New almost-planar crossing-critical graph families

Petr Hliněný*

Faculty of Informatics, Masaryk University
Botanická 68a, 602 00 Brno, Czech Republic

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Abstract

We show that, for all choices of integers $k > 2$ and m, there are simple 3-connected k-crossing-critical graphs containing more than m vertices of each even degree $\leq 2k - 2$. This construction answers one half of a question raised by Bokal, while the other half asking analogously about vertices of odd degrees at least 5 in crossing-critical graphs remains open. Furthermore, our constructed graphs have several other interesting properties; for instance, they are almost planar and their average degree can attain any rational value in the interval $\left[4, 6 - \frac{8}{k+1}\right)$.

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1 Introduction

We assume that the reader is familiar with basic terms of graph theory. In a *drawing* of a graph G in the plane the vertices of G are points and the edges are simple curves joining their endvertices. Moreover, it is required that no edge passes through a vertex (except at its ends), and no three edges cross in

*`hlineny@fi.muni.cz`. Supported by grant GAČR 201/05/0050.

a common point. The *crossing number* $\mathrm{cr}(G)$ of a graph G is the minimum number of crossing points of edges in a drawing of G in the plane.

For $k \geq 1$, we say that a graph G is *k-crossing-critical* if $\mathrm{cr}(G) \geq k$ but $\mathrm{cr}(G - e) < k$ for each edge $e \in E(G)$. It is important to study crossing-critical graphs in order to understand structural properties of the crossing number problem. The only 1-crossing-critical graphs are, by the Kuratowski theorem, the subdivisions of K_5 and $K_{3,3}$. A construction of an infinite family of 2-crossing-critical simple 3-connected graphs was published by Kochol [4], improving previous construction by Širáň [8]. Many more crossing-critical constructions have appeared since. (The known constructions tend to have similar structure, and moreover all k-crossing-critical graphs are proved [3] to have path-width bounded in k.)

It has been noted by Bokal [1] that the (typical) known constructions of infinite families of simple 3-connected k-crossing-critical graphs create only bounded numbers (wrt. k) of vertices of degrees other than $3, 4$, or 6. His natural question thus was, what about occurence of other vertex degree values in infinite families of k-crossing-critical graphs? We positively answer one half of his question (see Theorem 3.1 and Proposition 2.1);

- namely we construct, for each $k > 2$, an infinite family of simple 3-connected k-crossing-critical graphs such that members of this family contain arbitrary numbers of vertices of each *even degree* $4, 6, 8, \ldots, 2k - 2$.

The analogous question about occurence of vertices of odd degrees ≥ 5 in k-crossing-critical graphs remains open, and it appears to be significantly harder than the even case. Actually one should note that the question about an existence of simple 5-regular crossing-critical graphs was first raised by Richter and Thomassen [5].

Usual constructions of crossing-critical graphs use an approach that can be described as a "Möbius twist"—they create graphs embeddable on a Möbius band which thus have to be "twisted" for drawing in the plane. We offer a quite different approach in Section 2, which extends our older construction [2], resulting in graphs that are *almost planar*, i.e. they can be made planar by deleting just one edge. As an easy corollary of this new and very flexible construction, we also produce almost-planar crossing-critical families of other interesting properties like prescribed average degree, as summarized in Section 4.

2 "Belt" construction

An illustrating example of crossing-critical graphs constructed in our old work [2] is shown in Figure 1. The construction in [2] used vertices of degrees 4 or 3, and now we generalize it to allow more flexible structure and, particularly, vertices of arbitrary even degrees.

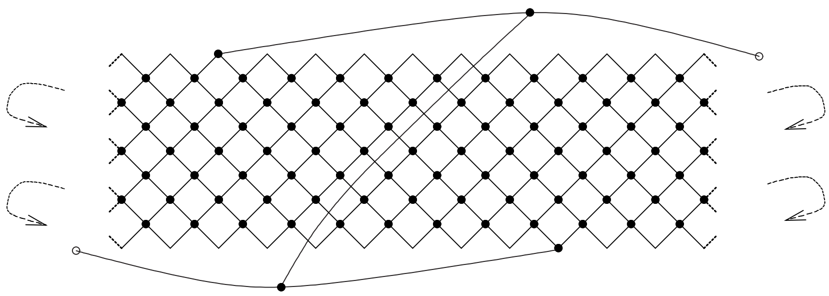

Figure 1: An example of a simple 3-connected almost planar 8-crossing-critical graph. (The "grid-belt" is winded around a cylinder.)

Having a path P in a graph G, we call a *reduced length* of P the number of internal vertices of P having degree greater than 2 in G plus one. The *reduced distance* in G is defined accordingly. (If $\delta(G) > 2$, then this parameter equals usual graph distance.) A path P is an *ear* of a subgraph F in G if both ends of P belong to F, but all internal vertices of P are disjoint from F. For easier notation, we (in the coming definitions) consider embeddings in the plane $\mathcal{P}$ with removed open set $\mathcal{X}$. We say that a closed curve (loop) γ is of *type-$\mathcal{X}$* if the homotopy type of γ in $\mathcal{P} \setminus \mathcal{X}$ is to "wind once around $\mathcal{X}$". Having two loops γ, δ of type-$\mathcal{X}$, we write $\gamma \preceq \delta$ if γ separates $\mathcal{X}$ from $\delta \setminus \gamma$ (meaning γ is "*nested*" inside δ).

Crossed belt graphs. A plane graph F_0 is a *plane k-belt* graph if it can be constructed as an edge-disjoint union of k embedded "*belt*" cycles $C_1 \cup C_2 \cup \cdots \cup C_k = F_0$, where all $C_1, \ldots, C_k$ are of type-$\mathcal{X}$ nested as $C_1 \preceq C_2 \preceq \cdots \preceq C_k$. Moreover, the following must be true:

(B1) F_0 contains $4k$ pairwise disjoint "*radial*" paths $R_1, R_2, \ldots, R_{4k}$, each one connecting a vertex of $V(C_1)$ to a vertex of $V(C_k)$. Their ends on C_1 have this cyclic order: $R_1, R_2, \ldots, R_{4k}$.

(B2) We call a vertex of F_0 *accumulation* if its degree is at least 6 in F_0, i.e. if it is contained in at least three of the cycles $C_1, \ldots, C_k$. There is no accumulation vertex on the cycle C_k.

(B3) Denote by s_1, t_1 the ends of R_k and R_{3k}, respectively, on C_1. Analogously denote by s_2, t_2 the ends of R_{2k} and R_{4k} on C_k. Then $s_1, t_1 \in V(C_1) \setminus V(C_2)$, $s_2, t_2 \in V(C_k) \setminus V(C_{k-1})$, and each of s_1, t_1, s_2, t_2 must have reduced distance at least k from every accumulation vertex in F_0.

A graph F is a *crossed k-belt* if it is $F = F_0 \cup S_0 \cup S_1 \cup S_2$, where

- F_0 is a planar k-belt graph as above;
- S_1, S_2 are disjoint ears of F_0 with ends s_1, t_1 for S_1 and s_2, t_2 for S_2;
- and path S_0, disjoint from F_0, connects a vertex of S_1 to a vertex of S_2.

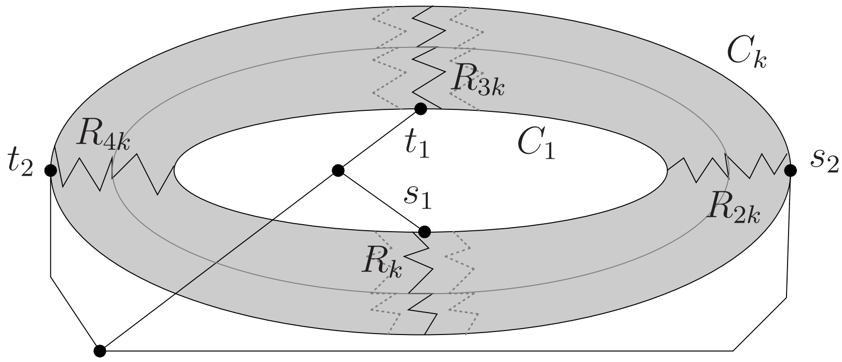

Figure 2: An illustration of the definition of a *crossed k-belt* graph.

This definition is illustrated in Figure 2. Notice that a crossed 1-belt graph is always a subdivision of $K_{3,3}$, and that removing an edge of S_0 from a crossed k-belt graph leaves it planar. Particularly, the graph in Figure 1 is a crossed 8-belt graph without accumulation vertices, and we call this special case a "*square-grid*" 8-belt graph. We aim to show that crossed k-belt graphs are k-crossing-critical with the exception of $k = 2$. (This exception is remarkable in view of successful research progress into the structure of 2-crossing-critical graphs.)

The cruical property which motivated our construction is stated here:

Proposition 2.1 *Let k be fixed. For every integer m there is a crossed k-belt graph which is simple 3-connected and which contains more than m vertices of each of degrees $\ell = 4, 6, 8, \ldots, 2k-2$.*

Proof. In this case a picture is worth more than thousand words. Figure 3 shows a local modification of a "square-grid" 8-belt graph which produces

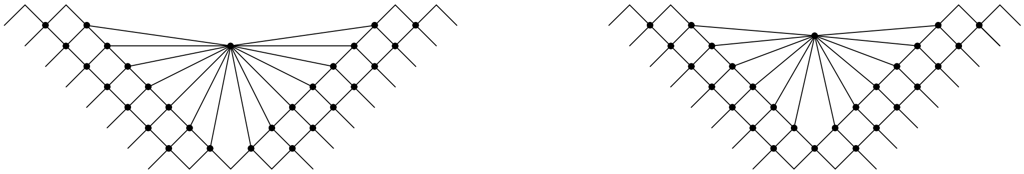

Figure 3:

accumulation vertices of degrees 14 and 12 while preserving its simplicity and connectivity. It is straightforward to generalize this picture to any $k > 3$ and all degrees $\ell = 6, 8, \ldots, 2k-2$. Starting from a sufficiently large "square-grid" k-belt graph F, we can produce in this way F' with arbitrarily many accumulation vertices of each degree $\ell = 6, 8, \ldots, 2k-2$, all of which are sufficiently far from the vertices s_1, t_1, s_2, t_2 as in the condition (B3). $\square$

3 Crossing-criticality

We continue to use the notation from the definition of k-belt graphs also in this section. Now we come to the main result of our paper.

Theorem 3.1 *For $k \geq 3$, every crossed k-belt graph is k-crossing-critical.*

Proof. Let F be our k-belt graph, considered with notation as in the definition above. In one direction, by a straightforward induction we argue that any crossed k-belt graph, $k \geq 1$, can be drawn such that the only crossings occur between the path S_0 and each of the belt cycles $C_1, \ldots, C_k$ once. This is trivial for $k = 1$. For $k > 1$, we draw a $(k-1)$-belt subgraph F' from Lemma 3.2 with $k-1$ crossings between S_0 and each of the belt cycles $C_2, \ldots, C_k$, in a way that one end of S_0 is inside the set $\mathcal{X}$ (see the definition of type-$\mathcal{X}$ in Section 2) and the other end of S_0 is in the face of C_k not with $\mathcal{X}$. By definition the remaining cycle C_1 is nested inside each cycle C_i, $i > 1$, and so to obtain an analogous drawing of (whole) F it is enough to add one more crossing of S_0 with C_1 since C_1 is also of type-$\mathcal{X}$. Furthermore, using analogous arguments, it is easy to verify that deleting any edge e of F allows us to draw $F - e$ with fewer than k crossings.

In the other direction, we assume an arbitrary drawing $\mathcal{F}$ of F, and we prove that $\mathcal{F}$ has at least k edge crossings. There are two possibilities—either

C_1 is drawn uncrossed in $\mathcal{F}$, or some edge of C_1 is crossed in $\mathcal{F}$. In the first case we will argue that $\mathrm{cr}(\mathcal{F}) \geq 2(k-1)$.

Let p_i and q_i, for $i = 1, 2, \ldots, k-1$, denote some vertices in the intersection of the belt cycle C_{i+1} with the radial paths R_{k-i} and R_{k+i}, respectively; and let a path Q_i be formed as a union of the subpath of R_{k-i} from its end on C_1 to p_i, the subpath of C_{i+1} between p_i and q_i intersecting R_k, and the subpath of R_{k+i} from q_i to its end on C_1. A path Q_{k+i-1}, for $i = 1, 2, \ldots, k-1$, is defined analogously from the radial paths R_{3k-i} and R_{3k+i} and the cycle C_{i+1}. Notice that all the defined paths $Q_1, \ldots, Q_{2k-2}$ are pairwise edge-disjoint by the construction of a plane k-belt and (B1), and since each Q_i of them is disjoint from s_1, t_1 on C_1 (B3), Q_i separates the vertices s_1 and t_1 from each other.

By (B2) the cycle C_k is disjoint from C_1 if $k \geq 3$. Hence by Jordan's curve theorem, if the cycle C_1 is not crossed in our drawing $\mathcal{F}$, then the whole component of $F - V(C_1)$ containing C_k must be drawn in one region of the drawing of C_1. This includes the path S_1 connected with C_k via $S_2 \cup S_0$, and all the paths $Q_1, \ldots, Q_{2k-2}$ connected with C_k via the radial path R_k. Thus, by Jordan's curve theorem again, the drawing of S_1 must cross the drawings of each of $Q_1, \ldots, Q_{2k-2}$, witnessing $\mathrm{cr}(\mathcal{F}) \geq 2k-2 > k$ if $k \geq 3$.

In the second possibility there is an edge f of C_1 which is crossed in $\mathcal{F}$. We apply Lemma 3.2 to F and f, so obtaining a crossed $(k-1)$-belt subgraph F' of $F - f$, and conclude by induction that $\mathrm{cr}(\mathcal{F}) \geq 1 + \mathrm{cr}(F') = 1 + (k-1) = k$ if the claim holds true in the base case $k = 3$. Hence it remains to consider $(k = 3)$ a crossed 3-belt graph F with an optimal drawing $\mathcal{F}$ such that both C_1, C_3 contain crossed edges f and f', respectively. (Since in the case $k = 3$ there can be no accumulation vertex in F, there is a symmetry between C_1 and C_3, and the case when C_1 is uncrossed has already been solved above.) We may easily choose f and f' with distinct crossings in $\mathcal{F}$, since even if C_1 crossed C_3, they would have to cross twice as disjoint cycles. Hence we can apply Lemma 3.2, with f and then with f', to obtain a 1-belt graph F'' (a subdivision of $K_{3,3}$), concluding that $\mathrm{cr}(\mathcal{F}) \geq 1 + 1 + \mathrm{cr}(F'') = 1 + 1 + 1 = 3 = k$. □

Lemma 3.2 *Let F be a crossed k-belt graph as above, and choose $f \in E(C_1)$. Then $F - f$ contains a crossed $(k-1)$-belt subgraph F' having $C_2, \ldots, C_k$ as its collection of belt cycles.*

Proof. We refer to the notation in the definition of belt graphs. Let s_1', t_1' denote vertices of $C_1 \cap C_2$ closest in $C_1 - f$ to s_1, t_1, respectively. So the reduced distance between s_1 and s_1' in $F - f$ is 1, and $s_1' \notin V(C_3)$ since (B3). The same holds for t_1'. Let $S_1' \subseteq S_1 \cup C_1 - f$ denote the ear of C_2 with ends s_1', t_1', and F_0' denote the subgraph of F induced on $V(C_2) \cup \cdots \cup V(C_k)$. We claim that $F' = F_0' \cup S_1' \cup S_2 \cup S_0$ is a crossed $(k-1)$-belt graph:

Concerning condition (B1), we restrict all the radial paths of F to F_0', and we "drop" $R_{k-1}, R_{k+1}, R_{3k-1}, R_{3k+1}$ of them. Then we form R_k' as an extension of R_k on C_2 with the end s_1'. (R_k' stays disjoint from both R_{k-2}, R_{k+2}.) Analogously we get R_{3k}' with the end t_1' on C_2. So $R_1, \ldots, R_{k-2}, R_k', R_{k+2}, \ldots, R_{3k-2}, R_{3k}', R_{3k+2}, \ldots, R_{4k}$ are desired $4(k-1)$ radial paths in F'. Condition (B2) is immediately inherited by F', and (B3) also follows for F' with parameter $k-1$ since the reduced distance of s_1' from s_1 is 1 and $s_1' \in V(C_2) \setminus V(C_3)$. □

4 Additional remarks

Although the main motivation for our k-belt construction of crossing-critical graphs was to answer a part of Bokal's [1, Section 6] question, the critical graph classes we obtain are so rich and flexible that they deserve further consideration and applications.

We look here at one particular question studied in a series of papers [7, 6, 1]: Salazar has constructed infinite families of k-crossing-critical graphs with average degree equal to any rational in the interval $[4, 6)$. Then Pinontoan and Richter [6] extended this to the interval $(3.5, 4)$, and finally Bokal [1] has found k-crossing-critical families for any rational average degree in the interval $(3, 6)$. (Average degrees ≤ 3 or > 6 cannot occur for infinite families, and the average degree 6 remains an open case.)

Using our construction and Theorem 3.1, we can essentially duplicate Salazar's result in Proposition 4.1. Although this brings nothing new for general crossing-critical graphs, we consider our addition worthy for two reasons—our construction is very simple and explicit, and it extends the result to the restricted subclass of almost-planar crossing-critical graphs.

Proposition 4.1 *For every odd $k > 3$ there are infinitely many simple 3-connected crossed k-belt graphs with the average degree equal to any given rational value in the interval $\left[4,\, 6 - \frac{8}{k+1}\right)$.*

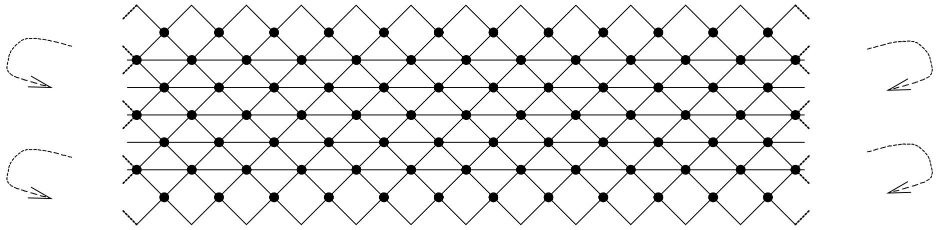

Figure 4: An approach to a plane 13-belt graph with accumulation vertices of degree 6.

Proof. Figure 4 illustrates a construction of a plane graph F_1 that fulfills all conditions of the definition of a plane 13-belt graph except (B3). *Splitting* of a vertex is a simple-graph inverse (not necessarily unique) of the edge-contraction operation. Figure 5 shows details of two "splitting" operations which can be applied to any accumulation vertex of F_1, and which preserve simplicity and 3-connectivity of F_1. Hence if we apply such splittings to all (at least) accumulation vertices of F_1 at reduced distance $<k$ from s_1, t_1, s_2, or t_2, we satisfy also (B3), and so construct a proper plane 13-belt graph F_0 and subsequently a crossed 13-belt graph F.

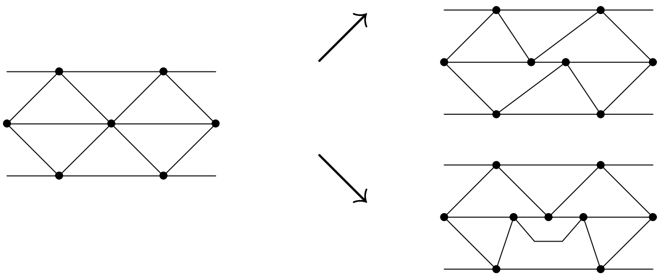

Figure 5: Details of *single-split* (top) and *double-split* (bottom) operations in the graph from Figure 4.

The construction F_1 from Figure 4 can easily be generalized for any odd $k > 3$. Let ℓ be the length of the C_1-cycle in F_1, and let the number of accumulation vertices from F_1 that are single-split during the construction of F_0 be m and the number of double-split accumulation vertices be m'. Admissible values of m and m' in our construction are at most the total number of accumulation vertices $m + m' \leq \ell(k-3)/2$, and at least $m \geq 4k^2$ since it is enough to split k^2 accumulation vertices from F_1 near each of s_1, t_1, s_2, t_2 to get a proper k-belt graph.

An easy calculation shows that F_0 has $\ell(k+1)/2+m+2m'$ vertices, and so F has $\ell(k+1)/2+m+2m'+6$ vertices. The average degree of F is

$$d_{avg}(F) = \frac{6k\ell - 2\ell + 4m + 12m' + 36}{k\ell + \ell + 2m + 4m' + 12} = 6 - \frac{8\ell + 8m + 12m' + 36}{k\ell + \ell + 2m + 4m' + 12}\,. \quad (1)$$

Now choose any rational $d_{avg} \in \left[4,\, 6-\frac{8}{k+1}\right)$. Then setting $d_{avg} = 6-\frac{p}{q} = 6-\frac{cp}{cq}$ in (1) gives a system of two linear equations in two unknowns ℓ, m and parameters k, c, m', which is nonsingular for each $k \neq 1$. Its solution is

$$\ell = \frac{c}{4k-4}(4q-p) - \frac{m'+3}{k-1}\,, \qquad m = \frac{cp}{8} - \frac{12(m'+3)}{8} - \ell\,.$$

The expressions show that choosing our parameters as $m'+3 = 2(k-1)$ and $c = c' \cdot 8(k-1)$ leads always to integer values of ℓ and m as

$$\ell = c'(8q-2p) - 2\,, \qquad m = c'\big((k+1)p - 8q\big) - 3k + 5\,. \quad (2)$$

By the choice $6-\frac{p}{q} \in \left[4,\, 6-\frac{8}{k+1}\right)$ it is easy to show in (2) that always $m + m' \leq \ell(k-3)/2 - 3$, and since $(k+1)p - 8q > 0$ it follows that for sufficiently large choices of c' we get also $m \geq 4k^2$. Thus we get from (2) an infinite sequence of admissible pairs ℓ, m (note fixed k and $m' = 2k-5$), defining each one a crossed k-belt graph F with average degree exactly $6-\frac{p}{q}$ as needed. This holds for any fixed odd $k > 3$. □

In connection with the above construction it appears interesting to ask:

Question 4.2 *Is there an infinite family of almost-planar simple* 3*-connected k-crossing-critical graphs with average degree equal to some* $d \in (3,4)$*?*

Acknowledgments

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