Matchings and non-rainbow colorings

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Abstract

We show that the maximum number of colors that can be used in a vertex coloring of a cubic 3-connected plane graph G that avoids a face with vertices of mutually distinct colors (a rainbow face) is equal to $\frac{n}{2} + \mu^* - 2$ where n is the number of vertices of G and μ^* is the size of the maximum matching of the dual graph G^*.

1 Introduction

Colorings of embedded graphs with face-constraints have recently drawn attention of several groups of researchers. The very first question that comes to one's mind in this area is the following:

Question 1. *What is the minimal number of colors needed to color an embedded graph in such a way that each of its faces is incident with vertices of at least two different colors, i.e., there is no monochromatic face?*

This problem can be found in work of Zykov [21] who studied the notion of *planar hypergraphs* and was further explored by Kündgen and Ramamurthi [14] for hypergraphs arising from graphs embedded in surfaces of higher genera. As an example of results obtained in this area, let us mention

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that every graph embedded on a surface of genus ε has a coloring with $O(\sqrt[3]{\varepsilon})$ colors [5] that avoids a monochromatic face.

An opposite type of question, motivated by results of anti-Ramsey theory, is the following:

Question 2. *What is the maximal number $\chi_f(G)$ of colors that can be used in a coloring of an embedded graph G with no rainbow face, i.e., a face with vertices of mutually distinct colors?*

In our further considerations, we call a vertex coloring of G with no rainbow face a *non-rainbow coloring* of G. Notice that, unlike in the case of ordinary colorings, the goal in this scenario is to *maximize* the number of used colors. Though it may take some time to digest the concept, the setting is so natural that it has recently appeared independently in papers of Ramamurthi and West [19] and of Negami [16] (see also [1, 2, 15] for some even earlier results of this favor). In fact, Negami addressed the following extremal-type question (equivalent to Question 2):

Question 3. *What is the smallest number $k(G)$ of colors such that every vertex-coloring of an embedded graph G with $k(G)$ colors contains a rainbow face?*

It is not hard to see that $\chi_f(G) = k(G) - 1$ and the results obtained in either of the scenarios translate smoothly to the other one.

We now briefly survey results obtained in the direction of Questions 2 and 3 for planar graphs. Ramamurthi and West [18] noticed that every plane graph G has a non-rainbow coloring with at least $\alpha(G) + 1$ colors, in particular, every plane graph G of order n has a coloring with at least $\lceil \frac{n}{4} \rceil + 1$ colors by the Four Color Theorem. Also, Grötzsch's theorem [7, 20] implies that every triangle-free plane graph has a non-rainbow coloring with $\lceil \frac{n}{3} \rceil + 1$ colors. It was conjectured [18] that this bound can be improved to $\lceil \frac{n}{2} \rceil + 1$. Partial results on this conjecture were obtained in [12] and the conjecture has eventually been proven in [10]. More generally, Jungić et al. [10] proved that every planar graph of order n with girth $g \geq 5$ has a non-rainbow coloring with at least $\left\lceil \frac{g-3}{g-2}n - \frac{g-7}{2(g-2)} \right\rceil$ colors if g is odd, and $\left\lceil \frac{g-3}{g-2}n - \frac{g-6}{2(g-2)} \right\rceil$ colors if g is even. All these bounds are the best possible.

Complementary to the lower bounds on $\chi_f(G)$ presented in the previous paragraph, there are also results on upper bounds on $\chi_f(G)$. Negami [16] investigated non-rainbow colorings of plane triangulations G and showed that

$\alpha(G)+1 \leq \chi_f(G) \leq 2\alpha(G)$. In [6], it was shown that $\chi_f(G) \leq \left\lfloor \frac{7n-8}{9} \right\rfloor$ for n-vertex 3-connected plane graphs G, $\chi_f(G) \leq \left\lfloor \frac{5n-6}{8} \right\rfloor$ if $n \not\equiv 3 \pmod 8$ and $\chi_f(G) \leq \left\lfloor \frac{5n-6}{8} \right\rfloor - 1$ if $n \equiv 3 \pmod 8$ for 4-connected plane graphs G, and $\chi_f(G) \leq \left\lfloor \frac{43}{100}n - \frac{19}{25} \right\rfloor$ for 5-connected plane graphs G. The bounds for 3- and 4-connected graphs are the best possible.

Besides results on non-rainbow colorings of graphs with no short cycles and non-trivially connected plane graphs, there are also results on specific families on plane graphs, e.g., the numbers $\chi_f(G)$ were also determined for all semiregular polyhedra [9].

Let us mention that there are also results on mixed types of colorings in which we require that there is neither a monochromatic nor a rainbow face, e.g., [4, 11, 13]. For instance, it is known that each plane graph with at least five vertices has a coloring with two colors as well as a coloring with three colors that avoid both monochromatic and rainbow faces [3, 17].

The quantity $\chi_f(G)$ is also related to several parameters of the dual graph of G. In particular, $\frac{n}{2} + \mu^* - 2 \leq \chi_f(G) \leq n - \alpha^*$ for connected cubic plane graphs G [8] where α^* is the independence number of the dual graph G^* of G and μ^* is the size of the largest matching of G^*. In fact, it was conjectured that the first inequality is always an equality if G is 3-connected:

Conjecture 1. *The maximum number of colors used in a non-rainbow coloring of a cubic* 3*-connected plane graph* G *is related to the size of a maximum matching of its dual as follows:*

$$\chi_f(G) = \frac{n}{2} + \mu^* - 2 \ .$$

We prove this conjecture. In our view, the fact that $\chi_f(G)$ only depends on the size of the largest matching of G^* in this specific case is quite surprising and deserves further investigation in more general setting.

As the first step towards proving the conjecture, we establish a lemma that guarantees the existence of a matching of large size in the dual of a 3-connected cubic plane graph (Lemma 2). With the help of this lemma, our main result is proven in Section 3. At the end of the paper, we briefly discuss generalizations and extensions of our results to cubic plane graph that need not to be 3-connected. In particular, we show that the assumption that G is 3-connected cannot be relaxed.

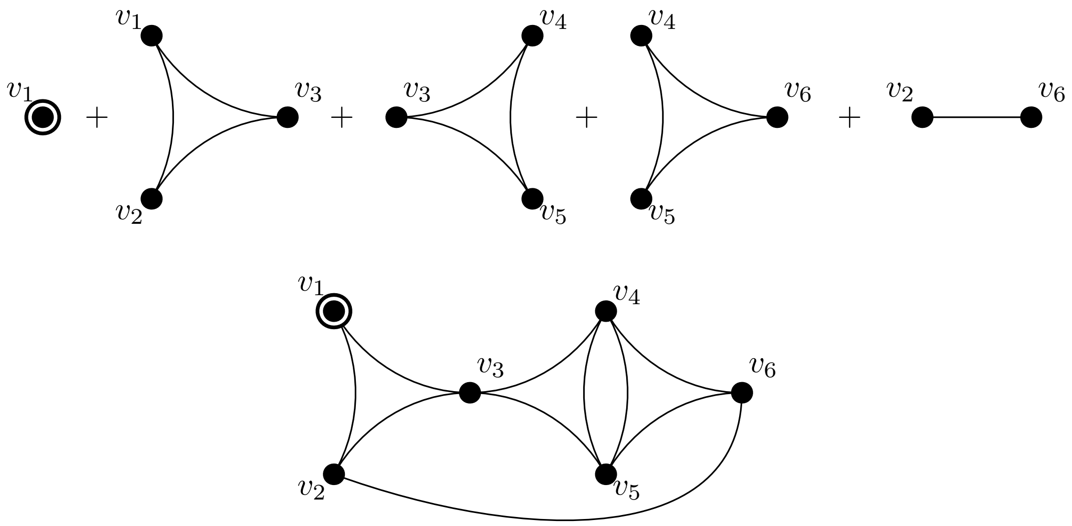

Figure 1: An example of a bigraph. The bigraph is formed by one singleton, one pair and three triples.

2 Auxiliary lemmas

In this section, we first introduce a special type of multigraphs that we call bigraphs and prove a lemma on the size of inclusion-wise maximal matchings in bigraphs. We then use this lemma to establish the existence of a large matchings in the dual of a cubic plane graph.

A *bigraph* is a connected graph that is obtained as a union of complete graphs of order one, two and three such that each vertex is contained in exactly two of the complete graphs (see Figure 1 for an example). We keep multiple edges present in a bigraph though replacing them with single edges would also be fine in our further considerations. The complete graphs forming a bigraph are referred as *singletons*, *pairs* and *triples* depending on their sizes. The bigraph obtained as a union of two triples turns out to be exceptional in our considerations: it is denoted by K_3^2 and can be found in Figure 2. A bigraph formed by α singletons, β pairs and γ triples, is an (α, β, γ)-*bigraph*. Observe that an (α, β, γ)-bigraph has $(\alpha + 2\beta + 3\gamma)/2$ vertices and $\beta + 3\gamma$ edges.

A *matching* of a bigraph is a collection of vertex-disjoint edges. A singleton, a pair or a triple of a bigraph is *covered* by a matching if all its vertices are incident with edges contained in the matching. The following lemma asserts that each bigraph contains a matching covering some of its single-

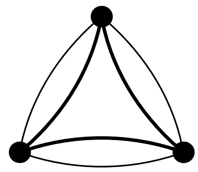

Figure 2: The bigraph K_3^2.

tons, pairs or triples. The proof of the lemma is quite straightforward but it involves distinguishing several cases.

Lemma 1. *Let G be an (α, β, γ)-bigraph. If $G \neq K_3^2$, G contains an inclusion-wise maximal matching M that covers α' singletons, β' pairs and γ' triples such that*

$$2\alpha + \beta + \alpha' + \beta' + \gamma' \geq 4 \ .$$

Proof. We distinguish several cases based on the values of α and β:

- $\alpha = 0$, $\beta = 0$ and every two triples share at most one vertex.
  Assume first that there are three triples T_1, T_2 and T_3 such that any two of them share a vertex, i.e., $T_i \cap T_j \neq \emptyset$ for all $i \neq j$. Let v_{ij} be the vertex of $T_i \cap T_j$, $i < j$, and let v_i be the remaining vertex of T_i, $i = 1, 2, 3$—see Figure 3. Finally, let T_0 be the other triple containing the vertex v_1 and w_1 and w_2 the vertices of T_0 different from v_1. If $\{w_1, w_2\} = \{v_2, v_3\}$, consider an inclusion-wise maximal matching M that contains the edges v_1v_{12}, $v_{13}v_{23}$ and v_2v_3. Since all the four triples T_0, T_1, T_2 and T_3 are covered by M, we obtain that $\gamma' \geq 4$ and the statement of the lemma now follows.

  If $\{w_1, w_2\} \cap \{v_2, v_3\} = \emptyset$, then we consider an inclusion-wise maximal matching M that contains the edges $v_{12}v_1$, $v_{23}v_2$, $v_{13}v_3$ and w_1w_2. Since M covers all the four triples T_0, T_1, T_2 and T_3, it holds that $\gamma' \geq 4$ and the inequality of the statement of the lemma is satisfied.

  If $|\{w_1, w_2\} \cap \{v_2, v_3\}| = 1$, we can assume by symmetry that $w_2 = v_3$ and $w_1 \neq v_2$. Let T_0' be the triple different from T_0 that contains the vertex w_1 and let w_1' and w_2' be the remaining vertices of T_0'. Since all the vertices v_1, $v_3 = w_2$, v_{12}, v_{13}, v_{23} and w_1 are contained in two of the triples T_0, T_0', T_1, T_2 and T_3, at least one of the vertices w_1' and w_2' is different from all these vertices as well as from v_2. By symmetry,

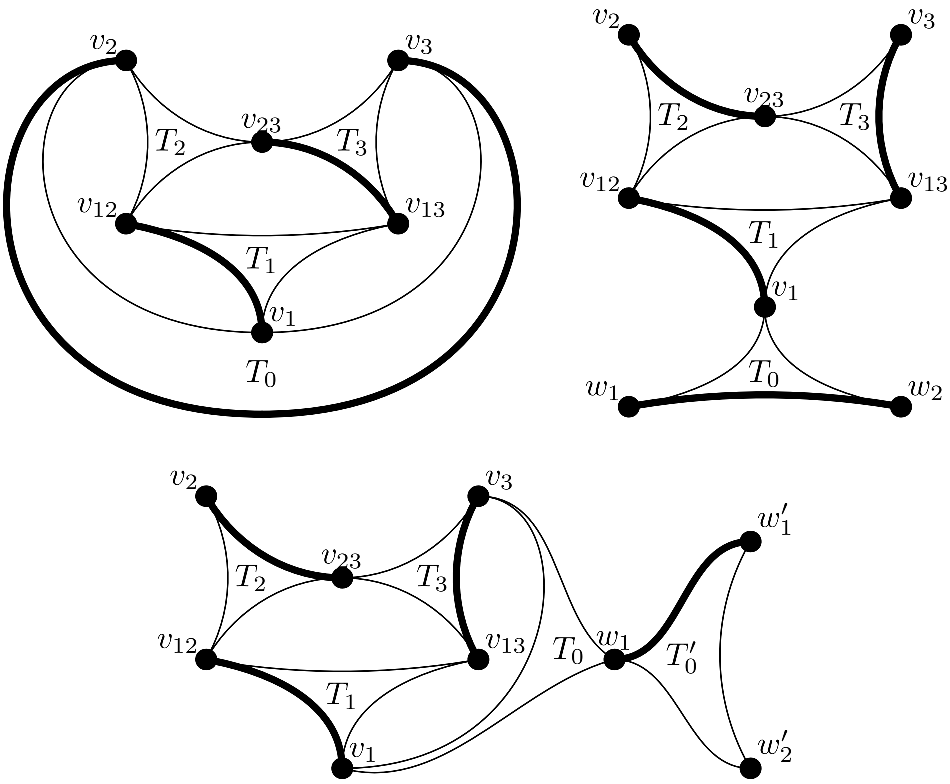

Figure 3: Possible configurations in the case that $\alpha = 0$, $\beta = 0$, there are no two triples sharing two vertices and there are three triples such that any two of them share a vertex.

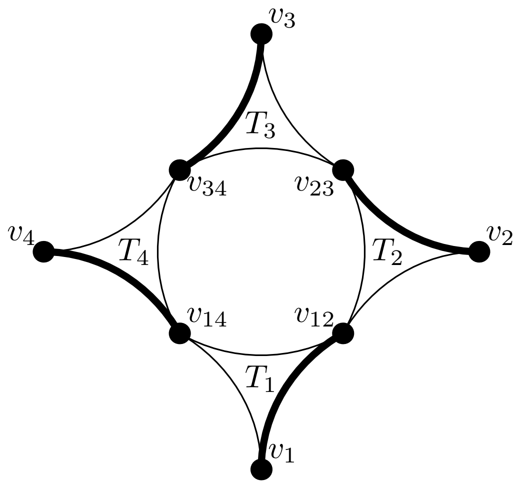

Figure 4: The configuration in the case that $\alpha = 0$, $\beta = 0$, there are no two triples sharing two vertices, there are no three triples such that any two of them share a vertex, and there are four triples (cyclically ordered) such that pairs of consecutive triples share a vertex.

we can assume that $w_1' \neq v_2$. Let us now consider an inclusion-wise maximal matching M that contains the edges $v_{12}v_1$, $v_{23}v_2$, $v_{13}v_3$ and w_1w_1'. Such a matching M covers all the triples T_0, T_1, T_2 and T_3, and thus $\gamma' \geq 4$. The inequality of the statement readily follows. This finishes the analysis of the case that there are three triples such that any two of them share a vertex. Hence, we assume in the rest that there are no three such triples.

Let us now assume that there are four triples T_1, T_2, T_3 and T_4 such that there exist $v_{12} \in T_1 \cap T_2$, $v_{23} \in T_2 \cap T_3$, $v_{34} \in T_3 \cap T_4$ and $v_{14} \in T_4 \cap T_1$ (see Figure 4). In addition, let v_i be the vertex of T_i, $i = 1, \ldots, 4$, different from the vertices v_{12}, v_{23}, v_{34} and v_{14}. The vertices v_1, v_2, v_3 and v_4 are mutually distinct—indeed, if $v_1 = v_2$, then the triples T_1 and T_2 share two vertices which is excluded by one of our assumptions. If $v_1 = v_3$, then any two of the triples T_1, T_2 and T_3 share a vertex which is also excluded by our assumptions. The other cases are symmetric. Let us now consider an inclusion-wise maximal matching M that contains the edges v_1v_{12}, v_2v_{23}, v_3v_{34} and v_4v_{14}. Since such a matching M covers all the triples T_1, T_2, T_3 and T_4, it holds that $\gamma' \geq 4$ and the inequality is satisfied. Hence, we can further assume that there are no four triples with the above property.

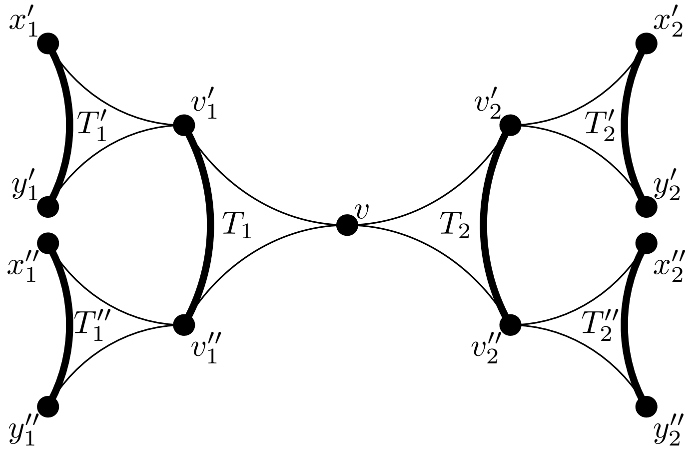

Figure 5: The configuration in the case that $\alpha = 0$, $\beta = 0$, there are no two triples sharing two vertices, there are no three triples such that any two of them share a vertex, and there are no four triples (cyclically ordered) such that each pair of consecutive triples share a vertex.

Consider now any two triples T_1 and T_2 that share a vertex. Let v be the vertex that they share, v_1' and v_1'' the other vertices of T_1, and v_2' and v_2'' the other vertices of T_2. Let T_i' be the other triple containing the vertex v_i' and T_i'' the other triple containing the vertex v_i'', $i = 1, 2$. Finally, let x_i' and y_i' be the other vertices of T_i', and let x_i'' and y_i'' be the other vertices of T_i'', $i = 1, 2$. See Figure 5 for illustration of our notation. By our assumptions, all the vertices v, v_1', v_1'', v_2', v_2'', x_1', x_1'', x_2', x_2'', y_1', y_1'', y_2' and y_2'' are mutually distinct since otherwise we would have three or four triples with the properties described before. Consider now an inclusion-wise maximal matching M that contains the edges $v_1'v_1''$, $v_2'v_2''$, $x_1'y_1'$, $x_1''y_1''$, $x_2'y_2'$ and $x_2''y_2''$. Such a matching M covers all the triples T_1', T_1'', T_2' and T_2''. Hence, $\gamma' \geq 4$. We conclude that the matching M satisfies the required inequality.

- $\alpha = 0$, $\beta = 0$ and there exist two triples that share two vertices.
  Let T_1 and T_2 be two triples sharing two vertices, let v_1 and v_2 be the two vertices they share and let w_1 and w_2 be the other vertices of T_1 and T_2, respectively. Note that $w_1 \neq w_2$ since $G \neq K_3^2$. Let T_1' and T_2' be the other triples that contain the vertices w_1 and w_2, respectively (see Figures 6 and 7).

  First assume that $T_1' = T_2'$. Let w be the other vertex contained in

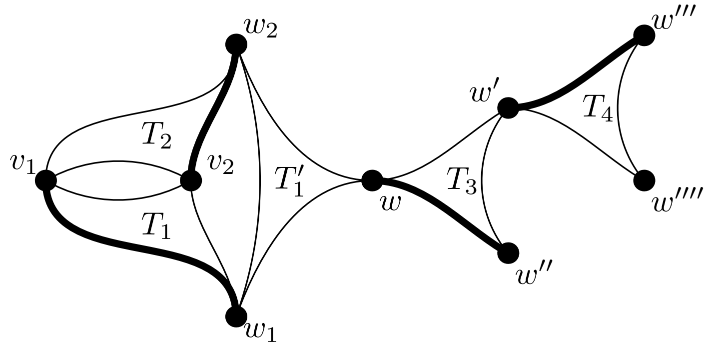

Figure 6: The configuration in the case that $\alpha = 0$, $\beta = 0$ and there are two triples sharing two vertices which share vertices with the same (third) triple.

$T'_1 = T'_2$ and T_3 the other triple containing w. Finally, let w' and w'' be the remaining vertices of T_3, T_4 the triple different from T_3 that contains w'. Finally, let w''' and w'''' be the remaining vertices of T_4. Clearly, both w''' and w'''' are different from the vertices v_1, v_2, w_1, w_2 and w. By symmetry, we can assume that $w''' \neq w''$. Consider now an inclusion-wise maximal matching M that contains the edges v_1w_1, v_2w_2, ww'' and $w'w'''$. Since $\gamma' \geq 4$ for such a matching M (M covers the triples T_1, T_2, $T'_1 = T'_2$ and T_3), the inequality holds for M.

Next, we assume that $T'_1 \neq T'_2$. Let w'_1 and w''_1 be the other vertices of T'_1, and w'_2 and w''_2 the other vertices of T'_2 (see Figure 7). If $\{w'_1, w''_1\} \cap \{w'_2, w''_2\} = \emptyset$, consider an inclusion-wise maximal matching M that contains the edges v_1w_1, v_2w_2, $w'_1w''_1$ and $w'_2w''_2$. If $\{w'_1, w''_1\} = \{w'_2, w''_2\}$, consider an inclusion-wise maximal matching M that contains the edges v_1w_1, v_2w_2 and $w'_1w''_1 = w'_2w''_2$. In both the cases, the triples T_1, T_2, T'_1 and T'_2 are covered and thus $\gamma' \geq 4$. It follows that the inequality holds.

It remains to consider the case $|\{w'_1, w''_1\} \cap \{w'_2, w''_2\}| = 1$. By symmetry, we can assume that $w''_1 = w''_2$ and $w'_1 \neq w'_2$. Let T_3 be the other triple containing w'_1 and w' and w'' the other vertices of T_3. Both w' and w'' are clearly different from the vertices v_1, v_2, w_1, w_2 and $w''_1 = w''_2$. By symmetry, we can also assume that $w' \neq w'_2$. Let us consider an inclusion-wise maximal matching M that contains the edges v_1w_1, v_2w_2, w'_1w' and $w'_2w''_1 = w'_2w''_2$. Such a matching M covers the triples T_1, T_2, T'_1 and T'_2 and thus $\gamma' \geq 4$. The inequality from the statement of the

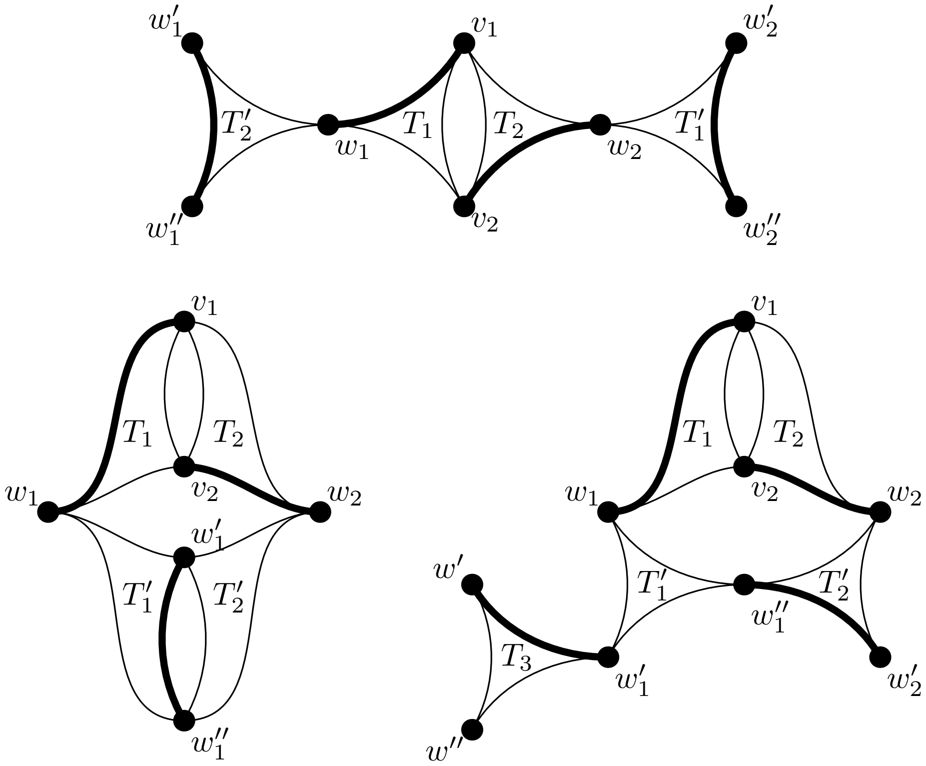

Figure 7: The configurations in the case that $\alpha = 0$, $\beta = 0$ and there are two triples sharing two vertices which do not share vertices with the same (third) triple.

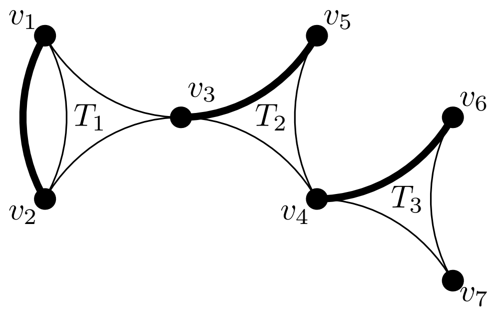

Figure 8: The configuration in the case $\alpha = 0$, $\beta = 1$ and the vertices of the pair are contained in the same triple.

lemma now follows.

- $\alpha = 0$ and $\beta = 1$
  Let v_1 and v_2 be the vertices contained in the pair. Assume first that there is a triple T_1 that contains both v_1 and v_2, and let v_3 be the other vertex contained in T_1 (see Figure 8). Since $\alpha = 0$ and $\beta = 1$, v_3 is contained in another triple, say T_2, and let v_4 and v_5 be the remaining vertices of T_2. Finally, let T_3 be the other triple that contains v_4 and v_6 and v_7 its vertices different from v_4. Clearly, both v_6 and v_7 are different from v_1, v_2 and v_3, and, by symmetry, we can assume that $v_6 \neq v_5$. Consider now an inclusion-wise maximal matching M that contains the edges v_1v_2, v_3v_5 and v_4v_6. Clearly, $\beta' = 1$ and $\gamma' \geq 2$ for such a matching M and the inequality from the statement of the lemma holds.

  The other case is that no triple contains both v_1 and v_2. Let T_1 and T_2 be the triples containing v_1 and v_2, respectively. Let v_1' and v_1'' be the other vertices of T_1 and v_2' and v_2'' the other vertices of T_2 (see Figure 9). If $\{v_1', v_1''\} \cap \{v_2', v_2''\} = \emptyset$, consider an inclusion-wise maximal matching M that contains the edges v_1v_2, $v_1'v_1''$ and $v_2'v_2''$. Since $\beta' = 1$ and $\gamma' \geq 2$ for such a matching M, the inequality from the statement of the lemma holds. If $\{v_1', v_1''\} = \{v_2', v_2''\}$, consider an inclusion-wise maximal matching M that contains the edges v_1v_2 and $v_1'v_1'' = v_2'v_2''$. Since $\beta' = 1$ and $\gamma' \geq 2$ for such a matching M, the inequality also holds. Hence, we can assume that $v_1' = v_2'$ and $v_1'' \neq v_2''$ in the rest.

  Let T_3 be the other triple that contains the vertex v_1'' and w_1 and w_2 the other vertices contained in T_3. Clearly, both w_1 and w_2 are

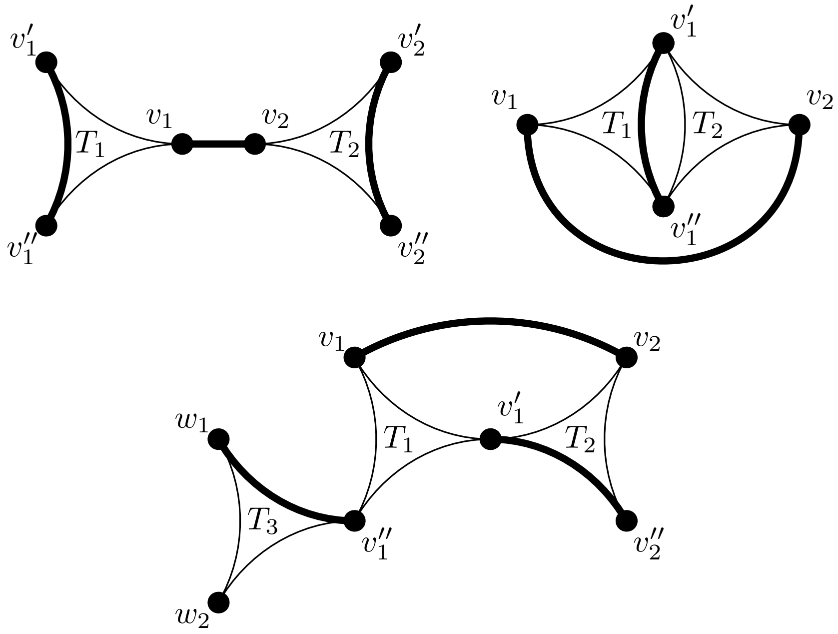

Figure 9: The configurations in the case $\alpha = 0$, $\beta = 1$ and the vertices of the pair are not contained together in any of the triples.

different from v_1, v_2 and $v_1' = v_2'$. By symmetry, we can assume that $w_1 \neq v_2''$. Let us now consider an inclusion-wise maximal matching M that contains the edges v_1v_2, $v_1'v_2'' = v_2'v_2''$ and $v_1''w_1$. Since $\beta' = 1$ and $\gamma' \geq 2$ for such a matching M, the inequality from the statement of the lemma holds.

- $\alpha = 0$ and $\beta = 2$

  If the two pairs are disjoint, consider an inclusion-wise maximal matching M that contains the edges corresponding to the pairs. Since $\beta' \geq 2$ for such a matching M, the matching M satisfies the inequality from the statement of the lemma.

  Next, we assume that the two pairs are not disjoint. If they completely coincide, then G is a bigraph formed by two parallel edges and a matching M containing one of the two edges is inclusion-wise maximal with $\beta' = 2$. In particular, M satisfies the inequality from the statement.

  Hence, it remains to consider the case that the two pairs share a single vertex: let v_1, v_2 and v_3 be the vertices such that v_1 and v_2 form one of the pairs and v_2 and v_3 form the other pair. Since $\alpha = 0$, the vertex v_3 is contained in a triple. Let w_1 and w_2 be the other vertices of the

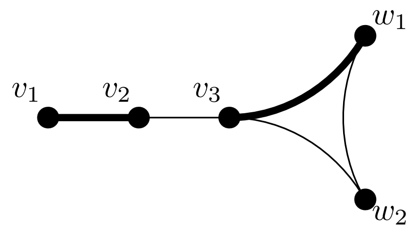

Figure 10: The last configuration considered in the case $\alpha = 0$ and $\beta = 2$.

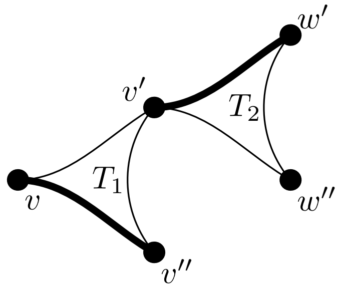

Figure 11: The configuration considered in the case $\alpha = 1$ and $\beta = 0$.

triple (see Figure 10). Clearly, both w_1 and w_2 are different from v_2. By symmetry, we can assume that $w_1 \neq v_1$. Let M be an inclusion-wise maximal matching that contains the edges v_1v_2 and v_3w_1. Since M covers both the pairs, $\beta' = 2$ and M satisfies the inequality from the statement of the lemma.

- $\alpha = 0$ and $\beta \geq 3$
  Let v and w be the vertices that form one of the pairs. If M is an inclusion-wise maximal matching that contains the edge vw, then $\beta' \geq 1$ and M satisfies the inequality.

- $\alpha = 1$ and $\beta = 0$
  Let v be the vertex of G contained in the singleton. Since G contains only one singleton and no pairs, v is also contained in a triple, say T_1 (see Figure 11). Let v' and v'' be the other vertices of T_1, T_2 the triple different from T_1 that contains v', and w' and w'' the remaining vertices of T_2. Clearly, both w' and w'' are different from v. By the symmetry, we assume that $w' \neq v''$.

  Consider now an inclusion-wise maximal matching M that contains the edges vv'' and $v'w'$. Since M covers the singleton formed by v and the triple T_1, it holds that $\alpha' = 1$ and $\gamma' \geq 1$. Hence, the inequality from the statement of the lemma holds.

- $\alpha = 1$ and $\beta \geq 1$
  Let v be the vertex of G contained in the singleton. Consider any inclusion-wise maximal matching M that contains an edge incident with v. Such a matching M clearly covers the singleton formed by v. Hence, $\alpha' \geq 1$ and the inequality from the statement of the lemma holds.

- $\alpha \geq 2$
  Any inclusion-wise maximal matching M has the property from the statement since $2\alpha \geq 4$.

□

We now derive from Lemma 1 the following lemma that relates the existence of matchings in the dual graph to the existence of certain vertex subsets of the original plane graph. Before stating the lemma, we need to introduce the following notation: if G is a plane graph and F is a set of its faces, then $G^*[F]$ is the subgraph of the dual graph G^* induced by the vertices corresponding to the faces of F.

Lemma 2. *Let G be a plane 3-connected cubic graph. If A is a non-empty subset of its vertices and F a (possibly empty) subset of its faces such that each face $f \in F$ is incident with at least two vertices of A, then the graph $G^*[F]$ contains a matching of size at least $|F| - |A| + 1$.*

Proof. If $F = \emptyset$, the statement of the lemma trivially holds. Hence, we assume $F \neq \emptyset$ in the remaining. For each face $f \in F$, choose arbitrarily two vertices v_f and v'_f that are contained in A and that are incident with f. These vertices are further called *representatives* of f. Observe that each vertex is a representative of at most three faces since G is cubic. Let further α be the number of vertices that are representatives of a single face, β the number of vertices that are representatives of two faces and γ the number of vertices that are representatives of three faces. Clearly, $\alpha + \beta + \gamma \leq |A|$.

We now construct a bigraph H whose vertices correspond to faces of F as follows: for each vertex v that is a representative of three faces, say f_1, f_2 and f_3, include the triple formed by f_1, f_2 and f_3 to H. For each vertex v that is a representative of two faces, say f_1 and f_2, include the pair formed by f_1 and f_2 to H. Finally, for each vertex v that is a representative of a single face f_1, include the singleton formed by f_1 to H. We do not include anything to H for those vertices that are representatives of none of the faces. Since each face has exactly two representatives, the graph constructed in this way is an (α, β, γ)-bigraph.

If $H = K_3^2$, then G would contain vertices v' and v'' such that the three faces incident with v' are the same faces as those incident with v''. However, v' and v'' would then form a vertex cut of G which is impossible since G is 3-connected. We infer that $H \neq K_2^3$. By Lemma 1, H has an inclusion-wise maximal matching that covers α' singletons, β' pairs and γ' of its triples such that

$$2\alpha + \beta + \alpha' + \beta' + \gamma' \geq 4 \tag{1}$$

Note that the matching M of H corresponds to a matching of $G^*[F]$ of the same size.

Let us now estimate the size of M. Clearly, any inclusion-wise maximal matching covers at least one vertex of each pair and at least two vertices of each triple (otherwise, we could add another edge to M). Hence, the number of vertices of H covered by M is

$$\frac{1}{2}\left(\alpha' + \beta + \beta' + 2\gamma + \gamma'\right) .$$

Note that we have to divide the above sum by two since each vertex of H is contained in exactly two of its singletons, pairs and triples. Consequently, the size of M is

$$\frac{1}{4}\left(\alpha' + \beta + \beta' + 2\gamma + \gamma'\right) .$$

Since the number of vertices of A is at least $\alpha + \beta + \gamma$ and the size of F is $\frac{1}{2}(\alpha + 2\beta + 3\gamma)$, we infer the following bound on $|F| - |A| + 1$:

$$|F| - |A| + 1 \leq \frac{1}{2}\left(\alpha + 2\beta + 3\gamma\right) - \alpha - \beta - \gamma + 1 \leq$$

$$\frac{1}{2}\left(\alpha + 2\beta + 3\gamma\right) - \alpha - \beta - \gamma + \frac{1}{4}\left(2\alpha + \beta + \alpha' + \beta' + \gamma'\right) =$$

$$\frac{1}{4}\left(\alpha' + \beta + \beta' + 2\gamma + \gamma'\right) = |M| .$$

Note that we have applied inequality (1) at the second step. The lemma now follows. $\square$

3 The formula

In this section, we prove our main result:

Theorem 3. *If G is a 3-connected cubic graph with n vertices and μ^* is the size of the maximum matching of G^*, then the following holds:*

$$\chi_f(G) = \frac{n}{2} + \mu^* - 2 \ .$$

Proof. We first prove that $\chi_f(G) \geq \frac{n}{2} + \mu^* - 2$. This inequality was proven in [8] but we decided to include its proof for the sake of completeness. Let M be a maximum matching of G^* and let E be the edges of G corresponding to those of M. Finally, let G_E be the subgraph of G with $V(G_E) = V(G)$ and $E(G_E) = E$. Clearly, G_E has at least $n - \mu^*$ components. Color now the vertices of each of the components of G_E with the same color and vertices of distinct components with distinct colors. Observe that all the faces of G covered by M are not rainbow. For each face f of G that is not covered by M choose any two vertices v and w incident with f and if they have different colors, recolor all the vertices that have the same color as w with the color of v. In this way, the number of colors is decreased by at most $f - 2\mu^*$ where f is the number of faces of G. Hence, the final number of colors used in the constructed coloring is at least

$$n - \mu^* - (f - 2\mu^*) = n + \mu^* - f \ .$$

By Euler's formula, we have that

$$n + f = \frac{3}{2}n + 2$$

since the graph G is cubic. We eventually infer the following lower bound on $\chi_f(G)$:

$$\chi_f(G) \geq n + \mu^* - f = n/2 + \mu^* - 2 \ .$$

Next, we prove the opposite inequality, i.e., $\chi_f(G) \leq \frac{n}{2} + \mu^* - 2$. Let us consider a vertex coloring of G with $\chi_f(G)$ colors that does not contain a rainbow face, and let $A_1, \ldots, A_{\chi_f(G)}$ be its color classes. For each face choose arbitrarily two vertices incident with it that have the same color and color this face with the color of the chosen vertices. Let F_i be the set of the faces colored with the i-th color. By Lemma 2, the subgraph of G^* induced by the vertices corresponding to the faces of F_i has a matching of size at least $|F_i| - |A_i| + 1$. Hence, the size of the maximum matching of G^* is at least:

$$\mu^* \geq \sum_{i=1}^{\chi_f(G)} (|F_i| - |A_i| + 1) = \chi_f(G) + f - n = \chi_f(G) - \frac{n}{2} + 2 \; .$$

This finishes the proof of the theorem. □

4 2-connected cubic graphs

Let us first describe a construction of cubic 2-connected plane graphs G_ℓ, $\ell \geq 0$, which are our example graphs. Start with two paths $v_0 u_1 v_1 u_2 v_2 \ldots u_\ell v_\ell$ and $v'_0 u'_1 v'_1 u'_2 v'_2 \ldots u'_\ell v'_\ell$ and add the edges $u_i u'_i$ for $i = 1, \ldots, \ell$. Next, add an edge ab and join both a and b to both v_0 and v'_0. At the other ends of the paths, add an edge $a'b'$ and join both a' and b' to v_ℓ and v'_ℓ. Finally, add the edges $v_i v'_i$, $i = 1, \ldots, \ell - 1$, in such a way that they are drawn in the outer face. The graphs G_1, G_2, G_3 and G_4 can be found in Figure 12. Observe that the graph G_ℓ is a 2-connected cubic graph with $n = 4\ell + 6$ vertices and $f = 2\ell + 5$ faces. Also observe that the maximum matching of the dual graph G^*_ℓ has size $\ell + 2$.

Theorem 4. *For every integer m, there exists a 2-connected cubic plane graph with n vertices such that*

$$\chi_f(G) > \frac{n}{2} + \mu^* - 2 + m \; .$$

Proof. Consider a graph G_ℓ for $\ell = 3m + 1$ and color the following pairs of vertices with the same color (distinct pairs with distinct colors): a and b, a' and b', u_i and u'_i for $i = 1, 4, \ldots, \ell$, and v_i and v'_i for $i = 2, 5, \ldots, \ell - 2$. Each of the remaining vertices gets a unique color. In this way, we construct a non-rainbow coloring of G with $4\ell + 6 - (2m + 3) = 10m + 7$ colors. Hence, we have the following:

$$\chi_f(G) - \frac{n}{2} - \mu^* + 2 \geq 10m + 7 - (6m + 5) - (3m + 3) + 2 = m + 1 \; .$$

The statement of the lemma now follows. □

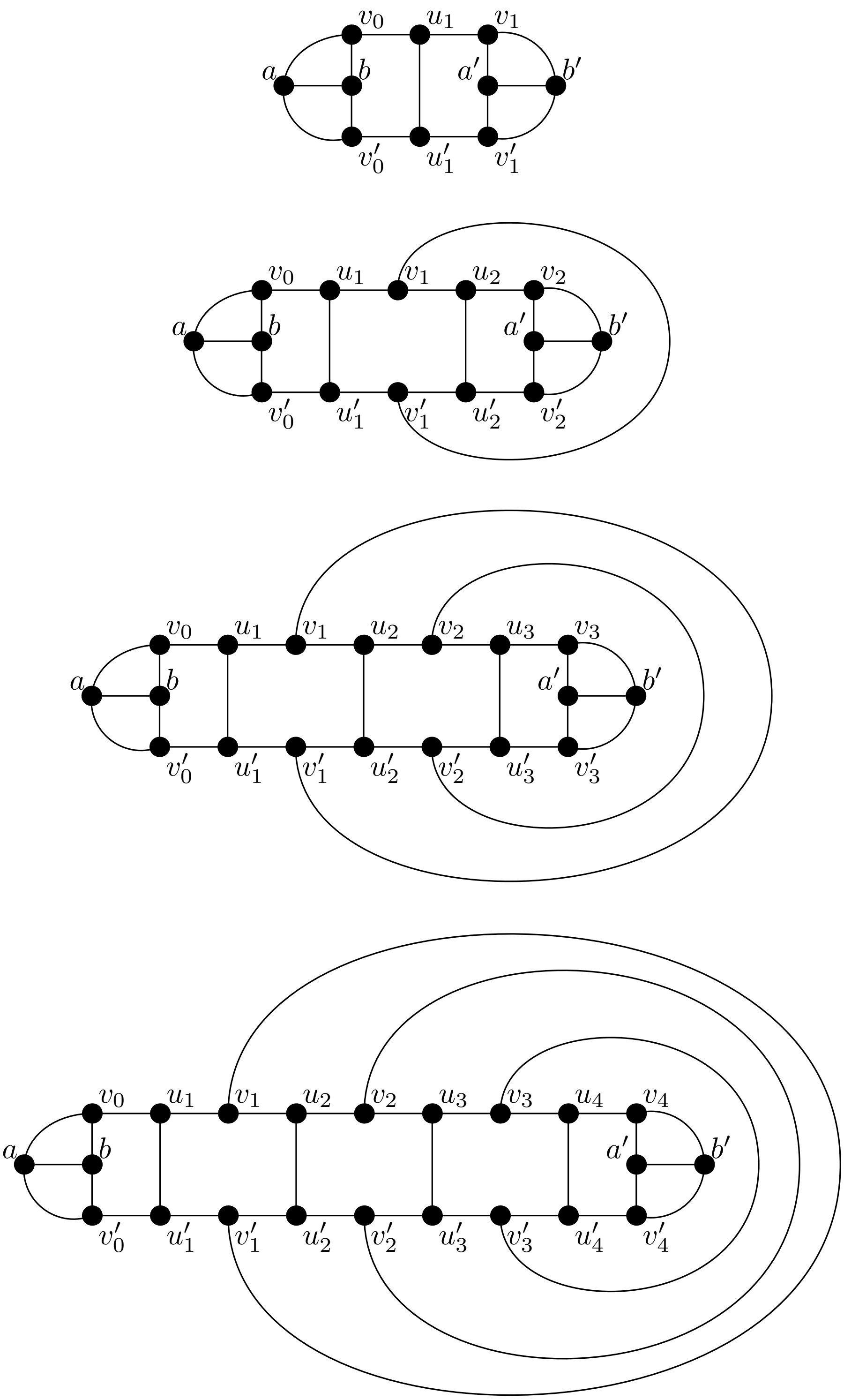

Figure 12: The graphs G_1, G_2, G_3 and G_4.

5 Concluding remarks

As shown in Theorem 4, the statement of Theorem 3 cannot be extended to all cubic plane graphs without any further assumptions. Though, it can be possible to efficiently describe the quantity $\chi_f(G)$ given a cubic plane graph G, in particular, to determine $\chi_f(G)$ algorithmically in polynomial time. It seems natural to consider a dynamic programming approach based on the structure of cuts of sizes one and two in the graph G. Such an algorithm can utilize the following generalization of Theorem 3:

Theorem 5. *If G is a plane 3-connected cubic graph and F a subset of its faces, then*

$$\chi_f^F(G) = n + \mu^* - |F|$$

where $\chi_f^F(G)$ is the maximum number of colors that can be used in a coloring such that no face of F is rainbow, and μ^* is the size of a maximum matching of $G^*[F]$.

The proof of Theorem 5 follows the lines of the proof of its counterpart. Though we believed that this approach should have led to a polynomial-time algorithm for determining $\chi_f(G)$ of all cubic graphs, we were not able to obtain such an algorithm and we suspect the problem could be NP-complete.

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