then $\Delta_M(H) \xrightarrow{hom} H$. By Corollary 6 from $H' \xrightarrow{TT_M} H$ we deduce that $\Delta_M(H') \xrightarrow{hom} \Delta_M(H)$. By composition,

$$\Delta_M(H') \xrightarrow{hom} \Delta_M(H) \xrightarrow{hom} H \xrightarrow{hom} H',$$

hence H' is right M-homotens. □

Corollary 7. *Let H, H' be homomorphically equivalent graphs (that is $H \xrightarrow{hom} H'$ and $H' \xrightarrow{hom} H$). Then H is right M-homotens if and only if H' is right M-homotens.*

Note that TT_M-equivalence is not sufficient in Corollary 7: each graph H is TT_M-equivalent with $\Delta_M(H)$ and the latter is always a right M-homotens graph (for each M), as we will see from the next proposition. Also note that the analogy of Corollary 7 does not hold for left homotens graphs.

Next, we consider a class of right M-homotens graphs that is central to this topic. We will say that H is an *M-graph* if it is a Cayley graph on some power of M ($\mathbb{Z}_2$-graphs are also called cube-like graphs; they have been introduced by Lovász [7] as an example of graphs, for which every eigenvalue is an integer).

Proposition 8. *Any M-graph is right M-homotens.*

Proof. Let H be an M-graph. As $H \xrightarrow{hom} H$, by Lemma 11 we conclude that $\Delta_M(H) \xrightarrow{hom} H$. □

In analogy with the chromatic number we define the TT_M number $\chi_{TT_M}(G)$ to be the minimum n for which there is a graph H with n vertices such that $G \xrightarrow{TT_M} H$. As any homomorphism induces a TT_M mapping, we see that $\chi_{TT_M}(G) \le \chi(G)$ for every graph G. Continuing our project of finding similarities between TT_M mappings and homomorphisms, we prove that for finite M the TT_M number cannot be much smaller than the chromatic number.

Corollary 8. *Let G be arbitrary graph. If M is a finite ring of characteristic p then $\chi(G)/\chi_{TT_M}(G) < p$.*

Moreover, $\chi(G)/\chi_{TT_\mathbb{Z}}(G) < 2$.

Proof. First we prove that $\chi(G) < m \cdot \chi_{TT_M}(G)$ for any finite ring M of size m. To this end, consider a Cayley graph on M^k with the generating set $M^k \setminus \{\vec{0}\}$—that is a complete graph K_{m^k} with every edge in both orientations. This is an M-graph, hence by Proposition 8 it is right M-homotens.

Now, choose k so that $m^{k-1} < \chi_{TT_M}(G) \leq m^k$. It follows that $G \xrightarrow{TT_M} K_{m^k}$, and as K_{m^k} is right M-homotens, $G \xrightarrow{hom} K_{m^k}$. Therefore, $\chi(G) \leq m^k < m \cdot \chi_{TT_M}(G)$.

Next, if p is the characteristic of M, this means that M contains $\mathbb{Z}_p$ as a subring. This by Lemma 16 implies that any TT_M mapping is $TT_{\mathbb{Z}_p}$, thus $\chi_{TT_M}(G) \geq \chi_{TT_{\mathbb{Z}_p}}(G)$, and the result follows. For the second part we use Lemma 16 again to infer that any $TT_{\mathbb{Z}}$ mapping is $TT_{\mathbb{Z}_2}$. □

How good is the bound given by Corollary 8 is an interesting and difficult question. Even in the simplest case $M = \mathbb{Z}_2$ this is widely open; perhaps surprisingly this is related with the quest for optimal error correcting codes. For details, see [19, 18] Another corollary of Proposition 8 is a characterization of right homotens graphs.

Corollary 9. *A graph is right M-homotens if and only if it is homomorphically equivalent to an M-graph.*

Proof. The 'if' part follows from Lemma 7 and Statement 8. For the 'only if' part, notice that $\Delta_M(H)$ is a M-graph, $H \subseteq \Delta_M(H)$, and if H is right M-homotens then $\Delta_M(H) \xrightarrow{hom} H$. □

Corollary 9 is not very satisfactory, as it does not provide any useful algorithm to verify if a given graph is right homotens. Indeed, it is more a characterization of graphs that are hom-equivalent to some M-graph, than the other way around: Suppose we are to test if a given graph is hom-equivalent to some (arbitrarily large) M-graph. It is not obvious if there is a finite process that decides this; however Corollary 9 reduces this task to decide if $\Delta_M(H) \xrightarrow{hom} H$. The latter condition is easily checked by an obvious brute-force algorithm.

We hope that a more helpful characterization of right homotens graphs will result from considering the core of a given graph. As a core of a graph H is hom-equivalent with H, it is right homotens if and only if H is. Therefore, we attempt to characterize right homotens cores, leading to an easy proposition and an adventurous question. We note that one part of the proof of the proposition is basically the folklore fact that the core of a vertex-transitive graph is vertex-transitive, while the other part is a generalization of an argument used by [6] to prove that K_n is right $\mathbb{Z}_2$-homotens if and only if n is a power of 2. However, we include the proof for the sake of completeness.

Proposition 9. *Let H be a right M-homotens graph that is a core. Then*

- $|V(H)|$ *is a power of* $|M|$*, and*

- H *is vertex transitive. If* $M = \mathbb{Z}_2$*, then for every two vertices of* H*, there is an automorphism exchanging them.*

Proof. For a function $g \in M^{V(H)}$ we let H_g denote the subgraph of $\Delta_M(H)$ induced by the vertex set $\{g + e_v; v \in V(H)\}$. Observe that each H_g is isomorphic with H. Let $f : \Delta_M(H) \to H$ be a homomorphism and for each $u \in V(H)$, define $V_u = \{v \in V(\Delta_M(H)); f(v) = u\}$. Now f restricted to H_g is a homomorphism from H_g to H. As H is a core, every homomorphism from H to H is a bijection. Consequently, for every g the graph H_g contains precisely one vertex from each V_u. By considering all graphs H_g we see that all sets V_u are of the same size $|M|^{|V(H)|}/|V(H)|$. Therefore, $|V(H)|$ is a power of $|M|$.

For the second part let u, v be distinct vertices of H. We know $\Delta_M(H) \xrightarrow{hom} H$. As $H \simeq H_{\vec{0}}$ ($\vec{0}$ being the identical zero), we have a homomorphism $f : \Delta_M(H) \xrightarrow{hom} H_{\vec{0}}$. As H is a core, we know that f restricted to $H_{\vec{0}}$ is an automorphism of $H_{\vec{0}}$. By composition with the inverse automorphism, we may suppose that f restricted to $H_{\vec{0}}$ is an identity. Next, consider the isomorphism $\varphi : \Delta_M(H) \xrightarrow{hom} \Delta_M(H)$ given by $g \mapsto g + e_v - e_u$. A composed mapping $f \circ \varphi$ is a homomorphism $H_{\vec{0}} \xrightarrow{hom} H_{\vec{0}}$ (therefore an automorphism) that maps u to v. Moreover, if $M = \mathbb{Z}_2$ then $f \circ \varphi$ maps v to u as well. $\square$

The previous proposition suggests that a stronger result might be true, and that this may be a way to a characterization of right homotens graphs. In particular, we ask the following.

Question 3.

1. Suppose H is a right M-homotens graph and a core. Is H an M-graph?

2. Is the core of each M-graph an M-graph?

We note that even the (perhaps easier to understand) case $M = \mathbb{Z}_2$ is open. But one can see easily that 1 and 2 in Question 3 are equivalent: If H is a right M-homotens core, then H is the core of the M-graph $\Delta(H)$; hence 2 implies 1. Conversely, let K be an M-graph and H its core. By Proposition 8, K is right M-homotens, therefore by Corollary 7 H is right M-homotens. If 1 is true, then H is an M-graph, as claimed.

6 Density

In this section we compare homomorphisms and tension-continuous mappings from a different perspective: we prove that partial orders defined by existence of a homomorphism (a TT_M mapping respectively) share an important property, namely the *density*. To recall, we say that a partial order $<$ is dense, if for every A, B satisfying $A < B$ there is an element C for which $A < C < B$.

It is known [8, 16] that the homomorphism order (with all hom-equivalence classes of finite graphs as elements and with the relation $\prec_h$) is dense, if we do not consider graphs without edges. The parallel result for the order defined by TT_M mappings is given by the following theorem. In fact we prove a stronger property (proved in [8] for homomorphisms) that every finite antichain in a given interval can be extended; density is the special case $t = 0$.

Theorem 6. *Let M be a ring, let $t \geq 0$ be an integer. Let G, H be graphs such that $G \prec_M H$ and $E(G) \neq \emptyset$. Let G_1, G_2, ..., G_t be pairwise incomparable (in $\prec_M$) graphs satisfying $G \prec_M G_i \prec_M H$ for every i. Then there is a graph K such that*

1. *$G \prec_M K \prec_M H$,*
2. *K and G_i are TT_M-incomparable for every $i = 1, \ldots, t$.*

If in addition $G \preccurlyeq_h H$ then we have even $G \prec_h K \prec_h H$. If we consider undirected graphs, then we get undirected graph K.

This theorem was proved in a previous paper [19] by the authors, here we present a much shorter proof. The key of the proof is the use of graphs $\Delta_M(G)$ for a new proof of Lemma 14. From this, Theorem 6 follows directly.

Proof (Theorem 6—sketch). We use the next lemma for graphs G, G_1, ..., G_t. and we let G' be the graph, that this lemma ensures. Put $K = G + G'$. For details, see [19]. $\square$

Lemma 14 (Sparse incomparability lemma for TT_M)**.** *Let M be an abelian group (not necessarily a finitely generated one), let l, $t \geq 1$ be integers. Let G_1, G_2, ..., G_t, H be (finite directed non-empty[4]) graphs such that $H \overset{TT_M}{\not\longrightarrow} G_i$ for every i. Then there is a graph G such that*

[4] that is with non-empty edge set

1. *$g(G) > l$ (that is G contains no circuit of size at most l),*

2. *$G \prec_h H$,*

3. *$G \overset{TT_M}{\not\longrightarrow} G_i$ for every $i = 1, \dots, t$.*

(For undirected graphs we get undirected graph G.)

In the proof we will use a variant of Sparse incomparability lemma for homomorphisms in the following form (it has been proved for undirected graphs in [17], the version we present here follows by the same proof).

Lemma 15 (Sparse incomparability lemma for homomorphisms)**.** *Let l, $t \geq 1$ be integers, let H, G_1, …, G_t be (finite directed non-empty) graphs such that $H \overset{hom}{\not\longrightarrow} G_i$ for every i. Let c be an integer. Then there is a (directed) graph G such that*

- *$g(G) > l$ (that is G contains no circuit of size at most l),*
- *$G \prec_h H$, and*
- *$G \overset{hom}{\not\longrightarrow} G_i$ for every i.*

(For undirected graphs we get undirected graph G.)

Before we start the proof, we summarize necessary results about influence of ring M on the existence of TT_M mappings. The following summarizes results that appear as Theorem 4.4 in [3], and as Lemma 14 and 17 in [19].

Lemma 16. *Let G, H be graphs, $f : E(G) \to E(H)$ any mapping.*

1. *If f is $TT_{\mathbb{Z}}$ then it is TT_M for any group M.*

2. *Let M be a subring of N. If f is TT_N then it is TT_M.*

3. *Let G, H be* finite *graphs. Then $G \preccurlyeq_n H$ holds either for finitely many n or for every n. In the latter case $G \preccurlyeq_{\mathbb{Z}} H$ holds.*

Proof (Lemma 14). First, suppose that M is a finite ring; by Lemma 12 we know that $H \overset{hom}{\not\longrightarrow} \Delta_M(G_i)$ for every i. Therefore, we may use Lemma 15 to obtain G' of girth greater than l such that $G' \preccurlyeq_h H$ and $G' \not\preccurlyeq_h \Delta_M(G_i)$. Consequently $G' \overset{TT}{\not\longrightarrow}_M G_i$ for every i.

Next, let M be an infinite, finitely generated group, that is a ring. Then $M \simeq \mathbb{Z}^\alpha \times \prod_{i=1}^k \mathbb{Z}_{n_i}^{\beta_i}$, for some integers k, n_i, β_i, α. As M is infinite, we have $\alpha > 0$, therefore $M \geq \mathbb{Z}$. By Lemma 16 we conclude that for any mapping it is equivalent to be TT_M and to be $TT_\mathbb{Z}$, hence we may suppose $M = \mathbb{Z}$. By Lemma 16, there is only finitely many integers n for which holds $H \xrightarrow{TT_n} G_i$ for some i or $H \xrightarrow{TT_n} \overrightarrow{K}_2$. Pick some n for which neither of this holds. By the previous paragraph for ring $\mathbb{Z}_n$ we find a graph G' such that $G' \not\xrightarrow{TT_n} G_i$ for every $i = 1, \dots, t$. It follows from Lemma 16 that also $G' \not\xrightarrow{TT_M} G_i$.

Finally, let M be a general abelian group. For each mapping $f : E(H) \to X$ (where $X \in \{G_1, \dots, G_t\}$) there is an M-tension φ_X on X which certifies that f is not a TT_M mapping. Let $A = \big\{\varphi_X(e) \mid e \in E(X), X \in \{G_1, \dots, G_t\}\big\}$ be the set of all elements of M that are used for these certificates. Let M' be the subgroup of M generated by A; by the choice of A we have $H \not\xrightarrow{TT_{M'}} G_i$. By the previous paragraph there is a graph G' that meets conditions 1, 2, and $G' \not\xrightarrow{TT_{M'}} G_i$ for every i. Consequently, $G' \not\xrightarrow{TT_M} G_i$ for every i, which concludes the proof. □

Let us add a remark that partially explains the way we conducted the above density proof. Standard proofs of density of the homomorphism order rely on the fact, that the category of graphs and homomorphisms has products. We prove next, that this is not true for TT_M mappings; therefore another approach is needed. In [19] we developed a new structural Ramsey-type theorem to overcome the non-existence of products; here we used the construction Δ_M for much shorter proof.

Proposition 10. *Category $\mathcal{G}_{TT_M}$ of (directed or undirected) graphs and TT_M mappings does not have products for any ring M.*

Proof. We will formulate the proof for the undirected version, although for the directed version the same proof goes through. We show that there is no product $C_3 \times C_3$. Suppose, to the contrary, that P is the product $C_3 \times C_3$. Let π_1, π_2 : $P \xrightarrow{TT} C_3$ be the projections, let $E(C_3) = \{e_1, e_2, e_3\}$.

We look first at mappings $f_i : \overrightarrow{K}_2 \to C_3$ sending the only edge of $\overrightarrow{K}_2$ to e_i. If we consider mapping f_i to the first copy of C_3 and f_j to the second one, by definition of the product there is exactly one edge $e \in E(P)$ such that $\pi_1(e) = e_i$ and $\pi_2(e) = e_j$. We let $e_{i,j}$ denote this e. So, $E(P)$ consists of nine edges $e_{i,j}$, for $1 \leq i, j \leq 3$.

As π_1, π_2 are TT mappings, by Lemma 6 there are no loops in P. There are no parallel edges either: suppose e, f are parallel edges in P. Then without loss of generality $\pi_1(e) \neq \pi_1(f)$, hence we get a contradiction by Lemma 6.

Finally, for a $\rho \in S_3$ let $f_\rho : C_3 \to C_3$ send e_i to $e_{\rho(i)}$. Using the definition of product for mapping f_{id} and f_ρ, Lemma 6, and the fact that there are no parallel edges in P we find that $E_\rho = \{e_{1,\rho(1)}, e_{2,\rho(2)}, e_{3,\rho(3)}\}$ are edges of a cycle. Considering $\rho = \mathrm{id}$ and $\rho = (1, 3, 2)$ we find that part of P looks as in the Figure 5 (in the directed case, the orientation may be arbitrary, if $M = \mathbb{Z}_2^k$).

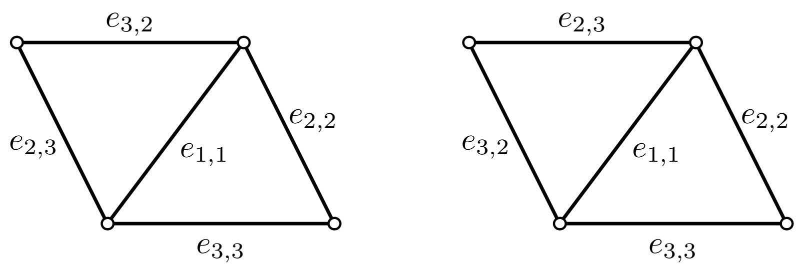


Figure 5: Proof of Proposition 10.

Consider the first case. As E_ρ is a cycle for $\rho = (2, 3, 1)$, the edges $e_{1,2}$ and $e_{2,3}$ are adjacent. By taking $\rho = (2, 1, 3)$, we find that $e_{1,2}$ and $e_{2,3}$ are adjacent. As there are no parallel edges in P, we have $e_{1,2} = xy$ or $e_{1,2} = yx$. Hence, $e_{1,2}$, $e_{2,3}$, $e_{2,2}$ forms a cycle. As π_1 is TT mapping, we obtain a contradiction by Lemma 6. In the second case we proceed in the same way with edge $e_{2,1}$, we prove that it is adjacent with $e_{3,2}$ and $e_{3,3}$ and yield a contradiction with π_2 being a TT mapping. $\square$

7 Remarks

7.1 Broader context (Jaeger's project)

Tension-continuous mappings were defined in [3, 19] in a broader context of three related types of mappings: FF (lifts flows to flows), FT (lifts tensions to flows), and TF (lifts flows to tensions). In [3, 18] these mappings are studied in more detail, in particular their connections to several classical conjectures (Cycle Double Cover conjecture, Tutte's 5-flow conjecture, and Berge-Fulkerson matching conjecture) are explained.

The universality and density of TT mappings shows that the Jaeger's project of characterizing "atoms" of a partial order defined by flow-continuous mappings has no dual analogue (for TT mappings). It follows from Theorem 6 that each of the quasiorders $\preccurlyeq_M$ is everywhere dense for the class of directed graphs. Graphs $\overrightarrow{K}_2$ and the loop graph are the minimal and the maximal element of these orders.

Particularly, there cannot be any atom (the contrary is conjectured for the flow-continuous order in [3, 9]). This is also in sharp contrast with the homomorphism order of oriented graphs where the homomorphism order $\preccurlyeq_h$ contains many gaps of a complicated structure. (These gaps are characterized by [16].) Another consequence of Theorem 6 is that each of the orders $\preccurlyeq_M$ contains an infinite antichain, a property which is presently open for M-flow-continuous mappings for every M, in particular for cycle-continuous mappings; see [3].

7.2 TT-perfect graphs

For every graph G, its chromatic number $\chi(G)$ is at least as big as the size of its largest clique, $\omega(G)$. Recall, that a graph G is called *perfect* if $\chi(G') = \omega(G')$ holds for every induced subgraph G' of G. A graph is called *Berge* if for no odd $l \geq 5$ does G contain C_l or $\overline{C}_l$ as an induced subgraph. It is easy to see that being perfect implies being Berge; the so-called Strong Perfect Graph Conjecture (due to Claude Berge) claims that the opposite is true, too. Perfect graphs have been a topic of intensive research that recently lead to a proof [2] of the Strong Perfect Graph Conjecture.

As a humble parallel to this development we define a graph G to be TT-*perfect*[5] if for every induced subgraph G' of G we have $\chi_{TT_2}(G') \leq \omega(G')$ (definition of $\chi_{TT_2}(G')$ appears before Corollary 8). Equivalently, G is TT-perfect if each of its induced subgraphs G' admits a TT_2 mapping to its maximal clique.

Note that we cannot ask for $\chi_{TT}(G') = \omega(G')$ since $K_4 \xrightarrow{TT} K_3$, and therefore $\chi_{TT}(K_4) = 3$, while $\omega_{TT}(K_4) = 4$.

As any homomorphism induces a TT mapping (see Lemma 7), $\chi_{TT}(G') \leq \chi(G')$ holds for every graph G'. Consequently, every perfect graph is TT perfect. The converse, however, is false. For example, let $G = \overline{C}_7$. Graph G itself is not perfect. On the other hand $\chi_{TT}(G) = 3$ and every induced subgraph of G is Berge, hence perfect, hence TT-perfect. Let us study TT-perfect graphs in a similar manner as Strong Perfect Graph Theorem does for perfect graphs. To this end, we define a graph G to be *critical* if G is not TT-perfect, but each induced subgraph of G is. We start our approach by a technical lemma.

Lemma 17. *Let $l \geq 3$ be odd. Cycle C_l is not TT-perfect. Graph $\overline{C}_l$ is TT-perfect if and only if $l = 7$.*

[5] more precisely, TT_2-perfect, but we will not consider $M \neq \mathbb{Z}_2$ in this section

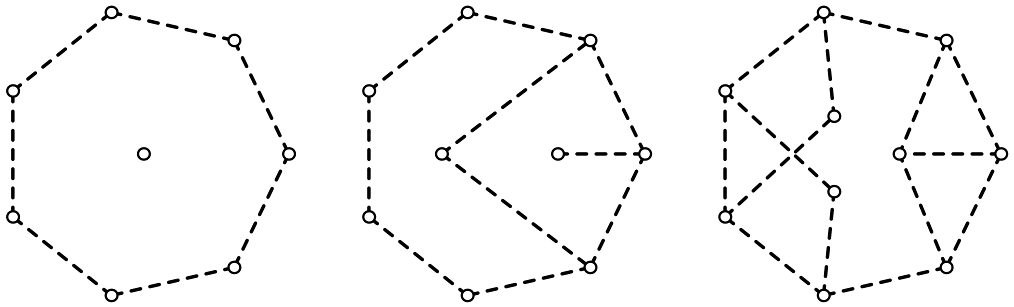


Figure 6: Several critical graphs that are not cycles neither complements of cycles. The dashed lines denote precisely the **non-edges** of the graph.

Proof. Clearly $\chi_{TT}(C_l) = 3 > \omega(C_l)$. Graph $\overline{C}_7$ was discussed above, $\overline{C}_5$ is isomorphic to C_5. As $\chi(\overline{C}_9) = 5$ and as K_4 is right $\mathbb{Z}_2$-homotens, being a $\mathbb{Z}_2$-graph, we have $\chi_{TT}(\overline{C}_9) = 5 > \omega(\overline{C}_9)$. It is easy to verify that graphs $\overline{C}_l$ for $l \geq 13$ are nice. Thus they are homotens and not TT-perfect, since they are not perfect. The only remaining case is the graph $\overline{C}_{11}$. This is not nice, on the other hand, every edge is contained it a K_5 and all K_5's are 'connected'—there is a chain of all 11 copies of K_5 such that neighboring copies intersect in a K_4. It follows that $\overline{C}_{11}$ is homotens, in particular $\overline{C}_{11} \stackrel{TT}{\not\longrightarrow} K_5$. □

Corollary 10. *For every odd $l > 3$ graph C_l is critical; if $l \neq 7$ then $\overline{C}_l$ is critical, too. Moreover graphs G_1, G_2, and G_3 in Figure 6 are critical.*

Proof. We sketch the proof of G_1 being critical. We have $\chi(G_1) = 1+\chi(\overline{C}_7) = 5$, therefore Corollary 8 implies $\chi_{TT}(G_1) = 5 > \omega(G_1)$ and G_1 is not TT-perfect. Let G' be an induced subgraph of G_1. If $G' = \overline{C}_7$ then G' is TT-perfect; otherwise, it is a routine to verify that G' is Berge, consequently perfect and TT-perfect. □

We do not know how many other critical graphs there are, not even if there is an infinite number of them.

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