Total-colouring of plane graphs with maximum degree nine

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Abstract

The central problem of the total-colourings is the Total-Colouring Conjecture, which asserts that every graph of maximum degree Δ admits a $(\Delta + 2)$-total-colouring. Similarly to edge-colourings—with Vizing's edge-colouring conjecture—this bound can be decreased by one for plane graphs of higher maximum degree. More precisely, it is known that if $\Delta \geq 10$ then every plane graph of maximum degree Δ is $(\Delta + 1)$-totally-colourable. On the other hand, such a statement does not hold if $\Delta \leq 3$. We prove that every plane graph of maximum degree 9 can be 10-totally-coloured.

1 Introduction

Given a graph $G = (V, E)$ and a positive integer k, a *k-total-colouring of G* is a mapping $\lambda : V \cup E \to \{1, 2, \ldots, k\}$ such that

(i) $\lambda(u) \neq \lambda(v)$ for every pair u, v of adjacent vertices;

(ii) $\lambda(v) \neq \lambda(e)$ for every vertex v and every edge e incident to v;

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(*iii*) $\lambda(e) \neq \lambda(e')$ for every pair e, e' of incident edges.

This notion was independently introduced by Bezhad [3] in his doctoral thesis, and Vizing [15]. It is now a prominent notion in graph colouring, to which a whole book is devoted [17]. Both Bezhad and Vizing made the celebrated *Total-Colouring Conjecture*, stating that every graph of maximum degree Δ admits a $(\Delta + 2)$-total-colouring. Notice that every such graph cannot be totally-coloured with less than $\Delta + 1$ colours, and that a cycle of length five cannot be 3-totally-coloured. The best general bound so far has been obtained by Molloy and Reed [10], who established that every graph of maximum degree Δ can be $(\Delta + 10^{26})$-totally-coloured. Moreover, the conjecture has been shown to be true for several special cases, namely for $\Delta = 3$ by Rosenfeld [11] and Vijayaditya [14], and then for $\Delta \in \{4, 5\}$ by Kostochka [9].

Another natural subclass to consider is the one of planar graphs. It attracted considerable attention, and several results were obtained. First, Borodin [5] proved that if $\Delta \geq 9$ then every plane graph of maximum degree Δ fulfils the conjecture. This result can be extended $\Delta = 8$ by the use of the Four Colour Theorem [1, 2], and Vizing's Theorem about edge colouring—the reader can consult the book by Jensen and Toft [8] for more details. Sanders and Zhao [12] solved the case $\Delta = 7$ of the Total-Colouring Conjecture for plane graphs. So the only open case regarding plane graphs is $\Delta = 6$. Interestingly, $\Delta = 6$ is also the only remaining open case for Vizing's edge-colouring conjecture, after Sanders and Zhao [13] resolved the case $\Delta = 7$.

However, plane graphs with high maximum degree allow a stronger assertion. More precisely, Borodin [5] showed that if $\Delta \geq 14$ then every plane graph with maximum degree Δ is $(\Delta+1)$-totally-colourable, and asked whether 14 could be decreased. Borodin, Kostochka and Woodall extended this result to the case where $\Delta \geq 12$ [6], and later to $\Delta = 11$ [7]. Recently, Wang [16] established the result for $\Delta = 10$. On the other hand, this bound is not true if $\Delta \leq 3$. The complete graphs K_2, K_4 and the cycles of length $3k + 2$ with $k \geq 1$ are examples of plane graphs that cannot be $(\Delta + 1)$-totally-coloured. We continue along those lines, and establish the following theorem.

Theorem 1. *Every plane graph of maximum degree* 9 *is* 10*-totally-colourable.*

So, the values of Δ for which it is not known whether all plane graphs of maximum degree Δ are $(\Delta + 1)$-totally-colourable are now $4, 5, 6, 7$ and 8. Recall that the case where $\Delta = 6$ is even open for the Total-Colouring Conjecture. We also note that if $\Delta \geq 3$, then every outerplane graph with maximum degree Δ can be $(\Delta+1)$-totally-coloured [19]. Another result of the same type is that every Halin graph of maximum degree 4 admits a 5-total-colouring [18]. Note also that the complete r-partite balanced graph K_{r*n}, whose maximum degree Δ is $n(r-1)$, admits a $(\Delta+2)$-total-colouring, and the cases where this bound can be decreased by 1 have been characterised [4].

We prove Theorem 1 by contradiction. From now on, we let $G = (V, E)$ be a minimum counter-example to the statement of Theorem 1, in the sense that the quantity $|V|+|E|$ is minimum. In particular, every proper subgraph of G is 10-totally-colourable. First, we establish various structural properties of G in Section 2. Then, relying on these properties, we use the Discharging Method in Section 3 to obtain a contradiction.

In the sequel, a vertex of degree d is called a *d-vertex*. A vertex is a *$(\leq d)$-vertex* if its degree is at most d; it is a *$(\geq d)$-vertex* if its degree is at least d. If f is a face of G, the *degree of f* is its length, i.e. the number of its incident vertices. The notions of *d-face*, *$(\leq d)$-face* and *$(\geq d)$-face* are defined analogously as for the vertices. Moreover, if a vertex v is adjacent to a d-vertex u, we say that u is a *d-neighbour* of v. A cycle of length 3 is called a *triangle*. For integers a, b, c, a *$(\leq a, \leq b, \leq c)$-triangle* is a triangle xyz of G with $\deg(x) \leq a, \deg(y) \leq b$ and $\deg(z) \leq c$. The notions of $(a, \leq b, \leq c)$-triangles, $(a, b, \geq c)$-triangles and so on, are defined analogously.

2 Reducible configurations

In this section, we establish some structural properties of the graph G. We prove that some plane graphs are *reducible configurations*, i.e. they cannot be subgraphs of G.

For convenience, we sometimes define configurations by depicting them in figures. In all the figures of this paper, 2-vertices are represented by small black bullets, 3-vertices by black triangles, 4-vertices by black squares, and white bullets represents vertices whose degree is at least the one shown on the figure.

Let λ be a (partial) 10-total-colouring of G. For each element $x \in V \cup E$, we define $\mathcal{C}(x)$ to be the set of colours (with respect to λ) of vertices and edges incident or adjacent to x. Also, we set $\mathcal{F}(x) := \{1, 2, \ldots, 10\} \setminus \mathcal{C}(x)$. If $x \in V$ we define $\mathcal{E}(x)$ to be the set of colours of the edges incident to x. Moreover, λ is *nice* if only some (≤ 4)-vertices are not coloured. Observe that every nice colouring can be greedily extended to a 10-total-colouring of G since for each (≤ 4)-vertex v, $|\mathcal{C}(v)| \leq 8$, i.e. v has at most 8 forbidden colours. Therefore, in the rest of the paper, we shall always suppose that such vertices are coloured at the very end. More precisely, every time we consider a partial colouring of G, we uncolour all (≤ 4)-vertices, and implicitely colour them at the very end of the colouring procedure of G. We make the following observation about nice colourings.

Observation. Let uv be an edge with $\deg(v) \leq 4$. There exists a nice colouring λ of $G-e$, in which u is coloured and v is uncoloured. Moreover, it then suffices to properly colour the edge e with a colour from $\{1, 2, \ldots, 10\}$ to extend λ to a nice colouring of G.

We will use this observation implicitely throughout the paper.

Lemma 2. *The graph G has the following properties.*

- *(i) The minimum degree is at least two;*
- *(ii) if vu is an edge with $\deg(v) \leq 4$ then $\deg(u) \geq 11 - \deg(v)$;*
- *(iii) a 9-vertex is adjacent to at most one 2-vertex;*
- *(iv) a triangle incident to a 3-vertex must also contain a 9-vertex;*
- *(v) there is no $(4, \leq 7, \leq 8)$-triangle;*
- *(vi) a triangle contains at most one (≤ 5)-vertex.*

Proof. (i). Suppose that v is a 1-vertex, and let u be its neighbour. By the minimality of G, the graph $G - v$ admits a nice colouring in which u is coloured. Since the degree of u in $G - v$ is at most 8, we obtain $|\mathcal{C}(vu)| \leq 9$. Thus, the edge vu can be properly coloured, which yields a nice colouring of G.

(ii). Suppose that $vu \in E$ with $\deg(v) \leq 4$ and $\deg(u) \leq 10 - \deg(v)$. There exists a nice colouring of $G' := G - vu$, in which u is coloured and v is uncoloured. Therefore, $|\mathcal{C}(vu)| \leq \deg(v) - 1 + \deg(u) - 1 + 1 \leq 9$. Hence we can colour properly the edge vu, thereby obtaining a nice colouring of G.

(iii). Suppose that v is a 9-vertex adjacent to two 2-vertices x and y. Let x' be the neighbour of x different from v, and let y' be the neighbour of y different from v. Notice that we may have $x' = y'$. By the previous assertion, x' and y' are 9-vertices. It is enough to consider the following two possibilities.

v is adjacent to neither x' nor y'. Then, we construct the graph G' by first removing x and y, and then adding the edge vx'. If $y' \neq x'$, we additionally add the edge vy'. Note that G' is a simple plane graph of maximum degree 9 with fewer vertices and edges than G. Therefore, it

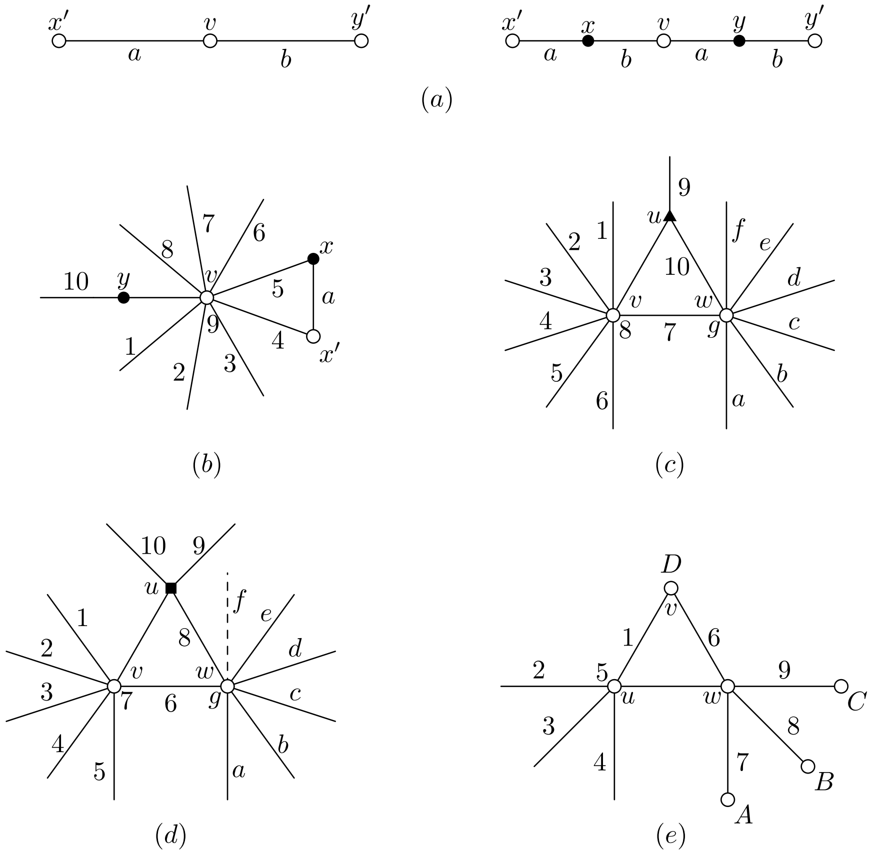


Figure 1: Configurations for the proof of Lemma 2.

admits a nice colouring λ by the minimality of G. We easily modify λ to obtain a nice colouring of G. First, put $\lambda(xx') := \lambda(vy) := \lambda'(vx')$. Now, if $x' \neq y'$ then we put $\lambda(vx) := \lambda(yy') := \lambda'(vy')$. See Figure 1($a$) for an illustration. And, if $x' = y'$ then we note that each of the edges yy' and vx has at most 9 forbidden colours. Thus, both of them can be coloured and the obtained 10-total-colouring of G is nice.

v is adjacent to x'. Thus vxx' is a triangle. Consider a nice colouring of $G - vy$. To extend it to G, it suffices to properly colour the edge vy. If this cannot be done greedily, then $|\mathcal{C}(vy)| = 10$, and up to a permutation of the colours, we can assume that the colouring is the one shown in Figure 1(b). If $a \neq 10$, then recolour vx with 10 and colour vy with 5 to obtain a nice colouring of G. And if $a = 10$, then we interchange the colours of vx' and xx', and afterwards colour vy with 4.

(iv). By (ii), a 3-vertex has only ($\geq$8)-neighbours. Thus we may suppose that vwu is a $(3, 8, 8)$-triangle, with u being the 3-vertex. Consider a nice colouring of $G-vu$. To extend it to G, again it suffices to properly colour the edge vu. If we cannot do this greedily, it means that $|\mathcal{C}(vu)| = 10$. Thus, up to a permutation of the colours, the colouring is the one shown in Figure 1(c). If the edge wu can be properly recoloured, then we do so, and afterwards colour the edge vu with 10, which gives a nice colouring of G. So we deduce that $|\mathcal{C}(wu)| = 9$. Consequently, $\{a, b, c, d, e, f, g\} = \{1, 2, 3, 4, 5, 6, 8\}$. Thus we obtain $9 \notin \mathcal{C}(vw)$. So, we can recolour vw with 9 and colour vu with 7 to conclude the proof.

(v). By (ii), it is enough to prove that there is no $(4, 7, \delta)$-triangle in G for $\delta \in \{7, 8\}$. Suppose that vwu is such a triangle with w having degree δ and u degree 4. Consider a nice colouring of $G - vu$. It is sufficient to properly colour the edge vu to obtain a nice colouring of G. Again, $|\mathcal{C}(vu)| = 10$, so up to a permutation of the colours, we assume that the colouring is the one of Figure 1(d). If the edge wu can be properly recoloured, then do so, and colour vu with 8 to obtain a nice colouring of G. Thus, we deduce that $|\mathcal{C}(wu)| = 9$. Therefore, $\{1, 2, 3, 4, 5, 7\} \subset \{a, b, c, d, e, f, g\}$. From this we infer that $|\mathcal{C}(vw)| \leq 6 + \delta - 6 = \delta \leq 8$. Thus, the edge vw can be properly recoloured, and so the edge vu can be coloured with 6, yielding a nice colouring of G.

(vi). Let vuw be a triangle with $\deg(u) = \deg(w) = 5$. Consider a total-colouring of $G - uw$, and uncolour the vertex w. Observe that $|\mathcal{F}(uw)| \geq 1$ and $|\mathcal{F}(w)| \geq 1$. Furthermore, these two sets must actually be equal and of size one, otherwise we can extend the colouring to G. Up to a permutation of the colours, the colouring is the one shown in Figure 1(e), with $\{A, B, C, D\} = \{1, 2, 3, 4\}$. Notice that the colours of the edges vu and vw can be safely interchanged. Now, the vertex w can be properly coloured with 6, and the edge uw with 10.

□

Lemma 3. *For the graph G, the following assertions hold.*

(i) *There is no $(5, 6, 6)$-triangle.*

(ii) *A 6-vertex has at most two 5-neighbours.*

(iii) *Suppose that v is a 7-vertex, and let x_1 be one of its neighbours. If v and x_1 have at least two common neighbours, then at most one of them has degree 4.*

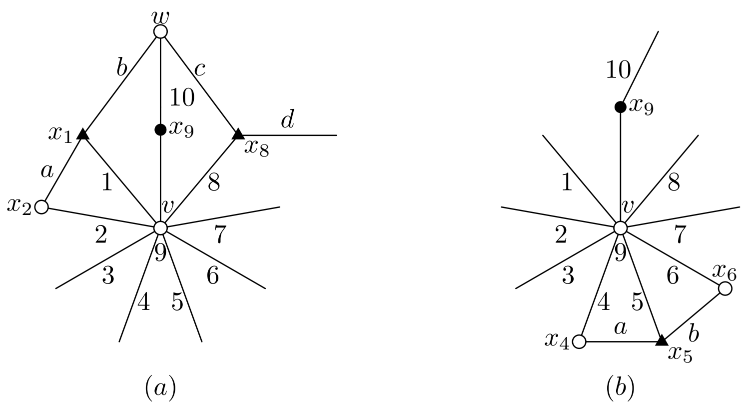


Figure 2: Reducible configurations of Lemma 3(vi) and (vii).

(iv) *Suppose that* vwu *and* vwu' *are two triangles with* $\deg(u) = 2$. *Then,* $\deg(u') \geq 4$.

(v) *Suppose that* v *is a* 9*-vertex incident to a* $(2, 9, 9)$*-triangle. Then it is not incident to a* $(\leq 3, \geq 8, 9)$*-triangle.*

(vi) *The configuration of Figure* 2*(a) is reducible.*

(vii) *The configuration of Figure* 2*(b) is reducible.*

Proof. (i). Suppose on the contrary that G contains a $(5, 6, 6)$-triangle uvw with u being of degree 5. The proof is in two steps. In the first step, we prove the existence of a 10-total-colouring of G in which only u is uncoloured. And in the second step, we establish that such a colouring can be extended to G. Consider a nice colouring of $G - vu$, and uncolour the vertex u. Our only goal in the first step is to properly colour the edge vu. If we cannot do this greedily, then $|\mathcal{C}(vu)| = 10$, and thus we can assume that the colouring is the one of Figure 3(a). We infer that $\{6, 7, 8, 9, 10\} = \{a, b, c, d, e\}$, otherwise we can choose a colour $\alpha \in \{6, 7, 8, 9, 10\} \setminus \{a, b, c, d, e\}$, recolour uw with α and colour vu with 4. Consequently, we have $\mathcal{C}(vw) = \{4, 6, 7, 8, 9, 10\}$. Thus, we can recolour vw with 1, and colour vu with 5.

For the second step, consider a partial 10-total-colouring of G such that only u is not coloured. If we cannot greedily extend it to G, then without loss of generality the colouring is the one of Figure 3(b). Note that if $|\mathcal{C}(vu)| \leq 8$, then we can recolour vu and colour u with 5. Thus, we infer that $\{a, b, c, d, e\} \supset \{7, 8, 9, 10\}$. Similarly, $\{e, f, g, h, i\} \supset \{6, 8, 9, 10\}$. Observe that $|\mathcal{C}(v)| = 9$, otherwise we just properly recolour v and colour u with 6.

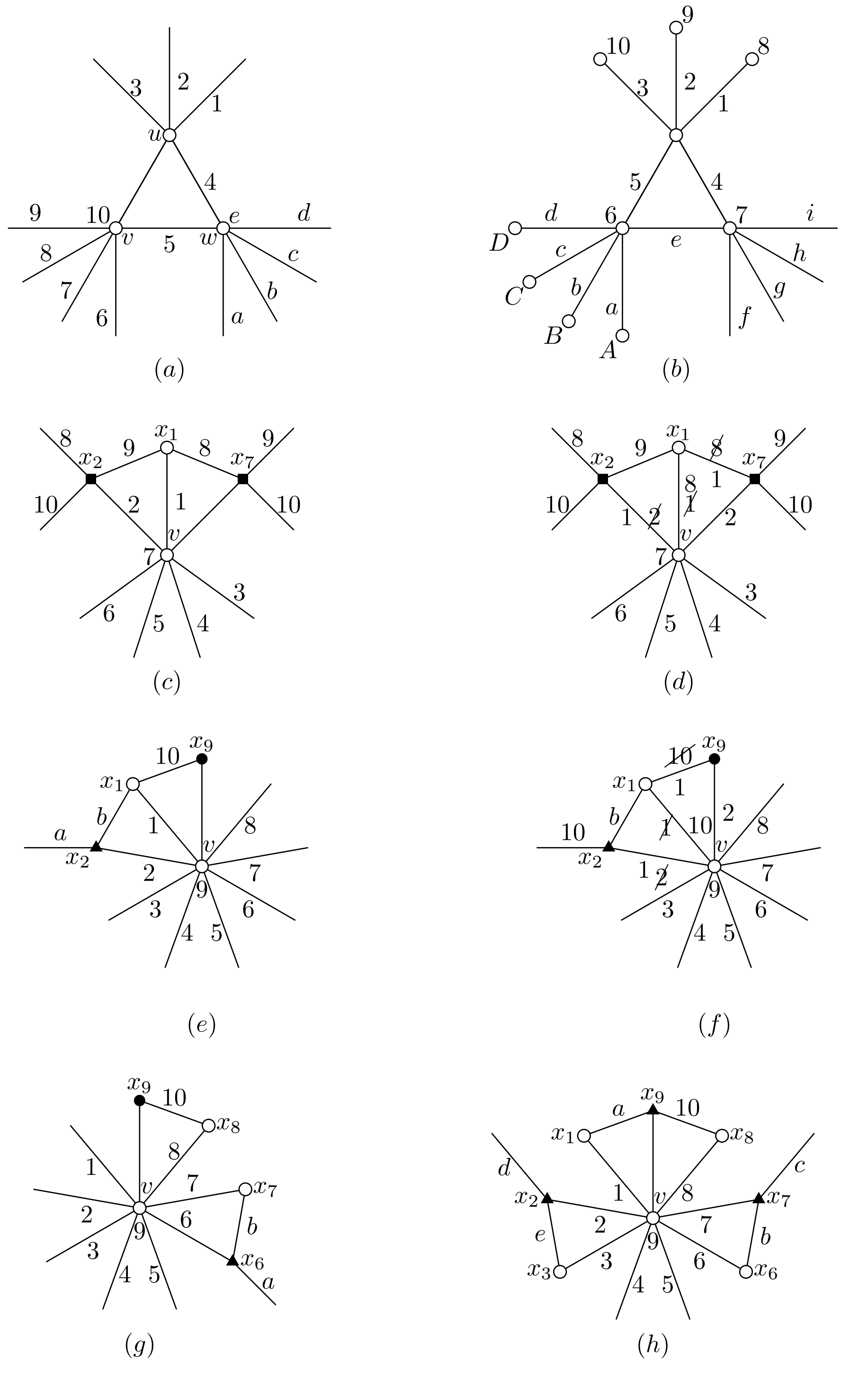


Figure 3: Configurations for the proof of Lemmas 3 and 4.

We assert that we can assume that $e \in \{1,2,3\}$. If it is not the case, then $e \in \{8,9,10\}$, say $e = 10$. By what precedes, $|\mathcal{C}(vw)| \leq 12-4 = 8$ and $\{4,5,6,7,8,9\} \subset \mathcal{C}(vw)$. Thus at least one colour among $1,2,3$ can be used to recolour vw, which proves the assertion. Therefore, $\{a,b,c,d\} = \{7,8,9,10\}$ and $\{f,g,h,i\} = \{6,8,9,10\}$. Thus vw can be recoloured by every colour of $\{1,2,3\}$. So, if there exists a colour $\alpha \in \{1,2,3\} \setminus \{A,B,C,D\}$, we can recolour vw with a colour of $\{1,2,3\}$ different from α, recolour v with α and colour u with 6. Hence $\{1,2,3\} \subseteq \{A,B,C,D\}$. Now, recall that $|\mathcal{C}(v)| = 9$, thus $4 \in \{A,B,C,D\}$. Consequently, we can interchange safely the colours of vu and wu, recolour v with 5, and finally colour u with 6.

(ii). Suppose that v is a 6-vertex with three 5-neighbours x_1, x_2, x_3. By Lemma 2(vi), these three vertices are pairwise non-adjacent. Let λ be a nice colouring of $G - vx_1$, and uncolour the edges vx_2 and vx_3 as well as the vertices v, x_1, x_2 and x_3. Notice that for each $i \in \{1,2,3\}$, $|\mathcal{C}(x_i)| \leq 8$ and $|\mathcal{C}(vx_i)| \leq 7$. Moreover, $|\mathcal{C}(v)| \leq 6$. Recall that $\mathcal{F}(x) := \{1,2,\ldots,10\} \setminus \mathcal{C}(x)$ for every $x \in V \cup E$. Observe that for each $i \in \{1,2,3\}$, we have $\mathcal{F}(v) \cap \mathcal{F}(x_i) \subseteq \mathcal{F}(vx_i)$. Hence, we infer that $|\mathcal{F}(v) \cap (\mathcal{F}(vx_i) \cup \mathcal{F}(x_i))| = |\mathcal{F}(v) \cap \mathcal{F}(vx_i)| \leq 3$. Consequently, there exists a colour $\alpha \in \mathcal{F}(v)$ that does not belong to $\mathcal{F}(x_3) \cup \mathcal{F}(vx_3)$. Set $\lambda(v) := \alpha$. If we colour properly x_1, vx_1, x_2 and vx_2, then we will be able to colour greedily vx_3 and x_3 and hence the proof would be complete. Observe that if α does not belong to $\mathcal{F}(x_1)$ or to $\mathcal{F}(vx_2)$, then the colouring can be extended greedily to x_1, x_2, vx_1, vx_2—just colour x_1 or vx_2 last, respectively. Therefore we assume that α belongs to these two lists. Uncolour v and colour x_1 and vx_2 with α. With respect to this colouring, note that $|\mathcal{F}(vx_1)| \geq 2$, $|\mathcal{F}(v)| \geq 3$, $|\mathcal{F}(x_2)| \geq 1$, $|\mathcal{F}(vx_3)| \geq 3$ and $|\mathcal{F}(x_3)| \geq 2$. Hence, we can colour x_2. Now, if there exists $\beta \in \mathcal{F}(vx_1) \cap \mathcal{F}(x_3)$, then we let $\lambda(vx_1) := \lambda(x_3) := \beta$, and afterwards greedily colour v and vx_3.

So, $\mathcal{F}(vx_1) \cap \mathcal{F}(x_3) = \emptyset$. If there exists $\kappa \in \mathcal{F}(v) \cap \mathcal{F}(x_3) \neq \emptyset$, then we set $\lambda(v) := \kappa$, and afterwards we greedily colour x_3, vx_3 and vx_1 in this order. Otherwise, greedily colouring vx_1, v, vx_3 and x_3 in this order yields a nice colouring of G.

(iii). Suppose that the statement is false, so the graph G contains the configuration of Figure 3(c). Consider a nice colouring λ of $G - vx_7$. If it cannot be extended to G, then $|\mathcal{C}(vx_7)| = 10$. Furthermore, $|\mathcal{C}(vx_2)| = 9$, otherwise we can colour the edge vx_7 with $\lambda(vx_2)$ and greedily recolour the edge vx_2, thereby obtaining a nice colouring of G. Therefore, we can assume that the colouring is the one shown in Figure 3(c). Then a nice colouring of G is obtained by interchanging the colours of the edges x_7x_1 and vx_1, recolouring vx_2 with 1 and colouring vx_7 with 2, as shown in Figure 3(d).

(iv). Suppose on the contrary that G contains the configuration of Figure 3(e). Consider a nice colouring of $G - vx_9$. If the edge vx_9 cannot be greedily coloured, then $|\mathcal{C}(vx_9)| = 10$. Thus we may assume that the colouring is the one shown in Figure 3(e). Notice that $a = 10$, otherwise we recolour vx_2 with 10 and colour vx_9 with 2. So, the recolouring in Figure 3(f) is nice.

(v). Suppose that G contains the configuration of Figure 3(g), and consider a nice colouring λ of $G - vx_9$. Without loss of generality, we may assume that it is the one of Figure 3(g). Observe that $10 \in \{a, b\}$, otherwise we obtain a nice colouring of G by setting $\lambda(vx_6) := 10$ and $\lambda(vx_9) := 6$. Now, we consider two cases regarding b.

$b = 10$. If $a \neq 7$ then we can interchange the colours of the edges x_6x_7 and vx_7, and colour vx_9 with 7 to obtain a nice colouring of G. And if $a = 7$, then we interchange the colours of the edges x_9x_8 and vx_8, and then we let $\lambda(vx_6) := 8$ and $\lambda(vx_9) := 6$.

$b \neq 10$. In this case, $a = 10$. We interchange the colours of x_9x_8 and vx_8. Similarly as before, we deduce that $b = 8$. Now, the previous case applies with 8 playing the role of colour 10.

(vi). Suppose on the contrary that G contains the configuration of Figure 2(a). Up to a permutation of the colours, every nice colouring of $G - vx_9$ is as the one of the figure. Note that $d = 10$, otherwise recolour vx_8 with 10 and colour vx_9 with 8. Similarly, $a = 10$. Now, interchange the colours of the edges x_1x_2 and vx_2. If $b \neq 2$, the obtained colouring extends to G by colouring vx_9 with 2. If $b = 2$, then interchange the colours of the edges x_9w and x_1w thereby obtaining a nice colouring of $G - vx_9$. Since $d = 10 \neq 2$, observe that we can extend it to G as before, i.e. we recolour vx_8 with 2 and colour vx_9 with 8.

(vii). Suppose that G contains the configuration of Figure 2(b). Consider a nice colouring of $G - vx_9$. Without loss of generality, we may assume that it is the one of the figure. Note that $10 \in \{a, b\}$, otherwise recolour vx_5 with 10 and colour vx_9 with 5. By symmetry, we can assume that $a = 10$. Interchange the colours of the edges x_5x_4 and vx_4. If $b \neq 4$, we have a nice colouring of $G - vx_9$, and we extend it to G by colouring vx_9 with 4. Otherwise, $b = 4$, we interchange the colours of the edges x_5x_6 and vx_6, and colour vx_9 with 6, which yields a nice colouring of G. □

Lemma 4. *The configuration of Figure* 3*(h) is reducible.*

Proof. Consider a nice colouring of $G-vx_9$. If it cannot be greedily extended to G then $|\mathcal{C}(vx_9)| = 10$, and so we can assume that the colouring is the one of Figure 3(h). First, we note that if $a \neq 7$ then $10 \in \{b, c\}$, otherwise we recolour vx_7 by 10 and colour vx_9 with 7. Similarly, if $a \neq 2$ then $10 \in \{d, e\}$. We now split the proof into three cases.

$a \notin \{6, 8\}$. Since a is different from either 2 or 7, we may assume that $a \neq 7$. As mentioned above, we must have $10 \in \{b, c\}$. Moreover, if we interchange the colours of the edges x_9x_8 and vx_8, we deduce as before that $8 \in \{b, c\}$, the colour 8 playing the role of colour 10. Hence $\{b, c\} = \{8, 10\}$. Now, interchange the colours of the edges x_7x_6 and vx_6, and colour vx_9 with 6. If $b = 10$, the obtained colouring is proper, and if $b = 8$ then we additionally interchange the colours of the edges x_9x_8 and vx_8 to obtain the desired colouring.

$a = 8$. In this case $10 \in \{b, c\}$. By interchanging the colours of the edges x_9x_8 and vx_8, and also of x_9x_1 and vx_1, we infer that $1 \in \{b, c\}$. Hence $\{b, c\} = \{1, 10\}$. Similarly as in the previous case, interchange the colours of x_7x_6 and vx_6, and afterwards colour vx_9 with 6. If $b = 10$, the obtained colouring of G is proper, and if $b = 1$ then it suffices to additionally interchange the colours of the edges x_9x_8 and vx_8, and also of x_9x_1 and vx_1 to obtain a nice colouring of G.

$a = 6$. Then, $10 \in \{d, e\}$. Note that the colours of the edges x_9x_8 and vx_8 can be interchanged safely, because $a \neq 8$. Therefore, as $a \neq 2$, we infer that $8 \in \{d, e\}$, and hence $\{d, e\} = \{8, 10\}$. We interchange now the colours of the edges x_2x_3 and vx_3 and colour vx_9 with 3. If $e = 10$, the obtained colouring of G if proper. And, if $e = 8$, then it suffices to interchange the colours of the edges x_9x_8 and vx_8 to obtain a desired colouring.

□

Lemma 5. *If uvz is a triangle with an 8-vertex v and a 3-vertex u, then v has no 3-neighbour distinct from u.*

Proof. Suppose that v is an 8-vertex that contradicts the lemma. Let u and w be two 3-neighbours of v, and assume that vuz is a triangle. We consider a nice colouring of $G - vu$. If we cannot extend it to G, then without loss of generality, we may assume that the colouring is the one shown on Figure 4(a). Observe that $\{a, b\} = \{9, 10\}$, otherwise we obtain the desired colouring by recolouring vw with either 9 or 10, and colouring vu with 2. Now, as depicted in Figure 4(b), we interchange the colours of the edges uz and vz, recolour vw with 1, and colour vu with 2 to obtain the sought colouring. □

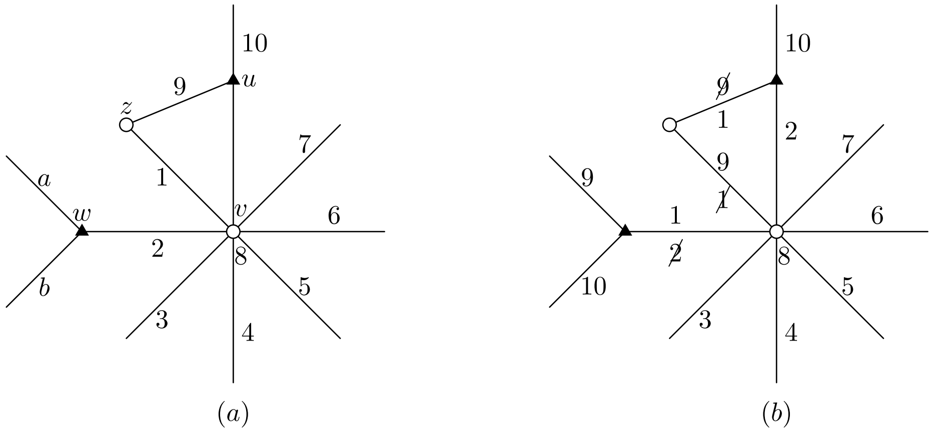


Figure 4: Colouring and recolouring for the proof of Lemma 5.

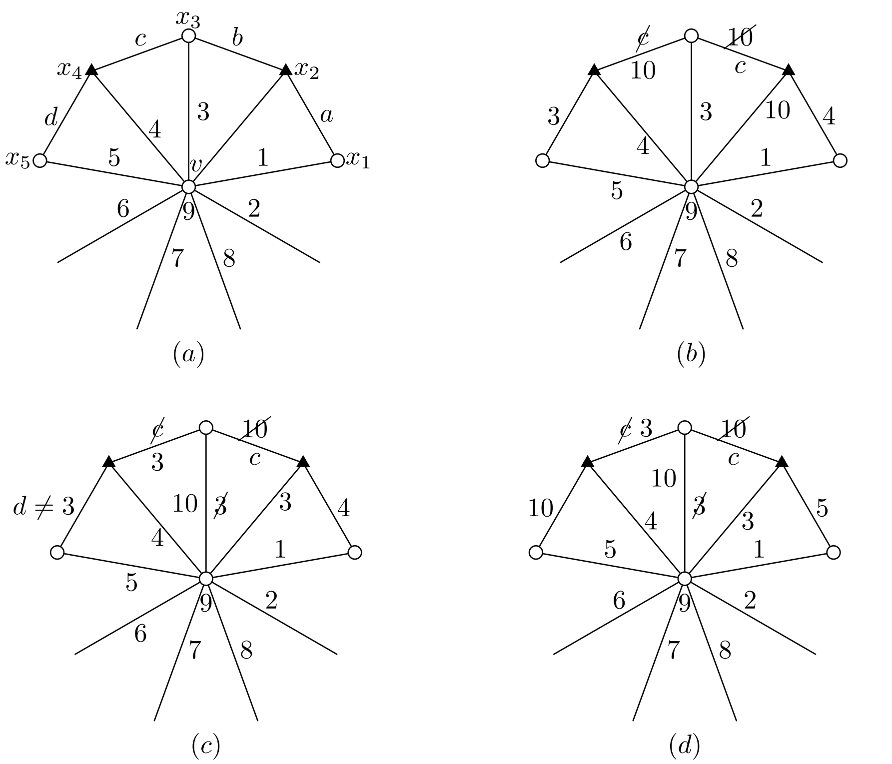


Figure 5: Configurations for Lemma 6.

Lemma 6. *The configuration of Figure* 5*(a) is reducible.*

Proof. Consider a nice colouring of $G - vx_2$. Up to a permutation of the colours, it is the one of Figure $5(a)$. Note that $10 \in \{a, b\}$, otherwise we obtain a nice colouring of G by colouring vx_2 with 10. We split the proof into two cases, regarding the value of b.

Case 1: $b = 10$. If $a = 4$, then apply the recolourings of Figure 5(b) and (c), regarding whether d is 3.

Suppose now that $a \neq 4$. In this case, we deduce that $d = 10$, otherwise we can recolour vx_4 with 10 and colour vx_2 with 4. If $c \neq 5$, then the desired colouring can be obtained as follows. If $a \neq 5$, interchange the colours of the edges x_4x_5 and vx_5, and colour vx_2 with 5, and if $a = 5$ then the recolouring of Figure 5(d) is nice.

We may assume now that $c = 5$. Interchange the colours of the edges x_4x_5 and vx_5, and also of the edges x_4x_3 and vx_3. If $a \neq 3$ then it suffices to colour vx_2 with 3. And, if $a = 3$, then additionally interchange the colours of the edges x_2x_1 and vx_1, recolour vx_4 with 1 and colour vx_2 with 4 to obtain the sought colouring.

Case 2: $b \neq 10$. Therefore, $a = 10$. First, note that $10 \in \{c, d\}$, otherwise we recolour vx_4 with 10 and colour vx_2 with 4. Either the obtained colouring of G is nice, or $b = 4$. In the latter case, we additionally interchange the colours of x_2x_3 and x_4x_3 to obtain the desired colouring.

Suppose now that $c = 10$. Then, $b = 4$ otherwise we uncolour vx_4, colour vx_2 with 4, and apply Case 1 to the obtained colouring with x_4 playing the role of the vertex x_2. Now, interchange the colours of x_4x_3 and vx_3. The obtained colouring is nice if $d \neq 3$, and we extend it to G by colouring vx_2 with 3. And, if $d = 3$, we additionally interchange the colours of x_4x_5 and vx_5 and colour vx_2 with 5.

Finally, assume that $c \neq 10$, and hence $d = 10$. Up to interchanging the colours of x_2x_3 and x_4x_3, we may assume that $b \neq 5$. Interchange the colours of x_4x_5 and vx_5. If $c \neq 5$, the obtained colouring is nice and we extend it to G by colouring vx_2 with 5. And, if $c = 5$, we additionally interchange the colours of x_4x_3 and vx_3, and colour vx_2 with 3.

□

Lemma 7. *The configuration of Figure* 6*(a) is reducible.*

Our proof of Lemma 7 uses the following result. Given a colouring λ and a vertex v, recall that $\mathcal{E}(v)$ is the set of colours assigned to the edges incident to v. Let $\mathcal{E}'(v) := \{1, 2, \ldots, 10\} \setminus (\mathcal{E}(v) \cup \{\lambda(v)\})$.

Lemma 8. *Suppose that G contains the configuration of Figure* 6*(b). Then, for every nice colouring λ of $G - vx_2$, it holds that $\mathcal{E}'(v) \cup \{\lambda(vx_6)\} \subseteq \mathcal{E}(x_2)$.*

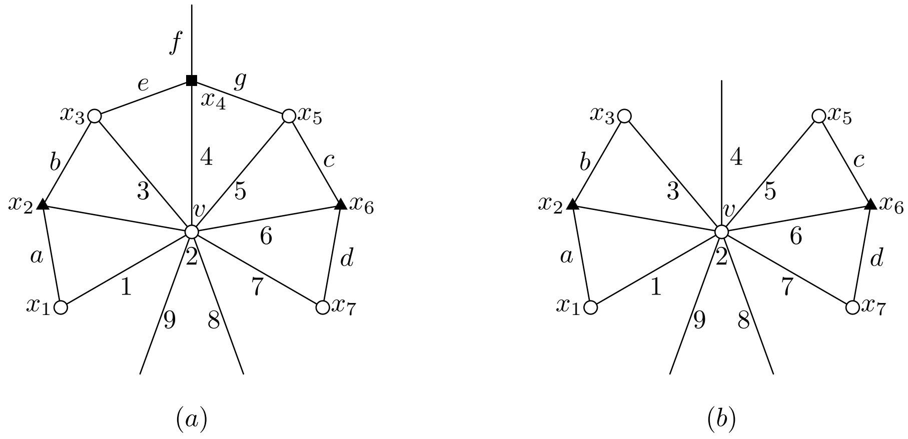


Figure 6: Configurations for Lemmas 7 and 8.

Proof. Up to a permutation of the colours, the colouring λ is the one of Figure 6(b). Notice that $\mathcal{E}'(v) = \{10\}$, $\lambda(vx_6) = 6$ and $\mathcal{E}(x_2) = \{a, b\}$. Clearly, $10 \in \{a, b\}$ otherwise we just colour vx_2 with 10. By symmetry, we may assume that $a = 10$. Thus, to finish the proof, it only remains to prove that $b = 6$. Suppose on the contrary that $b \neq 6$. Note that $10 \in \{c, d\}$ otherwise we recolour vx_6 with 10 and colour vx_2 with 6. By symmetry, we may assume that $d = 10$. We consider two possibilities, regarding the value of b.

$b = 1$: Interchange the colours of the edges x_6x_7 and vx_7. The obtained colouring of G is nice if $c \neq 7$, and if $c = 7$ we additionally interchange the colours of x_6x_5 and vx_5. Now, colouring vx_2 with 7 or 5 yields a nice colouring of G, a contradiction.

$b \neq 1$: In this case, $c = 1$. Indeed, if $c \neq 1$, we recolour vx_6 with 1, interchange the colours of x_2x_1 and vx_1 and colour vx_2 with 6 to obtain a nice colouring of G. Now, if $b \neq 7$ then interchange the colours of x_6x_7 and vx_7 and colour vx_2 with 7. And, if $b = 7$ then interchange the colours of x_6x_5 and vx_5, and also of x_2x_1 and vx_1, and colour vx_2 with 5.

□

Proof of Lemma 7. Consider a nice colouring λ of $G - vx_2$. Up to a permutation of the colours, we assume that the colouring is the one of Figure 6(a). By Lemma 8, we have $\{a, b\} = \{6, 10\}$. We consider two cases.

$a = 10$ *and* $b = 6$. If there exists a colour $\alpha \in \{1, 10\} \setminus \{e, f, g\}$, then recolour vx_4 with α and colour vx_2 with 4. The obtained colouring is nice if $\alpha = 10$. And, if $\alpha = 1$ it suffices to additionally interchange the colours of x_2x_1 and vx_1. Thus, $\{1, 10\} \subset \{e, f, g\}$.

Suppose that $6 \notin \{e, f, g\}$. We start by interchanging the colours of the edges x_2x_3 and x_4x_3. If $e = 10$, we additionally interchange the colours of x_2x_1 and vx_1. Observe that the obtained colouring does not fulfil the condition of Lemma 8, a contradiction. Hence, $\{e, f, g\} = \{1, 6, 10\}$ and so $e \in \{1, 10\}$. We interchange the colours of x_4x_3 and vx_3 and colour vx_2 with 3. Either this colouring of G is nice, or $e = 1$ and hence additionally interchanging the colours of x_2x_1 and vx_1 yields a nice colouring of G.

$a = 6$ *and* $b = 10$. If there exists $\alpha \in \{3, 10\} \setminus \{f, g\}$, then recolour vx_4 with α, and colour vx_2 with 4. If the obtained colouring is not nice, then $\alpha = 3$ and hence interchanging the colours of x_2x_3 and vx_3 yields a nice colouring of G, a contradiction. Observe that we may assume that $f = 3$ and $g = 10$. Indeed, if it is not the case, then we interchange the colours of x_2x_3 and vx_3 and obtain the desired condition, with 3 playing the role of colour 10.

Furthermore $e = 5$, otherwise we interchange the colours of x_4x_5 and vx_5 and colour vx_2 with 5. Now, observe that $d = 10$, otherwise we recolour vx_6 with 10, vx_4 with 6 and colour vx_2 with 4 to obtain a nice colouring of G. Finally, we interchange the colours of x_6x_7 and vx_7. If $c = 7$, we additionally interchange the colours of x_6x_5 and vx_5. Now, colouring vx_2 with 7 or 5 yields a nice colouring of G, a contradiction.

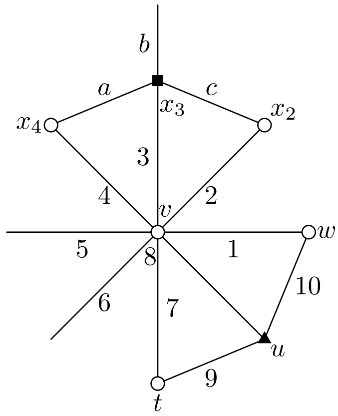
□

Lemma 9. *The configurations of Figure 7 are reducible.*

Proof. Consider a nice colouring of $G - vu$. We may assume that the colouring is the one of Figure 7. Let $\alpha \in \{1, 7, 9, 10\} \setminus \{a, b, c\}$. We recolour vx_3 with α and colour vu with 3. The obtained colouring of G is nice unless $\alpha \in \{1, 7\}$. If $\alpha = 7$ then we additionally interchange the colours of uw and vw. And if $\alpha = 1$, we interchange the colours of ut and vt. □

Lemma 10. *A 6-vertex incident to 6 triangles is not adjacent to two 5-vertices.*

Proof. Suppose that v is a 6-vertex. We let $x_1, x_2, \dots, x_6$ be its neighbours, such that x_i is adjacent to x_{i+1} if $i \in \{1, 2, \dots, 5\}$ and x_6 is adjacent to x_1. We also assume that x_6 is a 5-vertex, and we let w be the other 5-vertex. By symmetry and Lemma 2(vi), we may assume that $w \in \{x_2, x_3\}$. The



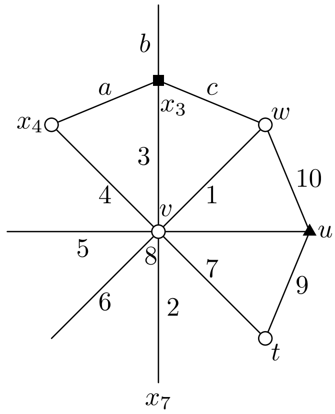


Figure 7: Reducible configurations of Lemma 9. We assume that the degree of v in G is 8.

proof is in two steps. In the first step, we show that there exists a partial 10-total-colouring of G in which only x_6 is uncoloured. In the second step, we show how to extend it to a 10-total-colouring of G.

Given a total-colouring and an element $x \in V \cup E$, recall that $\mathcal{C}(x)$ is the set of colours of all the elements of $V \cup E$ incident or adjacent to x. Recall also that if $x \in V$, $\mathcal{E}(x)$ is the set of colours of all the edges incident to x.

Let λ be a total-colouring of $G - vx_6$, in which furthermore we uncolour the vertex x_6. Our goal is to properly colour the edge vx_6. Note that $|\mathcal{C}(vx_6)| = 10$, otherwise the edge vx_6 can be greedily coloured. Without loss of generality, we may assume that the colouring is the one shown in Figure 8(a).

We want to colour vx_6 with $\lambda(vw)$. Recall that w is either x_2 or x_3. We set $\mathcal{E} := \mathcal{E}(w) \cup \{\lambda(w)\}$. If there exists a colour $\alpha \in \{7, 8, 9, 10\} \setminus \mathcal{E}$, then we set $\lambda(vx_6) := \lambda(vw)$ and $\lambda(vw) := \alpha$. Furthermore, if $1 \notin \mathcal{E}$, then interchange the colours of x_6x_1 and vx_1, colour vx_6 with $\lambda(vw)$ and recolour vw with 1. Thus, $1 \in \mathcal{E}$. Similarly, we deduce that $5 \in \mathcal{E}$. Finally, note that either 2 or 3 belongs to $\mathcal{E}$, according to whether w is x_2 or x_3. Consequently, this shows that $|\mathcal{E}| \geq 7$. But w has degree five, thus $|\mathcal{E}| = 6$, a contradiction. This concludes the first step.

Suppose now that we are given a partial 10-total-colouring of G in which only x_6 is not coloured. If we cannot extend it to G, then without loss of generality, we may assume that the colouring is the one shown in Figure 8(b). If there exists a colour $\alpha \in \{2, 4, 6, 10\} \setminus \{a, b, c, d, e\}$, then recolour vx_6 with α and colour x_6 with 7 to obtain a 10-total-colouring of G. Hence, $\{2, 4, 6, 10\} \subset \{a, b, c, d, e\}$. Suppose that $a \notin \{2, 4, 6\}$. In this case, $\{b, c, d, e\} = \{2, 4, 6, 10\}$, and thus $e \in \{2, 4, 10\}$. Interchange the colours of

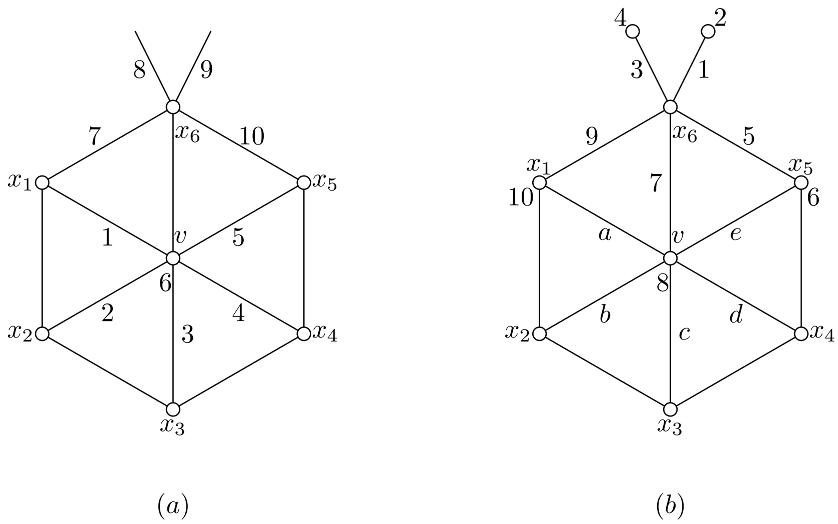


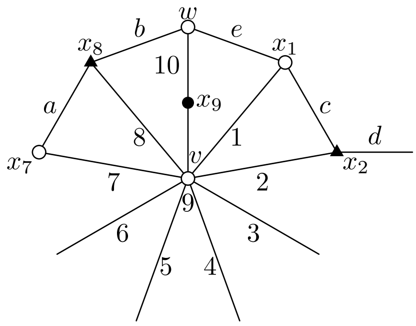
Figure 8: Proof of Lemma 10: (a) colouring of $G - vx_6$, (b) partial colouring of G in which x_6 is not coloured.

the edges x_6x_5 and vx_5. Now, if $a \neq 5$ then the obtained colouring is proper, and we extend it to G by colouring x_6 with 5. And, if $a = 5$, we additionally interchange the colours of x_6x_1 and vx_1, and colour v with 9. Consequently, we obtain $a \in \{2, 4, 6\}$.

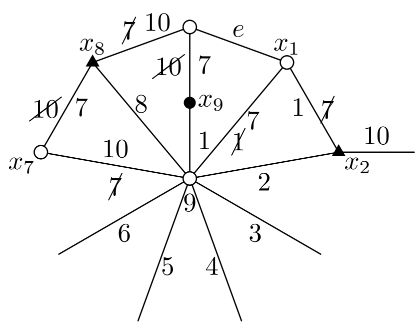
If $9 \notin \{b, c, d, e\}$, we can apply a similar recolouring. More precisely, interchange the colours of the edges x_6x_1 and vx_1. The obtained colouring is proper and can be extended to G by colouring x_6 with 9. So $9 \in \{b, c, d, e\}$, and hence $5 \notin \mathcal{E}(v)$. If $e = 9$, then $\{b, c, d\} \subset \{2, 4, 6, 10\}$. So, analogously to what precedes, it suffices to interchange the colours of x_6x_1 and vx_1, the colours of x_5x_6 and vx_5 and to colour v with 5. Therefore, we conclude that $e \in \{2, 4, 10\}$. We interchange the colours of x_6x_5 and vx_5, and colour x_6 with 5, thereby obtaining a nice colouring of G. □

Lemma 11. *The configuration of Figure* 9*(a) is reducible.*

Proof. Consider a nice colouring of $G - vx_9$. Without loss of generality, it is the one of Figure 9(a). First note that $a = 10$, otherwise we can recolour the edge vx_8 with 10 and colour vx_9 with 8. Next, we infer that $b = 7$, otherwise we can interchange the colours of x_8x_7 and vx_7, and colour vx_9 with 7. Now, observe that $10 \in \{c, d\}$, otherwise we recolour vx_2 with 10 and colour vx_9 with 2. Furthermore, $7 \in \{c, d\}$, otherwise we interchange the colours of x_8w and x_9w, and also of x_8x_7 and vx_7, recolour vx_2 with 7 and colour vx_9 with 10. Thus, $\{c, d\} = \{7, 10\}$. If $d = 7$ and $c = 10$, we just interchange the



(a)



(b)

Figure 9: Precolouring and recolouring for the proof of Lemma 11.

colours of the edges x_2x_1 and vx_1, and colour vx_9 with 1. And, if $d = 10$ and $c = 7$, the recolouring shown in Figure 9(b) is a nice colouring of G. $\square$

3 Discharging part

Recall that $G = (V, E)$ is a minimum counter-example to the statement of Theorem 1, in the sense that $|V| + |E|$ is minimum. We shall obtain a contradiction by using the Discharging Method. Here is an overview of the proof. We fix a planar embedding of G. Each vertex and face of G is assigned an initial charge. The total sum of the charges is negative by Euler's Formula. Then, some redistribution rules are applied, and vertices and faces send or receive some charge according to these rules. The total sum of the charges is not changed during this step, but at the end we infer that the charge of each vertex and face is non-negative, a contradiction.

Initial charge. We assign a charge to each vertex and face. For every $x \in V \cup F$, we define the initial charge $\mathrm{ch}(x)$ to be $\deg(x) - 4$, where $\deg(x)$ is the degree of x in G. By Euler's formula the total sum is

$$\sum_{v \in V} \mathrm{ch}(v) + \sum_{f \in F} \mathrm{ch}(f) = -8.$$

Rules. We need the following definitions to state the discharging rules. A 2-vertex is *bad* if it is not incident to a ($\geq$5)-face. A triangle is *bad* if it contains a vertex of degree at most 4. Recall that a triangle with vertices x, y and z, is a $(\deg(x), \deg(y), \deg(z))$-triangle.

Rule R0. *A* ($\geq$5)*-face sends* 1 *to each incident* 2*-vertex.*

Rule R1. *A* 5*-vertex* v *sends* 1/5 *to each incident triangle.*

Rule R2. *A* 6*-vertex sends* 13/35 *to each incident* $(5, 6, \geq 7)$*-triangle;* 1/3 *to each incident* $(6, 6, 6)$*-triangle; and* 2/7 *to each incident* $(6, \geq 6, \geq 7)$*-triangle.*

Rule R3. *A* 7*-vertex sends* 1/2 *to each incident bad triangle;* 3/7 *to each incident non-bad* $(\leq 7, \leq 7, 7)$*-triangle; and* 1/3 *to each incident non-bad triangle containing a* ($\geq$8)*-vertex.*

Rule R4. *A* 8*-vertex sends*

- (*i*) 1/3 *to each adjacent* 3*-vertex;*
- (*ii*) 1/2 *to each incident bad triangle;*
- (*iii*) 7/15 *to each incident* $(5, \leq 7, 8)$*-triangle and each incident* $(6, 6, 8)$*-triangle;*
- (*iv*) 2/5 *to each incident* $(5, \geq 8, 8)$*-triangle, each incident* $(6, 7, 8)$*-triangle and each incident* $(6, 8, 8)$*-triangle;*
- (*v*) 1/3 *to each incident* $(6, 8, 9)$*-triangle and each incident* $(\geq 7, \geq 7, 8)$*-triangle.*

Rule R5. *A* 9*-vertex sends*

- (*i*) 1 *to each adjacent bad* 2*-vertex and* 1/2 *to each adjacent non-bad* 2*-vertex;*
- (*ii*) 1/3 *to each adjacent* 3*-vertex;*
- (*iii*) 1/2 *to each incident bad triangle and each incident* $(5, \leq 7, 9)$*-triangle;*
- (*iv*) 3/7 *to each incident* $(6, 6, 9)$*-triangle;*
- (*v*) 2/5 *to each incident* $(5, \geq 8, 9)$*-triangle and each incident* $(6, \geq 7, 9)$*-triangle;*
- (*vi*) 1/3 *to each incident* $(\geq 7, \geq 7, 9)$*-triangle.*

In the sequel, we prove that the final charge $\mathrm{ch}^*(x)$ of every $x \in V \cup F$ is non-negative. Hence, we obtain

$$-8 = \sum_{v \in V} \mathrm{ch}(v) + \sum_{f \in F} \mathrm{ch}(f) = \sum_{v \in V} \mathrm{ch}^*(v) + \sum_{f \in F} \mathrm{ch}^*(f) \geq 0,$$

a contradiction. This contradiction establishes the theorem.

Final charge of faces. Let f be a d-face. Our goal is to show that $\mathrm{ch}^*(f) \geq 0$. By Lemma 2(*ii*) and (*iii*), f is incident to at most $\lfloor \frac{d}{3} \rfloor$ vertices of degree 2. Therefore, if $d \geq 5$ then by Rule R0 we obtain $\mathrm{ch}^*(f) \geq d - 4 - \lfloor \frac{d}{3} \rfloor = \lceil \frac{2d}{3} \rceil - 4 \geq 0$. A 4-face neither sends nor receives any charge, so its charge stays 0.

Finally, let $f = xyz$ be a triangle with $\deg(x) \leq \deg(y) \leq \deg(z)$. The initial charge of f is -1, and we assert that its final charge $\mathrm{ch}^*(f)$ is at least 0. We consider several cases and subcases according to the degrees of x, y and z.

$\deg(x) = 2$. Then both y and z have degree 9 by Lemma 2(*ii*), and hence f receives $1/2$ from each of y and z by Rule R5(*iii*).

$\deg(x) = 3$. In this case, by Lemma 2(*ii*) and (*iv*), we infer that $\deg(y) \geq 8$ and $\deg(z) = 9$. Thus, f receives $\frac{1}{2} + \frac{1}{2} = 1$ by Rules R4(*ii*) and R5(*iii*).

$\deg(x) = 4$. Then, by Lemma 2(*ii*) and (*v*), $\deg(y) \geq 7$ and $\deg(z) \geq 8$. Hence, by Rules R3, R4(*ii*) and R5(*iii*), f receives $\frac{1}{2} + \frac{1}{2} = 1$ from y and z.

$\deg(x) = 5$. According to Lemma 2(*vi*), $\deg(y) \geq 6$ and by Lemma 3(*i*), $\deg(z) \geq 7$. By Rule R1, f receives $1/5$ from x, so we only need to show that it receives at least $4/5$ from y and z together. Consider the following subcases.

$\deg(z) = 7$. By Rule R3, z sends $3/7$ to f, and by Rules R2 and R3, y sends at least $13/35$. Thus, f receives at least $\frac{13}{35} + \frac{3}{7} = \frac{4}{5}$ from y and z, as needed.

$\deg(z) = 8$. If $\deg(y) \leq 7$ then z sends $7/15$ to f by Rule R4(*iii*) and y sends at least $1/3$ by Rules R2 and R3. And, if $\deg(y) = 8$ then both y and z send $2/5$ to f by Rule R4(*iv*). So, in both cases f receives $4/5$ from y and z together.

$\deg(z) = 9$. Suppose first that $\deg(y) \leq 7$. Then, by Rule R5(*iii*), z sends $1/2$ to f. Moreover, by Rules R2 and R3, y sends at least

1/3 to f, which proves the assertion. Now, if $\deg(y) \geq 8$ then according to Rules R4(iv) and R5(v) f receives 2/5 from each of y and z, as needed.

$\deg(x) = 6$. First, if $\deg(z) = 6$ then f receives 1/3 from each of its vertices by Rule R2. So we assume that $\deg(z) \geq 7$. In this case, f receives 2/7 from x by Rule R2. Hence, we only need to show that y and z send at least 5/7 to f in total. We consider several cases, regarding the degree of z.

$\deg(z) = 7$. Then f receives 3/7 from z by Rule R3, and at least 2/7 from y by Rules R2 and R3, as desired.

$\deg(z) = 8$. If $\deg(y) = 6$, then z sends 7/15 by Rule R4(iii) and y sends 2/7 by Rule R2. And, if $\deg(y) \geq 7$ then y sends at least 1/3 by Rules R3 and R4(iv), and z sends at least 2/5 by Rule R4(iv).

$\deg(z) = 9$. If $\deg(y) = 6$ then f receives 2/7 from y by Rule R2 and 3/7 from z by Rule R5(iv). And, if $\deg(y) \geq 7$ then f receives at least 1/3 from y by Rules R3, R4(v) and R5(v), and at least 2/5 from z by Rule R5(v), which yields the result.

$\deg(x) \geq 7$. The assertion follows from Rules R3, R4(v) and R5(vi).

Final charge of vertices. Let v be an arbitrary vertex of G. We have $\deg(v) \geq 2$ by Lemma 2(i). For every positive integer d, we define v_d to be the number of d-neighbours of v, and f_d to be the number of its incident d-faces. Let $x_1, x_2, \ldots, x_{\deg(v)}$ be the neighbours of v in clockwise order. We prove that the final charge of v is non-negative. To do so, we consider several cases, regarding the degree of v.

If $\deg(v) = 2$, then its two neighbours are 9-vertices by Lemma 2(ii). If v is bad then it receives 1 from each of its two 9-neighbours by Rule R5(i), while otherwise it receives at least 1 from its incident faces by Rule R0, and 1/2 from each of its two 9-neighbours by Rule R5(i). Thus, in both cases, its final charge is at least 0.

If $\deg(v) = 3$, then all its neighbours have degree at least 8, so by Rules R4(i) and R5(ii) it receives 1/3 from each of its neighbours, setting its final charge to 0. If $\deg(v) = 4$, then it neither sends nor receives anything, so its charge stays 0. If v is a 5-vertex, then by Rule R1 it sends 1/5 to each of its at most five incident triangles, therefore its final charge is non-negative.

Suppose now that v is a 6-vertex. All its neighbours have degree at least 5 by Lemma 2(ii). Note that if $f_3 \leq 5$ then, according to Rule R2, $\mathrm{ch}^*(v) \geq$

$2-5\cdot\frac{13}{35}>0$. So, we assume now that $f_3=6$, i.e. v is incident to 6 triangles. Thus, we infer from Lemma 10 that $v_5\leq 1$. If $v_5=0$, then following Rule R2, v sends at most $6\cdot\frac{1}{3}=2$, so its final charge is at least 0. And, if $v_5=1$, then let x_1 be the unique 5-neighbour of v. By Lemma 3(i), $\deg(x_2)\geq 7$ and $\deg(x_6)\geq 7$. Consequently, vx_3x_2 and vx_5x_6 are two $(6,\geq 6,\geq 7)$-triangles. Thus, by Rule R2, $\mathrm{ch}^*(v)\geq 2-2\cdot\frac{13}{35}-2\cdot\frac{1}{3}-2\cdot\frac{2}{7}=\frac{2}{105}>0$.

Suppose that v is a 7-vertex. If $f_3\leq 6$ then $\mathrm{ch}^*(v)\geq 3-6\cdot\frac{1}{2}=0$ by Rule R3. So, we assume now that $f_3=7$. We consider several cases, according to the number of 4-neighbours of v. Note that, by Lemma 2(ii) and Lemma 3(iii), v has at most two such neighbours, i.e $v_4\leq 2$.

$v_4=0$. According to Rule R3, we have $\mathrm{ch}^*(v)\geq 3-7\cdot\frac{3}{7}=0$.

$v_4=1$. Let x_1 be this 4-neighbour. So, x_2 and x_7 both are 9-vertices by Lemma 2(v). According to Rule R3, v sends at most 1/3 to each of vx_2x_3 and vx_5x_6. Furthermore, v is incident to exactly two bad triangles, and sends at most 3/7 to each non-bad triangle. Therefore, we obtain $\mathrm{ch}^*(v)\geq 3-2\cdot\frac{1}{2}-3\cdot\frac{3}{7}-2\cdot\frac{1}{3}=\frac{1}{21}>0$.

$v_4=2$. Without loss of generality, we assume that x_1 has degree 4. According to Lemmas 2(ii) and 3(iii), the other 4-neighbour of v must be x_4 or x_5, say x_4 by symmetry. By Lemma 2(v), x_2,x_7,x_3 and x_5 all have degree 9. Note that x_6 has degree at least 5. Consequently, $\mathrm{ch}^*(v)\geq 3-4\cdot\frac{1}{2}-3\cdot\frac{1}{3}=0$.

Suppose now that v is an 8-vertex. If $v_3=0$, then $\mathrm{ch}^*(v)\geq 4-8\cdot\frac{1}{2}=0$ by Rule R4. Thus, we assume now that x_1 is a 3-vertex. Notice that, by Lemma 5, if a 3-neighbour of v is on a triangle then $v_3=1$. Therefore, $v_3+f_3\leq 9$. If $f_3\leq 6$, we obtain $\mathrm{ch}^*(v)\geq 4-6\cdot\frac{1}{2}-3\cdot\frac{1}{3}=0$. If $f_3=7$, we infer from Lemma 5 that $v_3\leq 1$, and so $\mathrm{ch}^*(v)\geq 4-7\cdot\frac{1}{2}-\frac{1}{3}=\frac{1}{6}>0$. Now we suppose that $f_3=8$, and thus $v_3=1$. Note that, according to Lemma 2(iv), $\deg(x_2)=9$ and $\deg(x_9)=9$. Moreover, by Lemma 9, all vertices but x_1 have degree at least five. Thus, by Rule R4, we infer that $\mathrm{ch}^*(v)\geq 4-\frac{1}{3}-2\cdot\frac{1}{2}-4\cdot\frac{7}{15}-2\cdot\frac{2}{5}=0$.

Finally, suppose that v is a 9-vertex. By Lemma 2(iii), v is adjacent to at most one 2-vertex. We consider two cases.

Case 1: $v_2=0$. Suppose first that v is incident to a (≥ 4)-face, i.e. $f_3\leq 8$. If $f_3=8$ then $v_3\leq 3$ by Lemmas 4 and 6, and hence $\mathrm{ch}^*(v)\geq 5-8\cdot\frac{1}{2}-3\cdot\frac{1}{3}=0$. If $f_3\leq 7$, then we assert that $f_3+v_3\leq 12$. Indeed, if $v_3\geq 6$ then, as two 3-vertices are not adjacent, we infer that $f_3\leq 2(9-v_3)$, which yields the assertion. So if $f_3\leq 6$, we obtain $\mathrm{ch}^*(v)\geq 5-6\cdot\frac{1}{2}-6\cdot\frac{1}{3}=0$. If

$f_3 = 7$ then we can see that $v_3 \leq 4$ by Lemmas 4 and 6. Consequently, $\mathrm{ch}^*(v) \geq 5 - 7 \cdot \frac{1}{2} - 4 \cdot \frac{1}{3} = \frac{1}{6} > 0$. Now assume that $f_3 = 9$. According to Lemma 4, $v_3 \leq 2$. If $v_3 \leq 1$, then $\mathrm{ch}^*(v) \geq 5 - 9 \cdot \frac{1}{2} - \frac{1}{3} = \frac{1}{6} > 0$. Assume now that $v_3 = 2$. Without loss of generality, say that x_1 is a 3-vertex, thus both x_2 and x_9 are ($\geq$8)-vertices. Note that by Lemma 6, both x_3 and x_8 are ($\geq$4)-vertices. Therefore, up to symmetry, it suffices to consider the following two cases.

x_4 is the second 3-neighbour. Then $\deg(x_3) \geq 8$, so by Rule R5(vi), v sends $\frac{1}{3}$ to vx_2x_3, so we infer that $\mathrm{ch}^*(v) \geq 5 - 2 \cdot \frac{1}{3} - 8 \cdot \frac{1}{2} - \frac{1}{3} = 0$.

x_5 is the second 3-neighbour. In this case, $\deg(x_4) \geq 8$ and $\deg(x_6) \geq 8$. Furthermore, by Lemma 7, $\deg(x_3) \geq 5$. Consequently, x_3x_2v and x_3x_4v both are $(\geq 5, \geq 8, 9)$-triangles hence, by Rule R5(v), v sends at most $\frac{2}{5}$ to each of them. So, $\mathrm{ch}^*(v) \geq 5 - 2 \cdot \frac{1}{3} - 7 \cdot \frac{1}{2} - 2 \cdot \frac{2}{5} = \frac{1}{30} > 0$.

Case 2: $v_2 = 1$. Let x_1 be the 2-neighbour. Observe that by Lemma 3(vii), v cannot have a 3-neighbour on two triangles. Moreover, x_1 cannot lie on two triangles, so $f_3 \leq 8$. We consider the following possibilities.

x_1 is on a triangle. Let this triangle be vx_1x_2. From Lemma 3(iv) and (v), we infer that $f_3 + v_3 \leq 8$. So, $\mathrm{ch}^*(v) \geq 5 - 1 - 8 \cdot \frac{1}{2} = 0$.

x_1 is bad but not on a triangle. In this case, x_1 is on two 4-faces, so in particular $f_3 \leq 7$. Note that by Lemma 3(vi), either one vertex among x_2, x_9 has degree at least 4, or $f_3 \leq 5$. Besides, according to Lemma 3(vii) there is no 3-neighbour on two triangles. Observe also that if both vx_2x_3 and vx_8x_9 are triangles, then Lemma 11 implies that $v_3 \leq 6$. Let us consider several cases regarding the value of f_3.

- $f_3 \leq 4$. Then $f_3 + v_3 \leq 10$, otherwise we obtain a contradiction by Lemma 3(vi) and (vii). Thus, $\mathrm{ch}^*(v) \geq 5 - 1 - 4 \cdot \frac{1}{2} - 6 \cdot \frac{1}{3} = 0$.
- $f_3 = 5$. Using Lemma 3(vi) and (vii), a small case-analysis shows that $v_3 \leq 5$. Moreover, if $v_3 = 5$ then the obtained configuration is the one of Lemma 11, which is reducible. And, if $v_3 \leq 4$ then we obtain $\mathrm{ch}^*(v) \geq 5 - 1 - 5 \cdot \frac{1}{2} - 4 \cdot \frac{1}{3} = \frac{1}{6} > 0$.
- $f_3 = 6$. In this case, $v_3 \leq 3$ by Lemma 3(vi) and (vii). Thus, $\mathrm{ch}^*(v) \geq 5 - 1 - 6 \cdot \frac{1}{2} - 3 \cdot \frac{1}{3} = 0$.
- $f_3 = 7$. By Lemma 3(vi) and (vii), v has at most one 3-neighbour, namely x_2 or x_9. Thus, $\mathrm{ch}^*(v) \geq 5 - 1 - 7 \cdot \frac{1}{2} - \frac{1}{3} = \frac{1}{6} > 0$.

x_1 is neither bad nor on a triangle. Notice that $f_3 \leq 7$. Let us again consider several cases regarding the value of f_3.

$f_3 \leq 5$. In this case, $f_3 + v_3 \leq 11$ by Lemma 3(*vii*). So, $\mathrm{ch}^*(v) \geq 5 - \frac{1}{2} - 5 \cdot \frac{1}{2} - 6 \cdot \frac{1}{3} = 0$.

$f_3 = 6$. Similarly as before, we infer that $v_3 \leq 4$, and hence $\mathrm{ch}^*(v) \geq 5 - \frac{1}{2} - 6 \cdot \frac{1}{2} - 4 \cdot \frac{1}{3} = \frac{1}{6} > 0$.

$f_3 = 7$. By Lemma 3(*vii*) the vertex v has at most two 3-neighbours, namely x_2 and x_9. Thus, $\mathrm{ch}^*(v) \geq 5 - \frac{1}{2} - 7 \cdot \frac{1}{2} - 2 \cdot \frac{1}{3} = \frac{1}{3} > 0$.

This establishes that the final charge of every vertex is non-negative, so the proof of Theorem 1 is now complete. □

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