Minors of simplicial complexes

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Abstract

We extend the notion of a minor from matroids to simplicial complexes. We show that the class of matroids, as well as the class of independence complexes, is characterized by a single forbidden minor. Inspired by a recent result of Aharoni and Berger, we investigate possible ways to extend the matroid intersection theorem to simplicial complexes.

1 Introduction

The concept of a minor plays a fundamental role in matroid theory. In this paper, we introduce minors in the more general context of simplicial complexes. The definition is both topologically natural and extends the matroid-theoretic definition. The latter is a substantial difference from the case of hypergraph minors (described, e.g., in [4, Section 2.2]).

Interestingly, it turns out that matroids (as a subclass of the class of simplicial complexes) can be characterized by a simple forbidden minor. The same holds for another natural class of simplicial complexes, the independence (or flag) complexes.

Are there any significant results in matroid theory that generalize to other classes of simplicial complexes without forbidden minors? One result we consider is the celebrated matroid intersection theorem [5], dealing with the rank of the intersection of two matroids. Its recent generalization that applies to a matroid and a general simplicial complex [2] suggests the question whether a further generalization to two arbitrary simplicial complexes is possible. Although this is not the case (see Section 6), we show that the result does hold true for other classes of simplicial complexes characterized by forbidden minors besides the class of matroids.

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2 Complexes

A *simplicial complex* (or just *complex*) K is a set system on a certain *ground set* such that if $B \in K$ and $A \subset B$, then $A \in K$. In the present paper, we are interested in *finite* complexes (i.e., those on a finite ground set), and we allow $A = \emptyset$ in the above definition. We recall the basic facts and definitions concerning simplicial complexes, referring the reader to [3] for more background.

The sets in a complex K are referred to as its *faces*; we use lowercase Greek letters to denote them. The *dimension* $\dim \sigma$ of a face σ is $|\sigma| - 1$, and the dimension of a complex K is the maximum dimension of a face in K. (Note that the dimension of a non-empty complex is at least -1; we define the dimension of the empty one to be -2.) A *facet* of K is any inclusionwise maximal face of K. The *d-skeleton* of K is the complex consisting of all the faces of K of dimension at most d. The d-skeleton of K is *complete* if every set $X \subseteq V(K)$ of size $d+1$ is a face of K.

Faces of dimension 0 are called *vertices*. We let $V(K)$ denote the set of vertices of K. In general, the ground set of K may contain elements that are not vertices of K. These are inessential for our purposes, and we consider complexes differing only in such elements as identical. Apart from this technical point, the notion of *isomorphism* of complexes is defined in the obvious way. If x is a vertex of a face σ of K, we write $\sigma \setminus x$ for $\sigma \setminus \{x\}$.

Complexes have a well-known topological interpretation: for each (finite) complex K, there is an associated topological space $\|K\|$ called the *space* of K. This is defined as follows. Assign to each vertex x of K a point $p(x)$ in $\mathbb{R}^n$, where $n = |V(K)|$, in such a way that the set $\{p(x) : x \in V(K)\}$ is affinely independent. For each non-empty $A \in K$, let $p(A)$ be the simplex whose vertices are the points $p(x)$ with $x \in A$. The space $\|K\|$ is defined as the union of all the simplices $p(A)$ such that $\emptyset \neq A \in K$, topologized as a subspace of $\mathbb{R}^n$.

An important concept in combinatorial topology is connectivity. A topological space X is *k-connected* ($k \geq 0$) whenever for each $d \leq k$, every continous mapping from a d-dimensional sphere S^d to X can be extended to a continuous mapping from the $(d+1)$-dimensional ball B^{d+1} to X. The *connectivity* of X, $\operatorname{conn}(X)$, is the maximum k such that X is k-connected. For a complex K, we define

$$\eta(K) = \operatorname{conn}(\|K\|) + 2$$

(the addition of the constant term makes the parameter more convenient to work with). If $\|K\|$ is k-connected for all k, we set $\eta(K) = \infty$. Following [2], we define $\bar{\eta}(K) = \min(\eta(K), \dim(K) + 1)$.

The *join* $K * L$ of two complexes K, L is defined as

$$K * L = \{\sigma \cup \tau : \sigma \in K \text{ and } \tau \in L\}.$$

The connectivity of a join is easily computed [2, Lemma 2.1]:

Lemma 1. *For complexes K and L,*

$$\eta(K * L) = \eta(K) + \eta(L).$$

3 Minors

Let K be a complex on a ground set V. The *induced subcomplex* $K|X$ on a set $X \subseteq V$ consists of all the faces of K contained in X. The *link* $\mathrm{lk}(\sigma)$ of a face $\sigma \in K$ is the complex

$$\mathrm{lk}(\sigma) = \{\tau : \sigma \cap \tau = \emptyset \text{ and } \sigma \cup \tau \in K\}.$$

The independent sets of any matroid form a complex. (See, e.g., [6, 8] for an introduction to matroid theory.) In fact, one may define a *matroid* to be a complex K such that for each $X \subseteq V$, all the maximal faces of $K|X$ have the same dimension (called the *rank* $\mathrm{rank}_K(X)$ of X in K). This definition, used, e.g., in [2], is equivalent to any of the standard definitions of matroids. Note that the independent sets of a matroid M are the faces of M viewed as a complex.

In matroid theory, the operations of contraction and deletion and the associated notion of a minor, play an important role. We now extend them to arbitrary complexes. If K is a complex and $\sigma \in K$, define

$$K \setminus \sigma = K|(V(K) \setminus \sigma),$$
$$K / \sigma = \mathrm{lk}(\sigma).$$

The first operation is the *deletion* of σ from K, the second one is the *contraction* of σ in K. A complex L is said to be a *minor* of K if L can be obtained from K by a succession of contractions and deletions of faces.

It is easy to see that if K is a matroid, then the above definitions of contraction and deletion of an independent set σ coincide with the matroid-theoretical contraction and deletion (which is essentially observed in [2, Remark 3.1]). Since any matroidal minor of K can be obtained by a series of contractions and deletions of independent sets, it follows that the above definition of a minor specializes to the matroidal one.

Observation 2. *A matroid L is a minor of a matroid K in the sense of the above definition if and only if L is a minor of K in the usual matroidal sense. In particular, any minor of a matroid is a matroid.*

Contraction and deletion in matroids are dual operations. This does not seem to be the case for complexes, at least for the straightforward extension of the matroidal duality.

4 Matroids

We now show that the class of matroids, as a subclass of the class of complexes, is characterized by a single forbidden minor. If F is a complex, then we say that another complex K *has no F-minor* if K contains no minor isomorphic to F.

Let T_1 be the complex on $\{1, 2, 3\}$ whose facets are $\{1, 2\}$ and $\{3\}$ (see Figure 1a).

Theorem 3. *A complex K is a matroid if and only if it does not contain a T_1-minor.*

Proof. If K is a matroid, then by Observation 2, any minor of K is a matroid. Since T_1 is clearly not a matroid, it cannot be contained in K as a minor.

Conversely, assume that K is not a matroid. Suppose that K contains no T_1-minor and choose K such that $|V(K)|$ is as small as possible. Let σ and τ be two facets of K such that $\dim \sigma < \dim \tau$. Note that $K\,/(\sigma \cap \tau)$ is not a matroid, since $\sigma \setminus \tau$ and $\tau \setminus \sigma$ are facets of $K\,/(\sigma \cup \tau)$ of different dimensions. Since K contains a T_1 minor whenever $K\,/(\sigma \cup \tau)$ does, the minimality of K implies that the faces σ and τ are disjoint.

Let $x \in \sigma$. By the minimality of K, the complex $K \setminus \{x\}$ is a matroid. It follows that $\sigma \setminus x$ is not a facet of $K \setminus \{x\}$, for

$$\dim{(\sigma \setminus x)} < \dim \sigma < \dim \tau \tag{1}$$

and τ is a facet. Thus, there is a face $\sigma' \supset \sigma \setminus x$ of $K \setminus \{x\}$ such that $\dim \sigma' = \dim \tau$. By (1),

$$\dim{(\sigma' \cap \tau)} \geq 1. \tag{2}$$

Consider now the complex $K' = K\,/(\sigma \setminus x)$. It contains x as an isolated vertex (i.e., $\{x\}$ is a facet) since σ is a facet of K. Furthermore, $\sigma' \cap \tau$ is a face of K' of dimension at least 1. It follows that the induced subcomplex of K' on a 3-vertex set X containing x and two vertices from $\sigma' \cap \tau$ is T_1. Since $K'|X$ is just $K' \setminus (V(K') \setminus X)$, T_1 is a minor of K. □

Note that by Observation 2, the characterization from Theorem 3 combines well with classical forbidden minor characterizations of various classes of matroids. For instance, Tutte's characterization of binary matroids [7] implies that a complex is a binary matroid if and only if it contains no minor from the set $\{T_1, U_{2,4}\}$, where $U_{2,4}$ is the uniform matroid of rank 2 on 4 elements.

5 Other classes characterized by forbidden minors

We turn to another naturally defined class of complexes. The *independence complex* $I(G)$ of a graph G is the complex whose ground set is the vertex set of G

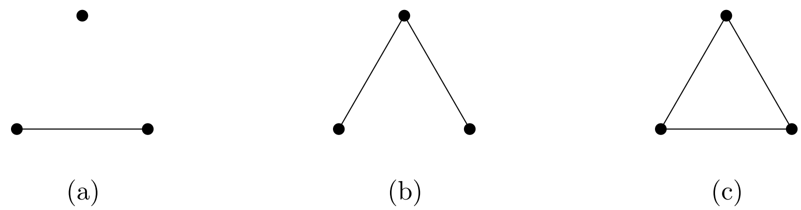

Figure 1: The complexes (a) T_1, (b) T_2 and (c) the uniform matroid $U_{2,3}$ (pictured as a complex).

and whose faces are all the independent sets in G. It turns out that the class of independence complexes can also be characterized by a single forbidden minor, namely the uniform matroid $U_{2,3}$ shown in Figure 1c:

Lemma 4. *Let K be a complex containing no $U_{2,3}$-minor. A set $\sigma \subseteq V(K)$ is a face of K if and only if each pair of vertices in σ forms a face in K.*

Proof. The 'only if' part is clear. Assume thus that each pair of vertices in σ is a face of K, but σ itself is not, and that σ is minimal with respect to this property. Thus, for each $x \in \sigma$, $\sigma \setminus x$ is a face of K, and by the assumption, $\dim \sigma \geq 2$. Let $\tau \subset \sigma$ be a set of dimension $\dim \sigma - 3$. Clearly, the contraction in K of τ is $U_{2,3}$. Consequently, $U_{2,3}$ is a minor of K. □

Theorem 5. *A complex K contains no $U_{2,3}$-minor if and only if there is a graph G such that $K = I(G)$.*

Proof. Assume that K contains no $U_{2,3}$-minor. Let G be the graph whose vertices are all the vertices of K, such that the vertices v, w form an edge in G if and only if $\{v, w\}$ is not a face of K. By Lemma 4, a set $\sigma \subseteq V(G)$ is a face of K if and only if each pair of vertices in σ is non-adjacent, which is the case if and only if σ is a face of $I(G)$. Thus, $K = I(G)$.

For the converse, let x be a vertex of a graph G. Note that

$$I(G) \setminus x = I(G - x),$$
$$I(G) / x = I(G \setminus N_G[x]),$$

where $N_G[x]$ is the closed neighborhood of x in G. It follows that every minor of an independence complex is an independence complex again. Since $U_{2,3}$ is clearly not an independence complex, the statement follows. □

We have seen that for $K \in \{T_1, U_{2,3}\}$, there is a natural characterization of the complexes with no K-minor. We now derive a similar characterization for $K = T_2$, the complex in Figure 1b.

Let $f : 2^V \to \mathbb{N}$ be a function assigning a nonnegative integer to each subset of a set V. Let us call f *decreasing* if for each pair of subsets $X \subseteq Y \subseteq V$,

$$f(X) \geq \min\{f(Y), |X|\}. \tag{3}$$

Any decreasing function f determines a complex $K(f)$ on V whose faces are all the sets $\sigma \subseteq V$ with the property that $f(\sigma) \geq |\sigma|$.

A function $f : 2^V \to \mathbb{N}$ is *admissible* if it is decreasing and for each $X, Y \subseteq V$,

$$f(X \cup Y) \geq \min\{f(X), f(Y), |X \cap Y| + 1\}. \tag{4}$$

Every complex with no T_2-minor determines an admissible function:

Lemma 6. *Let K be a complex on a set V containing no T_2-minor. The function $f_K : 2^V \to \mathbb{N}$, defined by*

$$f_K(X) = \max\{d \leq |X| : \textit{the } (d-1)\textit{-skeleton of } K|X \textit{ is complete}\}, \tag{5}$$

is admissible. Furthermore, $K = K(f_K)$.

Proof. Clearly, f_K is decreasing and $K = K(f_K)$. We prove that f_K is admissible. For the sake of contradiction, suppose that $X, Y \subseteq V$ and

$$f_K(X \cup Y) < \min\{f_K(X), f_K(Y), |X \cap Y| + 1\}. \tag{6}$$

Thus, for $d = f_K(X \cup Y) + 1$, there is a set $Z \subseteq X \cup Y$ of size d such that $Z \notin K$. It follows that $Z \not\subseteq X$ and $Z \not\subseteq Y$, and we may choose vertices $x \in X \setminus Y$ and $y \in Y \setminus X$. The intersection $X \cap Y \cap Z$ contains at most $d - 2$ vertices. By (6), $|X \cap Y| \geq d - 1$; we may therefore choose a vertex $z \in (X \cap Y) \setminus Z$. Set $\sigma = Z \setminus \{x, y\}$.

Since $\sigma \cup \{x, z\}$ is a subset of X of size d, it is a face of K. Similarly, $\sigma \cup \{y, z\}$ is a face of K. However, $\sigma \cup \{x, y\} = Z$ is not, so

$$(K|(Z \cup \{z\})) / \sigma \cong T_2.$$

Thus, K has a T_2-minor, a contradiction. □

Theorem 7. *A complex K on a set V has no T_2-minor if and only if there is an admissible function $f : 2^V \to \mathbb{N}$ such that $K = K(f)$.*

Proof. The 'only if' part follows directly from Lemma 6. To prove the 'if' part, let $z \in V(K)$ and define functions $f_1, f_2 : 2^{V \setminus z} \to \mathbb{N}$ by

$$f_1(X) = f(X),$$
$$f_2(X) = f(X \cup \{z\}) - 1$$

and observe that they are admissible. Furthermore, $K(f_1)$ coincides with $K \setminus z$, while $K(f_2)$ is K / z.

It follows that it suffices to prove that T_2 is not of the form $K(g)$ for any admissible g. Assume the contrary. Let the facets of T_2 be $\{a, b\}$ and $\{b, c\}$. By the definition, the value of g on each of $\{a, b\}$ and $\{b, c\}$ is at least 2. By (4), $g(\{a, b, c\}) \geq 2$. Inequality (3) implies that $g(\{a, c\}) \geq 2$, whence $\{a, c\}$ is a face of T_2. This is a contradiction. □

6 Intersection of complexes

The following result of Edmonds[5] is known as the matroid intersection theorem:

Theorem 8. *Let K and L be matroids on a common ground set V. K and L have a common independent set of size n if and only if for each $X \subseteq V$,*

$$\mathrm{rank}_K(X) + \mathrm{rank}_L(V \setminus X) \geq n. \tag{7}$$

Recently, Aharoni and Berger [2] proved an extension of (the nontrivial direction of) this theorem to a situation where one of the matroids is replaced by an arbitrary complex. In view of the following lemma [3], a natural replacement for the rank function is the parameter η or $\bar{\eta}$ defined in Section 2. (Recall that a *coloop* in a matroid is an element contained in every independent set.)

Lemma 9. *Let M be a matroid on a ground set V and $X \subseteq V$. Then*

$$\eta(M|X) = \begin{cases} \infty & \text{if } M \text{ contains a coloop,} \\ \mathrm{rank}_M(X) & \text{otherwise.} \end{cases}$$

In Aharoni and Berger's result, the 'if and only if' type condition of Theorem 8 is replaced by a sufficient condition:

Theorem 10. *Let K be a matroid on a ground set V and let L be a complex on V. If, for each $X \subseteq V$,*

$$\mathrm{rank}_K(X) + \eta(L|(V \setminus X)) \geq n,$$

then K has an n-element independent set belonging to L.

One may ask whether both of the matroids in Theorem 8 can be replaced by arbitrary complexes. It has been observed [1] that the straightforward generalization does not work, but we are not aware of any specific example in the literature.

In generalizing Theorem 8 to complexes, it seems more natural to replace the rank function by the parameter $\bar{\eta}$, rather than η. Consider, for example, complexes K and L, each of which is a simplex of large dimension, such that K and L intersect in one vertex. Then the sum

$$\eta(K|X) + \eta(L|(V \setminus X))$$

is infinite for each X, but the dimension of $K \cap L$ is 0.

The extension of Theorem 8 involving $\bar{\eta}$ also fails to work in general, but for less trivial reasons:

Theorem 11. *For any $n \geq 1$, there are complexes K and L on a set V of $5n$ vertices such that for each $X \subseteq V$,*

$$\bar{\eta}(K|X) + \bar{\eta}(L|(V \setminus X)) \geq 2n, \tag{8}$$

but $\dim(K \cap L) = n - 1$.

Proof. Let the graphs $G_1, \dots, G_n$ be disjoint copies of the circuit of length 5, and let V be the union of their vertex sets $V_i = V(G_i)$. Define complexes K and L on V by

$$K = I(G_1) * \cdots * I(G_n),$$
$$L = I(\overline{G_1}) * \cdots * I(\overline{G_n}),$$

where $\overline{G_i}$ is the complement of G_i (thus also a circuit of length 5). Note that $\dim(K \cap L) = n - 1$. To establish (8), we show that each G_i contributes at least 2 to the sum on the left-hand side of (8).

If $\eta(K|X)$ is finite, then by Lemma 1,

$$\bar{\eta}(K|X) = \sum_{i=1}^{n} \eta(K|(X \cap V_i)).$$

Note that the contribution of G_i to this sum is at least 1 whenever $X \cap V_i$ is nonempty.

If, on the other hand, $\eta(K|X)$ is infinite, then

$$\bar{\eta}(K|X) = \sum_{i=1}^{n} (\dim(K|(X \cap V_i)) + 1),$$

and again, G_i contributes at least 1 whenever $X \cap V_i$ is nonempty.

By symmetry, the only case where the contribution of G_i to the left hand side of (8) could be less than 2, is when either $X \cap V_i = \emptyset$ or $V_i \subseteq X$. However, since

$$\eta(K|V_i) = \eta(L|V_i) = 2$$

and

$$\dim(K|V_i) + 1 = \dim(L|V_i) + 1 = 2,$$

the contribution is at least 2 in this case as well. □

However, forbidding a minor (other than T_1) may ensure that an analogue of Theorem 8 holds. As a simple illustration, we consider T_2 as a forbidden minor. First, we need a lemma concerning the parameter $\bar{\eta}$.

Lemma 12. *Let K be a complex on a set V containing no T_2-minor. Then*

$$\bar{\eta}(K) = f_K(V(K)),$$

where $f_K : 2^V \to \mathbb{N}$ is defined as in Lemma 6.

Proof. Set $d = f_K(V(K))$. We claim that K is $(d-2)$-connected. It is well known that this is the case if and only if the $(d-1)$-skeleton is $(d-2)$-connected. The $(d-1)$-skeleton of K, being complete, coincides with the $(d-1)$-skeleton of a simplex Σ on $V(K)$. Since Σ is k-connected for all k, the claim follows.

By (5), the dimension of K is at least $d-1$. In combination with the above,

$$\bar{\eta}(K) \geq d.$$

It remains to show that $\bar{\eta}(K) \leq d$. If $\dim K = d-1$, then this holds trivially, so assume that $\dim K \geq d$. Let $X \subset V(K)$ be maximal such that $f_K(X) \geq d+1$. Since, clearly, $X \neq V(K)$, we may choose a vertex $z \notin X$ of K. Let σ be a d-element subset of X.

We claim that for any $x \in \sigma$, the set $(\sigma \setminus x) \cup \{z\}$ is not contained in any face τ with $|\tau| = d+1$. Assume the contrary. By the admissibility of f_K,

$$f_K(X \cup \tau) \geq \min\{f_K(X), f_K(\tau), |X \cap \tau| + 1\} \geq d+1, \tag{9}$$

contradicting the maximality of X.

Let $K' = K|(\sigma \cup \{z\})$. Since the $(d-1)$-skeleton of K' is complete, it consists of all the proper faces of a d-dimensional simplex on $\sigma \cup \{z\}$, and thus the space of K' is homeomorphic with the $(d-1)$-dimensional sphere S^{d-1}. Since, by the above, K' has faces that are not contained in any d-dimensional face of K, this homeomorphism clearly cannot be extended to a mapping from the d-dimensional ball B^d to K. We conclude that K is not $(d-1)$-connected and $\bar{\eta}(K) = d$ as claimed. □

We use Lemma 12 to prove the following stronger analogue of Theorem 8:

Theorem 13. *Let K and L be complexes on a ground set V without a T_2-minor. The dimension of $K \cap L$ is*

$$\dim(K \cap L) + 1 \geq \min_{X \subseteq V} (\bar{\eta}(K|X) + \bar{\eta}(L|(V \setminus X))). \tag{10}$$

In fact, the inequality holds even if the minimum is restricted to $X \in \{V(K), V(L)\}$.

Proof. Set $I = V(K) \cap V(L)$ and note that for $X = V(L)$,

$$\bar{\eta}(K|I) = \bar{\eta}(K|X) + \bar{\eta}(L|(V \setminus X)).$$

An analogous expression for $\bar{\eta}(L|I)$ is obtained by putting $X = V(K)$. Thus, to prove (10), it suffices to prove

$$\dim(K \cap L) + 1 \geq \min\{\bar{\eta}(K|I), \bar{\eta}(L|I)\}. \tag{11}$$

Let d denote the right hand side of (11). As we know from Lemma 12, the $(d-1)$-skeleton of both $K|I$ and $L|I$ is nonempty and complete. Consequently, K and L have a common face of dimension $d-1$. □

It would be interesting to determine other sets of forbidden minors for which an analogue of Theorem 8 is true.

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