An algorithm for two coloring of hypergraphs

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Abstract

Let $r > 1$ be a fix integer. Let $\mathcal{H}$ be an arbitrary r-uniform hypergraph. We give an algorithmic, elementary proof of the fact, that there exists a 2-coloring $f : V(\mathcal{H}) \to \{r, b\}$ of $V(\mathcal{H})$ with at most $2^{1-r} \cdot |\mathcal{H}|$ monochromatic edges. Our algorithm uses only $O(|\mathcal{H}|)$ steps.

Let $m(n, r)$ denote the minimal number of edges of an r-uniform hypergraph on n points, which is not 2-colorable. As an application, we obtain the following bounds for $m(n, r)$:

$$(1 - \frac{1}{2^{r-1}})\binom{n}{r} \leq m(n, r) \leq (1 - \frac{1}{r^2 \cdot 2^r - 1})\binom{n}{r}.$$

1 Introduction

First we introduce some notation. Let n be a positive integer and $[n]$ stand for the set $\{1, 2, \dots, n\}$.

Let X be a set, $k > 0$ be a positive integer. We denote by $\binom{X}{k}$ the family of all k element subsets of X.

In [3] P. Erdős proved the following Theorem with simple probabilistic arguments.

Theorem 1.1 *Let $r > 1$ be a fix integer. Let $\mathcal{H}$ be an arbitrary r-uniform hypergraph. Then there exists a 2-coloring $f : V(\mathcal{H}) \to \{r, b\}$ of $V(\mathcal{H})$ with at most $2^{1-r} \cdot |\mathcal{H}|$ monochromatic edges.*

In these notes we prove Theorem 1.1 using a simple, elemantary constructive algorithm. Our algorithm uses only $O(|\mathcal{H}|)$ steps.

Let $m(n, r)$ denote the minimal number of edges of an r-uniform hypergraph on n points, which is not 2-colorable. We obtain the following bounds for $m(n, r)$ using Theorem 1.1 and an upper bound for the Turán density of an arbitrary hypergraph.

Theorem 1.2

$$(1-\frac{1}{2^{r-1}})\binom{n}{r} \leq m(n,r) \leq (1-\frac{1}{r^2\cdot 2^r-1})\binom{n}{r}. \tag{1}$$

□

2 The algorithm

Proof of Theorem 1.1:

Let $\mathcal{H}$ be an arbitrary r-uniform hypergraph and define $V := V(\mathcal{H})$.

For each 3-coloring $c : V \to \{r, b, g\}$ we define a nonnegative weight $w(\mathcal{H}, c) \in \mathbb{R}$.

Let $h \in \mathcal{H}$ be a fixed edge. First we define the weight $w(h, c) \in \mathbb{R}$.

1. If there exist two points $v_1, v_2 \in h$, $v_1 \neq v_2$ such that $c(v_1) = r$, $c(v_2) = b$, then let $w(h, c) := 0$.

2. If $c(v) = g$ for each $v \in h$, then let $w(h, c) := 2^{1-r}$.

3. If

$$h = \overline{g} \cup \overline{r}$$

gives a disjoint decomposition of the set h such that $c(v) = r$ for each $v \in \overline{r}$ and $c(v) = g$ for each $v \in \overline{g}$ and $|\overline{r}| \geq 1$, then let $w(h, c) := 2^{|\overline{r}|-r}$.

4. Similarly, if

$$h = \overline{g} \cup \overline{b}$$

gives a disjoint decomposition of the set h such that $c(v) = b$ for each $v \in \overline{b}$ and $c(v) = g$ for each $v \in \overline{g}$ and $|\overline{b}| \geq 1$, then let $w(h, c) := 2^{|\overline{b}|-r}$.

Define

$$w(\mathcal{H}, c) := \sum_{h\in\mathcal{H}} w(h, c). \tag{2}$$

Let $G : V \to \{r, b, g\}$ denote the 3-coloring of the set V for which $G(v) = g$ for each $v \in V$.

By the definition of the weight function w

$$w(\mathcal{H}, G) = |\mathcal{H}| \cdot 2^{1-r}. \tag{3}$$

Proposition 2.1 *Let $c : V \to \{r, b, g\}$ be a fixed 3-coloring and $v \in V$ be an arbitrary point such that $c(v)$ is green. Then let c_{red} denote the following coloring: $c_{red}(v) := r$ and $c_{red}(v') := c(v')$ for each $v' \in V \setminus \{v\}$. Similarly, let c_{blue} denote the following coloring: $c_{blue}(v) := b$ and $c_{blue}(v') := c(v')$ for each $v' \in V \setminus \{v\}$.*

Then

$$w(\mathcal{H}, c) = \frac{w(\mathcal{H}, c_{red}) + w(\mathcal{H}, c_{blue})}{2}. \tag{4}$$

Proof.

It is enough to prove that

$$w(h, c) = \frac{w(h, c_{red}) + w(h, c_{blue})}{2} \tag{5}$$

for each $h \in \mathcal{H}$.

Let $h \in \mathcal{H}$ be a fixed edge.

1. Suppose that $v \notin h$. Then

$$w(h, c_{red}) = w(h, c_{blue}) = w(h, c).$$

We can assume in the following that $v \in h$.

2. Suppose that for each $v' \in h$, $c(v') = g$. Then

$$w(h, c_{red}) = w(h, c_{blue}) = w(h, c) = 2^{1-r}.$$

3. Suppose that there exist $v_1, v_2 \in h$, $v_1 \neq v_2$ such that $c(v_1) = r$ and $c(v_2) = b$. Then

$$w(h, c) = w(h, c_{blue}) = w(h, c_{red}) = 0.$$

4. Suppose that

$$h = \overline{r} \cup \overline{g} \tag{6}$$

gives a disjoint decomposition of the set h such that $c(v') = r$ for each $v' \in \overline{r}$ and $c(v') = g$ for each $v' \in \overline{g}$. Let $d := |\overline{r}| \geq 1$. Then

$$w(h, c) = 2^{d-r},$$

$$w(h, c_{blue}) = 0$$

and

$$w(h, c_{red}) = 2^{d-r+1}.$$

5. Finally, suppose that

$$h = \bar{b} \cup \bar{g} \tag{7}$$

gives a disjoint decomposition of the set h such that $c(v') = b$ for each $v' \in \bar{b}$ and $c(v') = g$ for each $v' \in \bar{g}$. Let $d := |\bar{b}| \geq 1$. Then

$$w(h, c) = 2^{d-r},$$

$$w(h, c_{blue}) = 2^{d-r+1}$$

and

$$w(h, c_{red}) = 0.$$

□

Our algorithm proceeds as follows: First order arbitrarily the points of $v \in V$: $v_1, \ldots, v_{|V|}$. For each $0 \leq i \leq |V|$ we define a 3-coloring $c_i : V \to \{r, b, g\}$.

The initial step: let $c_0 := G$, where $G(v) := g$ for each $v \in V$.

Let $1 \leq i \leq |V|$ be fixed. Suppose that we have defined a coloring $c_{i-1} : V \to \{r, b, g\}$ such that $c_{i-1}(v_j) \in \{r, b\}$ for each $1 \leq j \leq i - 1 < |V|$ and $c_{i-1}(v_k) = g$ for each $k \geq i$. Now our aim is to define a new coloring $c_i : V \to \{r, b, g\}$ such that $c_i(v_j) \in \{r, b\}$ for each $1 \leq j \leq i$ and $c_i(v_k) = g$ for each $k > i$.

Define the colorings c_{red} and c_{blue} as in Proposition 2.1. Proposition 2.1 implies that

$$w(\mathcal{H}, c) = \frac{w(\mathcal{H}, c_{red}) + w(\mathcal{H}, c_{blue})}{2}. \tag{8}$$

Hence either $w(\mathcal{H}, c_{red}) \leq w(\mathcal{H}, c)$ or $w(\mathcal{H}, c_{blue}) < w(\mathcal{H}, c)$.

First suppose that $w(\mathcal{H}, c_{red}) \leq w(\mathcal{H}, c)$. Then define $c_i := c_{red}$.

Now suppose that $w(\mathcal{H}, c_{blue}) < w(\mathcal{H}, c)$. Then let $c_i = c_{blue}$.

Clearly $c_i(v_j) \in \{r, b\}$ for each $1 \leq j \leq i$ and $c_i(v_k) = g$ for each $k > i$.

This algorithm yields finally to a 2-coloring $C := c_{|V|} : V \to \{r, b\}$. We claim, that if we color the points of the base set $V(\mathcal{H})$ with this coloring C, then we get at most $2^{1-r} \cdot |\mathcal{H}|$ monochromatic edges.

Namely by the definition of the weight function $w(\mathcal{H}, C)$,

$$w(\mathcal{H}, C) = |\{h \in \mathcal{H} : \ h \text{ is a monochromatic edge in the coloring } C\}|. \tag{9}$$

This is clear, since

$$w(\mathcal{H}, C) = \sum_{h \in \mathcal{H}} w(h, C)$$

and $w(h, C) = 1$, if h is a monochromatic edge, $w(h, C) = 0$ otherwise, because C was a 2-coloring.

It is easy to verify from the definition of the coloring $c_i : V \to \{r, b, g\}$ that

$$w(\mathcal{H}, c_i) \leq w(\mathcal{H}, c_{i-1})$$

for each $1 \leq i \leq |V|$, hence

$$w(\mathcal{H}, C) = w(\mathcal{H}, c_{|V|}) \leq w(\mathcal{H}, c_0) = w(\mathcal{H}, G). \tag{10}$$

We proved in (3) that

$$w(\mathcal{H}, G) = |\mathcal{H}| \cdot 2^{1-r}. \tag{11}$$

The equations (9), (10) and (11) proves the Theorem. □

3 Application

Given an r-uniform hypergraph $\mathcal{F}$, the Turán number of $\mathcal{F}$ is the maximum number of edges in an r-uniform hypergraph on n vertices that do not contain a copy of $\mathcal{F}$. We denote this number by $\mathrm{ex}(n, \mathcal{F})$. It is not hard to show that the limit $\pi(\mathcal{F}) = \lim_{n\to\infty} \mathrm{ex}(n, \mathcal{F})/\binom{n}{r}$ exists. It is usually called the *Turán density* of $\mathcal{F}$.

A general upper bound on Turán densities was obtained by de Caen [1], who showed

$$\pi(K_s^{(r)}) \leq 1 - \binom{s-1}{r-1}^{-1},$$

where $K_s^{(r)}$ denotes the complete r-uniform hypergraph on s vertices. Sidorenko gave a construction in [10] (see also [11]) showing that $\pi(K_s^{(r)}) \leq 1 - (\frac{r-1}{s-1})^{r-1}$. For a general hypergraph Sidorenko [8] obtained a bound for the Turán density in terms of the number of edges.

Theorem 3.1 *Let $\mathcal{F}$ be an arbitrary hypergraph, which has f edges. Then*

$$\pi(\mathcal{F}) \leq \frac{f-2}{f-1}. \tag{12}$$

Proof of Theorem 1.2:

First we prove the upper bound.

P. Erdős proved in [2] that there exists a $\mathcal{H} \subseteq \binom{[n]}{r}$ r-uniform hypergraph with

$$|\mathcal{H}| = (1 + o(1))\frac{e \cdot \ln 2}{4}r^2 2^r$$

and $\chi(\mathcal{H}) > 2$. Let $\mathcal{G}$ be an arbitrary r-uniform hypergraph with more than $(1 - \frac{1}{r^2 \cdot 2^r - 1})\binom{n}{r}$ edges. We can apply Theorem 3.1 for $\mathcal{F} := \mathcal{H}$ and $f := (1 - \frac{1}{r^2 \cdot 2^r})\binom{n}{r}$. We get that there exists a copy of $\mathcal{H}$ in $\mathcal{G}$, which shows that $\mathcal{G}$ is not 2-colorable.

Now we prove the lower bound.

By Theorem 1.1 there exists a 2-coloring $f : \binom{[n]}{r} \to \{r, b\}$ of the complete r-uniform hypergraph $\binom{[n]}{r}$ with at most $2^{1-r}\binom{n}{r}$ monochromatic edges. Fix such a coloring. Remove from this colored hypergraph all monochromatic edges. Then this coloring shows that after the removing we get a 2-colorable hypergraph with at least $\binom{n}{r}(1 - 2^{1-r})$ edges. $\square$

The algorithmic proof of Theorem 1.1 can be used to prove the following more general result.

Theorem 3.2 *Let $\mathcal{H}$ be an arbitrary r-uniform hypergraph. Then there exists an s-coloring of $\mathcal{H}$ with at most $|\mathcal{H}| \cdot s^{1-r}$ monochromatic edges.* $\square$

The modification of the proof of the lower bound in (1) gives the following Corollary.

Corollary 3.3 *There exists an r-uniform s-colorable hypergraph $\mathcal{H}$ on n points which has at least $(1 - s^{1-r})\binom{n}{r}$ edges.* $\square$

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