MORE ON CONFIGURATIONS IN PRIESTLEY SPACES, AND SOME NEW PROBLEMS

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Dedicated to Bernhard Banaschewski

Abstract. Prohibiting configurations (≡ induced finite connected posets) in Priestley spaces and properties of the associated classes of distributive lattices, and the related problem of configurations in coproducts of Priestley spaces, have been brought to satisfactory conclusions in case of configurations with a unique maximal element. The general case is, however, far from settled. After a short survey of known results we present the desired answers for a large (although still not complete) class of configurations without top.

Introduction

Priestley spaces are ordered compact topological spaces with a specific separation property (see 1.2 below). They can be viewed as a description of the structure of prime ideals of distributive lattices, which is the gist of the famous Priestley duality ([14], [15]; see [10] for an introductory treatment). Thus, by studying the order structure of Priestley spaces, and the appearance or non-appearance of various configurations therein, we are in effect analyzing the inclusion structures possible among the prime ideals. Hence it is not surprising that "geometrical" facts about (order) shapes in the spaces have algebraic

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connotations for the behaviour of distributive lattices. For instance, the flatness of the space, i.e., the non-existence of a chain $x_0 < x_1$, indicates that the corresponding lattice is Boolean; non-existence of a chain $x_0 < x_1 < x_2$ indicates the existence, for any a_0, a_1 in the lattice, of c_0, c_1 such that $a_0 \wedge c_0 = 0$, $a_1 \wedge c_1 \leq a_0 \vee c_0$, $a_1 \vee c_1 = 1$ ([1]; see also 2.1); non-existence of a V-shape indicates that the lattice is relatively normal ([13]).

In general, prohibiting a configuration P in spaces determines a class of distributive lattices, the properties of which have been the subject of intensive study in recent years. In particular it has been shown that if P is acyclic then the corresponding class is axiomatizable while, on the other hand, if P is cyclic with top or bottom, the class is not even first order definable. The situation of the cyclic configurations without top or bottom is more entangled. Here the non-definability has been established only for generic cycles, i.e., diamonds and k-crowns, ([6],[7]), while the general problem has resisted all efforts. The results in this paper are also incomplete: we present the solution for the class of all configurations containing diamonds, and for a large class of other cyclic configurations as well, but not for all configurations.

Another problem discussed is that of the order stucture of coproducts of Priestley spaces. By duality, since distributive lattices constitute a complete category, the category of Priestley spaces is cocomplete. In particular, there are coproducts of arbitrarily large collections. For infinite collections, the compacness of the coproduct forces the augmentation of the disjoint union by additional points; the question of the order structure of the remainder is unavoidable. Although it may not be apparent at first sight, this question is closely related to the forbidden configuration problem above. And the facts and the gaps are quite parallel: acyclic configurations cannot appear in a coproduct without appearing in the summands, while the cyclic ones with top (or bottom) can, and so also for the k-crowns. The partial solution of the general problem mentioned above applies to the behaviour of configurations in coproducts as well, which is to say that the P from the class of cyclic configurations mentioned above can appear in coproducts without appearing in the summands. And here also the general problem is still open.

In this context, another problem has appeared. When constructing coproducts with configurations that were not in the summands, one has always had cyclic summands. Can a cycle materialize in a coproduct of acyclic spaces? Certainly not when the summands are acyclic with top. But for the general case this may be an even harder problem than that of the prohibited configurations.

The paper is divided into six sections. The first contains the standard preliminaries, and in the second we present a survey of the facts so far established. Section 3 contains ground-clearing for the new results in the following two sections. Section 4 deals with general configurations containing diamonds, and in Section 5 we extend the facts to a fairly large class of additional cyclic configurations. Finally, in Section 6 we briefly discuss some further aspects of coproducts, and formulate the problem of coproducts of acyclic spaces.

1. First preliminaries

1.1. If M is a subset of a poset $(X, \leq)$ we write $\downarrow M$ for $\{x \mid x \leq m \in M\}$ and $\uparrow M$ for $\{x \mid x \geq m \in M\}$. If $M = \downarrow M$ (resp. $M = \uparrow M$) we say that M is a *down-set* (resp. *up-set*).

Immediate succession, that is, the fact that $x < y$ and if $x \leq z \leq y$ then $x = z$ or $y = z$, is indicated as

$$x \prec y.$$

We refer to such a pair as a *cover pair*.

1.2. Recall that a *Priestley space* $X = (X, \tau, \leq)$ is a compact ordered space such that whenever $x \not\leq y$ there is a clopen down-set $U \subseteq X$ such that $x \notin U \ni y$. *Priestley maps* are monotone continuous functions. The resulting category will be denoted by **PSp**.

In particular, finite posets with discrete topologies are Priestley spaces and will be viewed as such. Monotone maps between them, and more generally monotone maps from finite posets to general Priestley spaces, are automatically Priestley maps.

1.3. Further recall (see, e.g., [14],[15],[10]) the Priestley duality between **PSp** and the category **DLat** of bounded distributive lattices and 01-lattice homomorphisms; the equivalence functors are usually given as

$$\mathcal{P}(L) = (\{x \mid x \text{ is a prime ideal in } L\}, \subseteq, \tau), \quad \mathcal{P}(h)(x) = h^{-1}(x)$$

$$\mathcal{D}(X) = (\{a \mid a \text{ is a clopen down-set in } X\}, \cap, \cup), \quad \mathcal{D}(f)(a) = f^{-1}(a).$$

$\mathcal{P}(L)$ is endowed with a suitable topology and ordered by inclusion.

An *embedding* $f : X \to Y$ is a Priestley map such that $x \leq y$ in X iff $f(x) \leq f(y)$ in Y. The existence of such an embedding will be indicated as $X \hookrightarrow Y$ and non-existence of such a map as $X \not\hookrightarrow Y$.

It is a well-known fact that

1.3.1. *a Priestley map $f : X \to Y$ is an embedding if and only if its Priestley dual $\mathcal{D}(f) : \mathcal{D}(Y) \to \mathcal{D}(X)$ is onto.*

As a consequence, since obviously products $\prod h_i : \prod A_i \to \prod B_i$ of onto homomorphisms are onto, we have that

1.3.2. *if $f_i : X_i \hookrightarrow Y_i$ are embeddings then the coproduct map $\coprod f_i : \coprod X_i \to \coprod Y_i$ is an embedding as well.*

1.4. A *configuration* is an order-connected finite poset. In our context, the existence of top (≡ largest element) in a configuration makes everything much simpler; we then speak of the *topped case*, as opposed to the general one.

A *cycle* C in a configuration P is a sequence

$$x_0 < \cdots < x_{n_1} > \cdots > x_{n_2} < \cdots < x_{n_3} > \cdots \\ \cdots < x_{n_{2k-1}} > \cdots > x_{n_{2k}} = x_0$$

such that there are no comparabilities besides the indicated ones. In such a cycle, the k is called the *spike number*, and cycles with spike number $k \geq 2$ are called *k-crowns.* We speak of a *fine k-crown* if all the x_i, x_{i+1} resp. x_{i+1}, x_i are cover pairs. On the other hands, the k-crowns of the form

$$x_0 < x_1 > x_2 < \cdots < x_3 > \cdots\cdots < x_{2k-1} > x_{2k} = x_0$$

are said to be *simple.* A 2-crown $x_0 < x_1 > x_2 < x_3 > x_0$ is *proper* if there is no y such that $x_0, x_2 < y < x_1, x_3$.

The case of spike number 1 has to be treated separately since $x_0 < x_1 > x_0$, $x_0 < \cdots < x_{n_1} > x_0$ or $x_0 < x_1 > \cdots > x_2 = x_0$ do not behave combinatorially as cycles. A *diamond* is a cycle $x_0 < \cdots < x_{n_1} > \cdots > x_{n_2} = x_0$ such that $n_1 > 1$ and $n_2 > n_1 + 1$. Again, we speak of a *fine diamond* if all the x_i, x_{i+1} resp. x_{i+1}, x_i are cover pairs while *simple diamonds* will be the poset $a < b < c > d > a$ with b incomparable to d. The fact in common with the simple k-crowns is that they are the minimal genuine cycles with the given spike number.

1.5. Cyclic and acyclic configurations. On a configuration P consider the symmetric antireflexive relation

$$x \succ\!\!\prec y \quad \equiv_{\mathrm{df}} \quad x \prec y \text{ or } y \prec x.$$

The facts in the sequel are heavily dependent on the properties of the graph $(P, \succ\!\!\prec)$.

A configuration $(P, \leq)$ is said to be a *(combinatorial) tree*, or to be *acyclic*, if $(P, \succ\!\!\prec)$ is a tree; otherwise we speak of a *cyclic* configuration.

In the topped case the notion of a *tree* coincides with the standard use of the term describing special posets, and

a topped P is cyclic iff it contains no diamond.

The general case is more complex. Here

- *a configuration is cyclic iff it contains a diamond or a k-crown with $k \geq 3$ or a proper 2-crown.*

Note that unlike in the topped case, an acyclic configuration can have a cyclic subconfiguration (remove the y from the $(\{x_0, x_1, x_2, x_3, y\}, \leq)$ in the improper crown in 1.4).

The *spike number* of a cyclic configuration P, denoted s_P, is 1 if there is a diamond, or else the least k such that P contains a k-crown, with the understanding that the crown is proper if $k = 2$.

1.6. Coproducts. By duality, since **DLat** is complete **PSp** is cocomplete. In particular we will be interested in coproducts. They are suitable compactifications of the disjoint unions $\bigcup_{i\in J}^{\text{disj}} X_i$, and can themselves be organized as disjoint unions of closed subspaces (see [11])

$$\coprod_{i\in J} X_i = \bigcup_{i\in\beta J}^{\text{disj}} X_u.$$

The index set βJ is the Čech-Stone compactification of J, viewed as the set of all ultrafilters on J. In this setup

(1) X_i can be identified with the $X_{\widetilde{i}}$ for the corresponding centered ultrafilter $\widetilde{i} = \{M \mid i \in M\}$,
(2) $\bigcup_J X_i$ is dense in $\coprod_J X_i$,
(3) the X_u are order independent,
(4) and for each $u \in \beta J$, X_u is the Priestley dual of the ultraproduct $\prod_u \mathcal{D}(X_i)$.

2. Summary of known results

2.1. The famous Stone duality fits into the more general Priestley duality as the correspondence between the Priestley spaces with trivial order and Boolean algebras. This can be expressed by stating that

> *a distributive lattice L is a Boolean algebra iff the chain $\{0 < 1\}$ cannot be embedded into $\mathcal{P}(L)$.*

In [1] (1991) Adams and Beazer proved that, more generally,

> *the prohibition of the n-chain $\{0 < 1 < \cdots < n\}$ in $\mathcal{P}(L)$ characterizes the distributive lattices L such that for any $a_0, \ldots, a_n \in L$ there exist $c_0, \ldots, c_n \in L$ with $a_0 \wedge c_0 = 0$, $a_k \wedge c_k \leq a_{k-1} \vee c_{k-1}$ for $0 < k \leq n$, and $a_k \vee c_k = 1$.*

Much earlier (1974) Monteiro proved in [13] that *the prohibition of the V-configuration* $\{0 < 1,2\}$ *(*$1,2$ *incomparable) in* $\mathcal{P}(L)$ *characterizes the relatively normal lattices* L*, that is, those in which for any two* a_1, a_2 *there are* c_1, c_2 *such that* $a_1 \vee c_1 \geq a_2$*,* $a_2 \vee c_2 \geq a_1$ *and* $c_1 \wedge c_2 = 0$*.*

These facts give rise naturally to the following type of problems. For a configuration P define

$$\mathsf{Forb}(P) = \{L \mid P \not\hookrightarrow \mathcal{P}(L)\}.$$

(1) Is every $\mathsf{Forb}(P)$ axiomatizable?
(2) If not, is every $\mathsf{Forb}(P)$ with P acyclic axiomatizable?
(3) Or at least, is every $\mathsf{Forb}(P)$ with P topped and acyclic axiomatizable?

This will be discussed below. For the time being let us just say that the first of the questions resolved was (3) in [3] (positively), then there appeared a negative answer to (1) in [5], and later a positive answer to (2) in [6]. In the positive cases the proof produces an algorithm to construct the desired formulas.

2.2. The following problem may not seem on first sight to be closely related to those in 2.1. Recall the coproduct of Priestley spaces from 1.6. Since the X_u in

$$\coprod_{i\in J} X_i = \bigcup_{i\in\beta J}^{\text{disj}} X_u$$

are order independent, a configuration P can be embedded into $\coprod X_i$ only if it can be embedded into some of the X_u's. But does it have to be embedded into some of the X_i, $i \in J$? That is, does the implication

$$\text{(copr P)} \qquad P \hookrightarrow \coprod_J X_i \quad \Rightarrow \quad \exists i \in J,\ P \hookrightarrow X_i$$

hold?

Let us call a configuration *coproductive* if it does. Now: which configurations are coproductive?

2.3. We have the complete answers to the above problems in the topped case.

Theorem. ([3],[5]) *Let* P *be a configuration with top. Then the following statements are equivalent.*

(1) P *is acyclic.*
(2) P *is coproductive.*
(3) $\mathsf{Forb}(P)$ *is closed under products.*
(4) $\mathsf{Forb}(P)$ *can be characterized by first order formulas.*

(5) $\mathsf{Forb}(P)$ *is axiomatizable.*

Restricting Priestley duality to Heyting algebras and the corresponding spaces and maps we can proceed with

...
(6) $\mathsf{Forb}(P)$ *is a quasivariety.*
(7) $\mathsf{Forb}(P)$ *is a variety.*

2.4. In the general case we have

2.4.1. Theorem. ([6]) *If P is acyclic then P is coproductive and* $\mathsf{Forb}(P)$ *is axiomatizable.*

As first steps to the converse we have, for the critical cyclic configurations (recall 1.5)

2.4.2. Theorem. ([6],[7]) *Diamonds and k-crowns are not coproductive and the corresponding* $\mathsf{Forb}(P)$ *cannot be characterized by first order formulas.*

We conjecture that this can be extended to all cyclic configurations and that the general facts are in a complete agreement with the topped case. In this article we prove it, at least, for a large class of cyclic configurations.

3. Second preliminaries

In this section we will extend the construction from [6] and [7] to systems of exceptional cover pairs instead of just one cover pair.

3.1. Set $\widetilde{k} = \{1, 2, \dots, k\}$. Let R be an antireflexive binary relation on $\widetilde{k}$. A subset $J \subseteq \widetilde{k}$ is *independent* if $(J \times J) \cap R = \emptyset$. The *chromatic number* of a subset $A \subseteq \widetilde{k}$, denoted $\chi_R(A)$, is the least cardinality of a covering of A by independent sets. Obviously $\chi_R(A) \leq |A|$ (since each $\{x\}$ is independent) and

$$\chi_R(A \cup B) \leq \chi_R(A) + \chi_R(B). \tag{3.1.1}$$

An *increasingly colorful sequence* (briefly, *i.c.s.*) of relations is a system $(\widetilde{k}_n, R_n)$ such that $k_1 < k_2 < \dots$, and

$$\chi_{R_n}(\widetilde{k}_n) \to \infty.$$

For instance take $1 < 2 < 3 < \dots$ with

$$R'_n = \{(x, y) \mid x \neq y\}, \quad \text{or} \quad R''_n = \{(x, y) \mid x < y\}. \tag{3.1.2}$$

Obviously, only one-point sets are independent. An important feature of these relations is that

3.1.1. *the poset*

$$(\widetilde{k}_n \times \{0,1\}, \{((x,0),(y,1)) \mid xRy\})$$

contains in the former case no k-crown with $k \geq 4$, and in the latter case no 3-crown.
(See [5], but this is very easy to verify.)

In [7], based on a deep result of Alon ([2]), there is constructed a much more involved i.c.s. of relations $(\widetilde{k}_n, R'''_n)$ such that

3.1.2. *the poset*

$$(\widetilde{k}_n \times \{0,1\}, \{((x,0),(y,1)) \mid xR'''_n y\})$$

contains no 2-crown.

3.2. Consider an i.c.s. of relations $(\widetilde{k}_n, R_n)_n$ (in the constructions in 3.4 below it will be specified). On the set

$$I = \{(n,j) \mid n \in \mathbb{N},\ j \in \widetilde{k}_n\}$$

choose a free ultrafilter u and set (χ_n stands for χ_{R_n})

$$F = \{J \subseteq I \mid \exists m, \{n \mid \chi_n(\{j \mid (n,j) \notin J\}) \leq m\} \in u\}.$$

Obviously $I \in F$ as $\chi_n(\emptyset) = 0 \leq m$, and $\emptyset \notin F$ since $\{j \mid (n,j) \in I\} = \widetilde{k}_n$ and $\chi_n(\widetilde{k}_n) \leq m$ only for finitely many n. Since $J_1, J_2 \in F \Rightarrow J_1 \cap J_2 \in F$ by (3.1.1), F is a proper filter and hence we can choose an ultrafilter

$$v \supseteq F.$$

3.2.1. Lemma. *For $J \subseteq I$ set*

$$f(J) = \{(n,j) \mid \exists i,\ (n,i) \in J \text{ and } iR_n j\}.$$

Then $j \in v \Rightarrow f(J) \in v$.

Proof. Suppose $J \in v$ and $f(J) \notin v$ so that $I \setminus f(J) \in v$ and $K = J \cap (I \setminus f(J)) \in v$. Now $K \cap f(K) = \emptyset$ and hence, for any n, the set $\{j \mid (n,j) \in K\}$ is independent, and $\chi_n(\{j \mid (n,j) \notin I \setminus K\}) = \chi_n(\{j \mid (n,j) \in K\}) = 1$. Thus, $I \setminus K \in F \subseteq v$ and $K \notin v$. ☐

3.3. Let Q be a configuration. A non-void set $\mathcal{A}$ of pairs $(a \prec b) \in Q$ is said to be an *independent system of covers* (briefly, *ISOC*) if

> for any $p < q$ in Q there is at most one $(a \prec b) \in \mathcal{A}$ such that $p \leq a \prec b \leq q$.

For pairs $\alpha = (a_1 \prec a_2), \beta = (b_1 \prec b_2)$ define the distance $\mathsf{dist}(\alpha, \beta)$ as the minimum number of turning points in a path connecting some a_i to some b_j. An ISOC $\mathcal{A}$ is said to be *well-distanced* (in Q) if for any two distinct $\alpha, \beta \in \mathcal{A}$, $\mathsf{dist}(\alpha, \beta) > s_Q$.

3.4. In the following construction, the i.c.s. of relations $(\widetilde{k}_n, R_n)_n$ will be chosen according to the spike numbers of the configuration Q as follows. In the notation of 3.1,

$$\begin{aligned}&\text{for } s_Q = 1 \text{ or } s_Q \geq 4 \text{ we choose } R_n = R'_n,\\ &\text{for } s_Q = 3 \text{ we choose } R_n = R''_n, \text{ and}\\ &\text{for } s_Q = 2 \text{ we choose } R_n = R'''_n.\end{aligned}$$

Let $\mathcal{A}$ be an ISOC in Q. Define

$$Q(\mathcal{A}, n) = (Q \times \widetilde{k}_n, \leq)$$

by the following rules. (We write pi for $(p, i) \in Q \times \widetilde{k}_n$, and $[p, q]$ for the interval $\{x \mid p \leq x \leq q\}$.)

(r1) If $p \leq a \prec b \leq q$ in Q for some $(a \prec b) \in \mathcal{A}$ then
  (r11) if $[p, q] = [p, a] \cup [b, q]$ then $pi < qj$ exactly when iR_nj,
  (r12) if $[p, q] \neq [p, a] \cup [b, q]$ then $pi < qj$ exactly when iR_nj or $i = j$.

(r2) If $[p, q]$ contains no pair from $\mathcal{A}$ then $pi < qj$ exactly when $p < q$ and $i = j$.

3.4.1. Proposition. *$Q(\mathcal{A}, n)$ is a poset and $pi \prec qj$ in $Q(\mathcal{A}, n)$ iff either $(p, q) = (a, b) \in \mathcal{A}$ and iR_nj, or $(p \prec q) \notin \mathcal{A}$ and $i = j$. In other words,*

$$pi \prec qj \quad \textit{iff} \quad p \prec q \textit{ in } Q \textit{ and } pi < qj \textit{ in } Q(\mathcal{A}, n).$$

Proof. To verify the transitivity suppose $pi < qj < rl$. Since $\mathcal{A}$ is an ISOC and $[p, q]$ is an interval, rule (r1) cannot apply to both $pi < qj$ and $qj < rl$. Next, if (r2) applies to both the pairs, we have $p < q < r$ and $i = j = l$ and $pi < rl$ (by (r2) or by (r12)). Now suppose that $pi < qj$ by (r11) with $p \leq a \prec b \leq q$, and $q < r$ and $j = l$. Then (r1) applies to $pi < rl$, whether as (r11) or (r12). Secondly, if $pi < qj$ by (ri2) then there is a $c \in [p, q]$ with $c \not\leq a$ and $c \not\geq b$, and iR_nj or $i = j$. Then $pi < rl$ as $j = l$ and $c \in [p, q]$ remains incomparable with a and b. Similarly in the remaining cases.

Now let $pi \prec qj$ in $Q(\mathcal{A}, n)$. If this is so because of (r2) then $p \prec q$ and $i = j$. If this is because of (r1) it must be because of (r11) since (r12) is not applicable to $p \prec q$, and we have $p = a$ and $q = b$. $\square$

3.5. Recall the functors $\mathcal{D}$ and $\mathcal{P}$ from 1.3. Consider

$$A_n = \mathcal{D}(Q(\mathcal{A}, k_n)),$$

the bounded distributive lattice of all down-sets of $Q(\mathcal{A}, k_n)$. Set $A = \prod_{n=2}^{\infty} A_n$ so that

$$\mathcal{P}(A) = \coprod_{n=2}^{\infty} Q(\mathcal{A}, k_n).$$

Define

$$m : Q \to \mathcal{P}(A)$$

by setting

$$m(p) = \{\alpha = (\alpha_n)_n \in A \mid \{(n, j) \mid pj \notin \alpha_n\} \in v\}$$

where v is the ultrafilter from 3.2.

3.5.1. Lemma. *For each $p \in Q$, $m(p)$ is indeed a proper prime ideal of A.*

Proof. Obviously $m(p)$ is a down-set, and $A \notin m(p)$. Further, for $\alpha, \beta \in m(p)$,

$$\{(m, j) \mid pi \notin (\alpha \vee \beta)_n = \alpha_n \cup \beta_n\} = \{(m, j) \mid pi \notin \alpha_n\} \cap \{(m, j) \mid pi \notin \beta_n\}$$

is in v, veryfying that $\alpha \vee \beta \in m(p)$. If $\alpha \wedge \beta \in m(p)$ then v contains

$$\{(m, j) \mid pi \notin (\alpha \wedge \beta)_n = \alpha_n \cap \beta_n\} = \{(m, j) \mid pi \notin \alpha_n\} \cup \{(m, j) \mid pi \notin \beta_n\}$$

and the primeness of v forces one of the sets displayed on the right to be contained in v, and hence α or β is in $m(p)$. $\square$

3.5.2. Proposition. *m is an embedding.*

Proof. To show that m preserves order consider, first, $p < q$ by virtue of rule (r2). If $\alpha \in m(p)$ then $\{(n, j) \mid pj \notin \alpha_n\} \subseteq \{(n, j) \mid qj \notin \alpha_n\} \in v$ meaning $\alpha \in m(q)$.

Next consider $p \leq a \prec b \leq q$ with $(a \prec b) \in \mathcal{A}$. Since there cannot be a $(c \prec d) \in \mathcal{A}$ with $p \leq c \prec d \leq a$ or $b \leq c \prec d \leq q$ and because $m(p) \leq m(a)$ and $m(b) \subseteq m(q)$ have been already established, it suffices to prove that $m(a) \subseteq m(b)$. For that purpose consider $\alpha \in m(a)$ so that $\{(n, i) \mid ai \notin \alpha_n\} \in v$. Since α_n is a down-set, if $ai \notin \alpha_n$ and $iR_n j$ then $bj \notin \alpha_n$. Therefore (recall 3.2.1)

$$\{(n, j) \mid bj \notin \alpha_n\} \supseteq f(\{(n, i) \mid ai \notin \alpha_n\}) \in v$$

and $\{(n, j) \mid bj \notin \alpha_n\} \in v$ and $\alpha \in m(b)$.

Finally suppose that $p \not\leq q$. Define α^p by setting

$$\alpha^p_n = \{rj \mid r \not\geq p\} \text{ for all } n.$$

Then

$$\{(n,j) \mid pj \notin \alpha^p_n\} = \{(n,j) \mid p \geq p\} = I \in v, \text{ and}$$
$$\{(n,j) \mid qj \notin \alpha^p_n\} = \{(n,j) \mid q \geq p\} = \emptyset \notin v.$$

Thus, $\alpha^p \in m(p) \setminus m(q)$. □

4. Diamonds in the general case

In this section we will deal with the general (not necessarily topped) configurations containing diamonds. A direct analogue of the procedure used in [5] does not seem to be applicable in general; the situation appears considerably more complex.

4.1. Lemma. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be disjoint ISOCs in a configuration Q. Then the system*

$$\mathcal{A}_2 * \widetilde{k} = \{ai \prec bi \mid (a \prec b) \in \mathcal{A}_2, i \in \widetilde{k}\}$$

is an ISOC in $Q(\mathcal{A}_1, k)$.

Proof. By 3.4.1 and by the disjointness of $\mathcal{A}_1$ and $\mathcal{A}_2$, $\mathcal{A}_2 * \widetilde{k}$ is a set of cover pairs. Now let $a_1i_1 \prec a_1i_1$ and $a_2i_2 \prec b_2i_2$ distinct in $\mathcal{A}_2 * \widetilde{k}$ be such that $pk \leq a_1i_1, a_2i_2$ and $b_1i_1, b_2i_2 \leq ql$. Then $p \leq a_i \prec b_i \leq q$ in Q and hence $a_1 = a_2 = a$, $b_1 = b_2 = b$ and $i_i \neq i_2$. We thus have

$$pk \leq ai_j \prec bi_j \leq ql \quad \text{for} \quad j = 1, 2.$$

Let, say, $k \neq i_1$ so that to have $pk < ai_1$ in $Q(\mathcal{A}_1, k)$ we have to have a $(c \prec d) \in \mathcal{A}_1$ such that $p \leq c \prec d \leq a$. Similarly, however, there must be a $(c' \prec d') \in \mathcal{A}_1$ with $b \leq c' \prec d' \leq q$ contradicting the ISOC property of $\mathcal{A}_1$. Thus, in an interval of $Q(\mathcal{A}_1, k)$ there is at most one element of $\mathcal{A}_2 * \widetilde{k}$. □

4.2. Let $\mathcal{A}$ be an ISOC and let $\mathcal{C}$ be a set of cycles C in Q that pass through none of the $(a \prec b) \in \mathcal{A}$. Set

$$\mathcal{C} * \widetilde{k} = \{C * i \mid C \in \mathcal{C}\}$$

with $\{x_1, \ldots, x_n\} * i = \{x_1i, \ldots, x_ni\}$.

Lemma. *Let $\mathcal{C}$ be the set of all fine diamonds in Q and let $\mathcal{A}$ be an ISOC in Q. Set*

$$\mathcal{C}_1 = \{C \in \mathcal{C} \mid \textit{none of the pairs } (a \prec b) \in \mathcal{A} \textit{ occurs in } C\}.$$

Then $\mathcal{C}_1 * \widetilde{k}$ is the set of all fine diamonds in $Q(\mathcal{A}, k)$.

Proof. First, all the $C * i$ are fine diamonds in $Q(\mathcal{A}, k)$ since only the clause (r2) from 3.4 applies.

For the converse suppose that

$$C:\ p_1 i_1 \prec \cdots \prec p_t i_t \succ \cdots \succ p_m i_m = p_1 i_1$$

is a fine diamond in $Q(\mathcal{A}, k)$. The interval $[p_1, p_t]$ contains at most one member of $\mathcal{A}$. If there is none, $i_1 = i_2 = \cdots = i$ and C is in $\mathcal{C}_1 * \widetilde{k}$. Now an $(a \prec b) \in \mathcal{A}$ cannot occur in just one of the paths $p_1 i_1 \prec \cdots \prec p_t i_t$ or $p_t i_t \succ \cdots \succ p_m i_m = p_1 i_1$ since this would yield $i_1 = i_t \neq i_1$. Thus there remains the case of $j < t$ and $l \geq t$ such that $(a, b) = (p_j, p_{j+1}) = (p_{l+1}, p_l)$. Now since C is a fine diamond we can have $p_{j+1} = b = p_l$ only if $j + 1 = t = l$ (since $i_{j+1} = i_t = i_l$) and for the same reason $p_j = a = p_{l+1}$ yields $j = 1$ and $l + 1 = m$. But this results in C being $ai < bj > ai$ which is not a diamond. $\square$

4.3. Construction. Let $\mathcal{C} \neq \emptyset$ be the collection of all fine diamonds of a configuration P. This collection will be decomposed as follows.

First, choose any $C \in \mathcal{C}$ and a cover pair $a_1 \prec b_1$ participating in C. Set

$$\mathcal{C}_1 = \{C \in \mathcal{C} \mid a_1 \prec b_1 \text{ participates in } C\}.$$

If $\mathcal{C}_1, \ldots, \mathcal{C}_j$ together with cover pairs $a_i \prec b_i$, $i \leq j$, have already been defined so that each $a_i \prec b_i$ occurs exactly once in each of the $C \in \mathcal{C}_i$, and if $\bigcup\{\mathcal{C}_i \mid i \leq j\} \neq \mathcal{C}$, choose arbitrarily $C \in \mathcal{C} \setminus \bigcup\{\mathcal{C}_i \mid i \leq j\}$ and a cover pair $a_{j+1} \prec b_{j+1}$ in C. Then set

$$\mathcal{C}_{j+1} = \{C \in \mathcal{C} \setminus \bigcup\{\mathcal{C}_i \mid i \leq j\} \mid a_{j+1} \prec b_{j+1} \text{ participates in } C\}.$$

Continuing as long as possible we obtain a disjoint union

$$\mathcal{C} = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_m$$

and distinct cover pairs $a_i \prec b_i$, $i \leq m$, associated with the classes $\mathcal{C}_i$. Now (using the relation $R_n = \{(i, j) \mid i \neq j\}$ – recall 3.4) set, first,

$$\mathcal{A}_1 = \{(a_1 \prec b_1)\},\ P_{n_1} = P(\mathcal{A}_1, n_1).$$

By 4.1, $\mathcal{A}_2(n_1) = \{(a_2 \prec b_2)\} * \widetilde{n}_1$ is an ISOC in P_{n_1} and we can define

$$P_{n_1 n_2} = P_{n_1}(\mathcal{A}_2(n_1), n_2).$$

Assume that for $j \leq t$ we have already defined $P_{n_1 \ldots n_j}$ so that

$$P_{n_1 \ldots n_j} = P_{n_1 \ldots n_{j-1}}(\mathcal{A}_j(n_1, \ldots, n_{j-1}), n_j)$$

where

$$\mathcal{A}_j(n_1, \ldots, n_{j-1}) = (\ldots(\{(a_j \prec b_j)\} * \widetilde{n}_1) \ldots) * \widetilde{n}_{j-1}.$$

Then

$$\mathcal{B}_j(n_1, \ldots, n_{j-1}) = (\ldots(\{(a_{j+1} \prec b_{j+1})\} * \widetilde{n}_1) \ldots) * \widetilde{n}_{j-1}$$

is disjoint with $\mathcal{A}_j(n_1,\dots,n_{j-1})$ and hence, by 4.1, $\mathcal{A}_{t+1}(n_1,\dots,n_t) = \mathcal{B}_t(n_1,\dots,n_{t-1}) * \widetilde{n}_t$ is an ISOC in $P_{n_1\dots n_t}$ and we can set

$$P_{n_1\dots n_{t+1}} = P_{n_1\dots n_t}(\mathcal{A}_{t+1}(n_1,\dots,n_t), n_{t+1}).$$

4.4. Proposition. *None of the posets $P_{n_1\dots n_m}$ from 4.3 contains a diamond.*

Proof. Starting with the decomposition

$$\mathcal{C}_1^0 \cup \dots \cup \mathcal{C}_m^0 = \mathcal{C}_1 \cup \dots \cup \mathcal{C}_m$$

define inductively

$$(*) \qquad \mathcal{C}_{j+1}^j(n_1,\dots,n_j) \cup \dots \cup \mathcal{C}_m^j(n_1,\dots,n_j)$$

by setting $\mathcal{C}_i^j = \mathcal{C}_i^{j-1} * \widetilde{j}$. By Lemma 4.2 and by the choice of the $\mathcal{A}_j(n_1,\dots,n_{j-1})$ the system $(*)$ contains all the fine diamonds of $P_{n_1\dots n_j}$, and ultimately there is no diamond in $P_{n_1\dots n_m}$. □

4.5. Theorem. *No configuration P containing a diamond is coproductive.*

Proof. By 3.5.2 we have embeddings

$$P \hookrightarrow \coprod_{n_1} P_{n_1} \quad \text{and} \quad P_{n_1\dots n_{j-1}} \hookrightarrow \coprod_{n_j} P_{n_1\dots n_{j-1}n_j}.$$

By 1.3.2 we have, hence, embeddings

$$\coprod_{n_1,\dots,n_{j-1}} P_{n_1\dots n_{j-1}} \hookrightarrow \coprod_{n_1,\dots,n_{j-1}} \coprod_{n_j} P_{n_1\dots n_{j-1}n_j} \hookrightarrow \coprod_{n_1,\dots,n_j} P_{n_1\dots n_j}$$

and therefore

$$P \hookrightarrow \coprod_{n_1} P_{n_1} \hookrightarrow \coprod_{n_1 n_2} P_{n_1 n_2} \hookrightarrow \cdots \hookrightarrow \coprod_{n_1,\dots,n_{m-1}} P_{n_1\dots n_{m-1}} \hookrightarrow \coprod_{n_1,\dots,n_m} P_{n_1\dots n_m}$$

where the summands in the last coproducts contain, by 4.4, no diamond, and hence cannot contain P. □

5. Another class of non-coproductive configurations

5.1. Let P be a configuration and let $\mathcal{A}$ be an ISOC on P. The subposets of $P(\mathcal{A}, n)$ carried by $P \times \{i\}$, $1 \le i \le n$, will be called *pages*.

In this section, P will be a cyclic configuration with $s_P \geq 2$, that is, without diamonds.

5.2. On the set $\mathcal{C}$ of s_P-crowns of P consider the equivalence E obtained by transitivity from $C \sim C' \;\equiv\; C \cap C' \neq \emptyset$.

A *cloud* in P is a union of an E-class viewed as the induced subposet. It is said to be *critical* if it cannot be embedded into a non-isomorphic cloud in P.

The following condition on the configuration P will play a crucial role.

(WD) There exists a complete isomorphism class $X_1, \dots, X_k$ of critical clouds, and $a_i \prec b_i$ contained in suitable cycles C_i participating in X_i such that $\mathcal{A} = \{a_i \prec b_i \mid i = 1, \dots, k\}$ is a well-distanced ISOC.

5.3. Lemma. *Let P satisfy (WD). Then every s_P-crown in $P(\mathcal{A}, n)$ is contained in a page.*

Proof. Let

$$C: \; p_0 i_0, p_1 i_1, \dots, p_k i_k$$

be an s_p-crown. By the assumption on the distance of the individual $a_i \prec b_i$ in $\mathcal{A}$ we cannot have two distinct $a_1 \prec b_i$ in positions between some p_r, p_{r+1} and p_s, p_{s+1}. Let there be one, say $a \prec b$. To analyze the situation we have to write the cycle in more detail:

$$p_0 i_0 < p_1 i_1 < \cdots < p_{n_1} i_{n_1} > \cdots > p_{n_2} i_{n_2} < \cdots <$$
$$\cdots < p_{n_{2s-1}} i_{n_{2s-1}} < \cdots > p_0 i_0.$$

We cannot have the $a \prec b$ in every interval between the turning points (in view of the choices of R_n in 3.4 this would contradict 3.1.1 and 3.1.2). Therefore the longest path $p_r, \dots, p_{r+j}$ with $p_r \geq b$ and avoiding $a \prec b$ (and hence keeping the same i) contains more than one turning point, and the comparabilities $b \leq p_r$, $p_{r+j} \leq a < b$ show that the path is a crown. By the minimality of s_P, $p_r i_r, \dots, p_{r+j} i_{r+j}$ have to cover C and hence there is only one occurence of $a \prec b$. Then, however, there is only one switch of the i_j's, and C is not a cycle at all. Thus, there is no $(a \prec b) \in \mathcal{A}$ inserted into the intervals of C and all the i_j's coincide. □

5.3.1. Corollary. *Let P satisfy (WD), let C be a cloud of P and let it be embedded into $P(\mathcal{A}, n)$. Then the image is contained in a page.*

5.4. Theorem. *A cyclic configuration P satisfying (WD), in particular any P such that there is among the critical clouds one that is not isomorphic to any other, is not coproductive.*

Proof. By 3.5.2, P can be embedded into $\coprod_{n=2}^{\infty} P(\mathcal{A}, n)$. By 5.3.1, however, there is no copy of P in any of the $P(\mathcal{A}, n)$ with $n \geq 2$.

Indeed, each of the critical clouds would have to be represented in a single page. However, according to the choice of $\mathcal{A}$ (see the formulation of (WD)) all the critical clouds from the isomorphism class $X_1, \dots, X_k$ are broken in $P(\mathcal{A}, n)$ by the $a_i \prec b_i$ and there is no subposet of $P(\mathcal{A}, n)$ isomorphic to the X_i. $\square$

5.5. Note. Variants of Theorem 5.4 can be based on any easily identifiable class of minimal cycles.

For instance, for a cycle $C \in \mathcal{C}$ define $d(C)$ as the length of a longest chain $y_0 \prec y_1 \prec \cdots \prec y_d$ for which y_d is one of the minimal elements of C. Set

$$d_0 = \max\{d(C) \mid C \in \mathcal{C}\},\ \mathcal{C}' = \{C \in \mathcal{C} \mid d(C) = d_0\}.$$

Now, thus defined, d_0 does not increase in any of the $P(\mathcal{A}, n)$ and hence a $C \in \mathcal{C}'$ has to be represented in a page again with the same characteristics. Thus, if we restrict ourselves to forming clouds in $\mathcal{C}'$ only and defining critical clouds accordingly, we can repeat the procedure.

Note that 5.4 does not include the statement just made, and on the other hand this statement does not include Theorem 5.4.

5.6. Summary. The non-coproductivity has been proved, besides all the configurations with diamonds, also for quite large classes of configurations without them: in fact, constructing a cyclic configuration that does not belong would require considerable effort. We **conjecture**, of course, that

no cyclic configuration is coproductive.

This general statement remains an open problem.

6. Remarks on coproducts and another problem

6.1. The simplest case of infinite coproducts of Priestley spaces is that of a co-power of a finite P.

Fact. *For a finite poset P, the co-power ${}^J P$ is $P \times \beta J$ with the order given by $(x, u) \le (y, v)$ iff $u = v$ and $x \le y$ in P.*

(That is, all the X_u from 1.6 above are isomorphic to P.) This follows immediately by algebraic reasoning from [11], but a direct proof is also very easy (see [4]).

6.2. The first case next in simplicity is that of acquiring infinite chains from finite ones.

Proposition. *Let X_n be the chain $\{0 < 1 < \cdots < n\}$. Then the coproduct $\coprod_{n=1}^{\infty} X_n$ contains an infinite chain.*

Proof. Note that $\mathcal{D}(X_n) = X_{n+1}$ viewed as a lattice. Thus, we have

$$\coprod_{n=1}^{\infty} X_n \cong \mathcal{P}(\prod_{n=2}^{\infty} X_n).$$

Take the elements $p^n = (1, 2, \dots, n-1, n, n, n, ...)$ in the product $\prod X_n$. Choose a free ultrafilter u on $\mathbb{N}$ and set

$$J(n) = \{X \mid \{i \mid x_i \leq p_i^n\} \in u\}.$$

Obviously $J(n)$ is a down-set and $(i)_{i\in\mathbb{N}} \notin J(n)$. Since $\{i \mid (x \vee y)_i \leq p_i^n\} = \{i \mid x_i \leq p_i^n\} \cap \{i \mid y_i \leq p_i^n\}$ it is a proper ideal, and since $\{i \mid (x \wedge y)_i \leq p_i^n\} = \{i \mid x_i \leq p_i^n\} \cup \{i \mid y_i \leq p_i^n\}$ and u is an ultrafilter, $J(n)$ is prime. Finally, obviously $m \leq n \;\Rightarrow\; J(m) \subseteq J(n)$, and $p^{n+1} \notin J(n) \ni p^n$ so that $J(1) < J(2) < \cdots < J(n) < \cdots$ is an infinite chain. □

6.3. The example in 6.2 is not surprising. The more substantial question was, of course, whether there were finite shapes (configurations) appearing in coproducts without having appeared in the individual summands. The cyclic configurations (all the topped ones and – so far – these of sections 4 and 5 above) are an answer to that.

6.4. For a system $(X_i)_{i\in J}$ of Priestley spaces take the embedding $\iota : \bigcup_J^{\text{disj}} X_i \hookrightarrow \coprod_J X_i$ from 1.6 and the standard compactification embedding $\kappa : \bigcup X_i \to \beta(\bigcup X_i)$ (ignoring the order). Then there is the unique continuous

$$\widetilde{\iota} : \beta(\bigcup X_i) \to \coprod X_i$$

such that $\widetilde{\iota}\kappa = \iota$. This $\widetilde{\iota}$ is onto but not necessarily one-one. By ([11], 3.7) one has

Proposition. *The mapping $\widetilde{\iota}$ is a homeomorphism iff there is a finite n such that the lengths of all but finitely many of the X_i are bounded by n.*

In all the constructions of embeddings witnessing the non-coproductivity of acyclic configurations mentioned in preceding sections, the summands have been bounded in length. Thus, whatever happened with the order, the topology of the coproduct was that of the standard Čech-Stone compactification of $\bigcup X_i$. From this point of view, the topology of the easy example in 6.2 is more exotic.

6.5. As we have seen in 2.3 (just taken with the opposite order, which does not change the facts – see [3]), in particular the configuration

$$V = (\{0,1,2\}, \{(i,i),(0,1),(0,2)\})$$

(already mentioned in connection with the Monteiro's result on relative normality) is coproductive. Now one has $V \not\to X$ iff X is a forest, that is, iff all its finite connected subposets are topped trees. Thus,

arbitrary coproducts of forests are forests again.

Hence, further, in the topped case a new cyclic configuration in a coproduct can appear only as a result of other cycles in the summands, never from acyclic ones.

6.6. This observation can be easily extended to co-forests, and further trivially to coproducts of forests and coforests. This of course does not say anything about the acyclic configurations that have neither top nor bottom. There are, however, also other indications that encourage us to venture a **Conjecture** that

a coproduct of acyclic Priestley spaces is acyclic.

Note that this is not closely connected with the results (and the open problem) concerning coproductivity of configurations. The fact that forests are characterized by the prohibition of a co-tree has no counterpart in general acyclic spaces.

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