Perfect Matchings Extend to Hamilton Cycles in Hypercubes

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Abstract

Kreweras' conjecture [1] asserts that any perfect matching of the hypercube Q_d, $d \geq 2$, can be extended to a Hamilton cycle. We prove this conjecture.

1 Introduction

A set of edges $P \subset E$ of a graph $G = (V, E)$ is a *matching* if every vertex of G is incident with at most one edge of P. If a vertex v of G is incident with an edge of P, we say that v is *covered* by P. A matching P is *perfect* if every vertex of G is covered by P.

The d-dimensional hypercube Q_d is a graph whose vertex set consists of all binary vectors of length d, with two vertices being adjacent whenever the corresponding vectors differ at exactly one coordinate.

It is well known that Q_d is hamiltonian for every $d \geq 2$. This statement can be traced back to 1872 [3]. Since then the research on Hamilton cycles in hypercubes satisfying certain additional properties has received considerable attention. An interested reader can find more details about this topic in the survey of Savage [2], e.g. Dvořák [4] showed that any set of at most $2d-3$ edges of Q_d ($d \geq 2$) that induces vertex-disjoint paths is contained in a Hamilton cycle.

Kreweras [1] conjectured the following:

Conjecture. *Every perfect matching in the d-dimensional hypercube with $d \geq 2$ extends to a Hamilton cycle.*

We prove this conjecture.

Since the 2-dimensional hypercube has only one perfect matching and the 3-dimensional hypercube has two distinct perfect matchings up to isomorphism, the conjecture is easy to check for $d = 2$ and $d = 3$. Kreweras [1] proved the conjecture for $d = 4$.

2 Proof of the conjecture

Let us consider a perfect matching P of the hypercube Q_d which is contained in a Hamilton cycle C of Q_d. Let R denote the set of edges contained in the cycle C but not in P. Obviously, R is also a perfect matching of Q_d. Hence, Kreweras' conjecture can be restated in the following way:

For every perfect matching P of the hypercube Q_d, $d \geq 2$, there exists a perfect matching R such that $P \cup R$ is a Hamilton cycle of Q_d.

The crucial step of our proof lies in the following lemma. Let K_{Q_d} be the complete graph on the vertices of the hypercube Q_d. A forest is *linear*, if each component of it is a path.

Lemma. *Let P be a matching of K_{Q_d} that is not perfect. Then, there exists a perfect matching R of Q_d, $d \geq 2$, such that $P \cap R = \emptyset$ and $P \cup R$ is a linear forest.*

Before proof the lemma, let us first prove a theorem that readily implies Kreweras' conjecture – simply apply the theorem for P any perfect matching of Q_d. Then, $P \cup R$ is a Hamilton cycle of K_{Q_d} whose all edges belong to Q_d. Thus $P \cup R$ is a Hamilton cycle of Q_d.

Theorem. *For every perfect matching P of K_{Q_d} there exists a perfect matching R of Q_d, $d \geq 2$, such that $P \cup R$ is a Hamilton cycle of K_{Q_d}.*

Proof. Let $e = xy$ be an arbitrary edge of P. For the matching $P' = P \setminus \{e\}$, by Lemma, there exists a perfect matching R of Q_d such that $P' \cap R = \emptyset$ and $P' \cup R$ is a linear forest. If $e \in R$, then every vertex a graph $(P' \cup R) \setminus \{e\}$ has even degree, but $P' \cup R$ is a forest. Hence, $e \notin R$. Now, it easily follows that $P' \cup R$ is a Hamilton path of K_{Q_d} from x to y. Hence, $P \cup R$ is a Hamilton cycle of K_{Q_d}. □

Now, we prove the lemma:

Proof of Lemma. The proof proceeds by induction on d. The statement holds for $d = 2$. Let us suppose that the statement is true for every hypercube Q_k with $2 \leq k \leq d - 1$ and let us prove it for d.

We know that P is a matching which is not perfect. Hence, there must exist at least two vertices $u_1, u_2 \in V(Q_d)$ uncovered by P. We can divide the d-dimensional hypercube Q_d into two $(d-1)$-dimensional sub-hypercubes Q^1 and Q^2 such that $u_i \in V(Q^i)$ for $i \in \{1, 2\}$. Let $K^i = (V(Q^i), \binom{V(Q^i)}{2})$ and $P^i = P \cap E(K^i)$ for $i \in \{1, 2\}$.

The set of edges P^1 is a matching of K^1 which is not perfect since u_1 is not covered. Hence, there exists a perfect matching R^1 of Q^1 such that $R^1 \cap P^1 = \emptyset$ and $R^1 \cup P^1$ is a linear forest.

We would like to find a similar perfect matching R^2 of Q^2, that would join the perfect matching $R = R^1 \cup R^2$ of Q_d. However, we forbid some edges to be contained in R^2 which will preserve that $P \cup R$ is acyclic. The forbidden set of edges is

$$S = \left\{ xy \in E(K^2) \;\middle|\; \begin{array}{c} \exists x', y' \in V(Q^1) \text{ such that } xx', yy' \in P \\ \text{and there exists a path from } x' \text{ to } y' \text{ of } P^1 \cup R^1 \end{array} \right\}.$$

Every vertex v of a graph $(V(K^1), P^1 \cup R^1)$ has degree one, if and only if v is not covered by P^1. If there exists a path from x' to y' of $P^1 \cup R^1$ and $xx', yy' \in P$ and $xy \in E(K^2)$, then x' and y' are not covered by P^1 and x' and y' are vertices of both ends of a path of $P^1 \cup R^1$. Thus, the set of edges S is a matching of K^2. Moreover, the set of edges $P^2 \cup S$ is a matching of K^2 which is not perfect because S covers (not necessary all) vertices covered by P but not by P^2. Hence, there must exist a perfect matching R^2 of Q^2 by the induction such that $R^2 \cap (P^2 \cup S) = \emptyset$ and $R^2 \cup P^2 \cup S$ is a linear forest.

We show that the perfect matching $R = R^1 \cup R^2$ of Q_d satisfies the requirements of the lemma. For sake of contradiction, suppose that C is a cycle of $R \cup P$. Notice that C cannot belong to K^1 or to K^2. So C has edges in both K^1 and K^2. Now, we can shorten every path $xx' \cdots y'y$, such that $x, y \in V(Q^2)$, $x', y' \in V(Q^1)$, $xx', yy' \in P$ and $x' \cdots y'$ is a path of $P^1 \cup R^1$, by the edge $xy \in S$. Hence, we obtain a cycle of $R^2 \cup P^2 \cup S$, which is a contradiction. Thus, $P \cup R$ is a forest. Since every vertex in the graph $P \cup R$ has degree one or two, it is a linear forest. □

Riste Škrekovski [5] suggested that the following stronger form of Kreweras' conjecture could be true:

If every (not necessarily perfect) matching of Q_d, $d \geq 2$, extends to a Hamilton cycle of Q_d.

The statement can be shown to be true for $d = 2, 3, 4$. However, our approach does not seem to lead to proving this stronger statement.

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