Connectivity of Matching graph of Hypercube

Jiří Fink*

Department of Applied Mathematics
Faculty of Mathematics and Physics
Charles University
Malostranské náměstí 25
118 00 Prague 1
E-mail: fink@atrey.karlin.mff.cuni.cz

Abstract

The *matching graph* $\mathcal{M}(G)$ of a graph G has a vertex set of all perfect matchings of G, with two vertices being adjacent whenever the union of the corresponding perfect matchings forms a Hamiltonian cycle.

We prove that the matching graph $\mathcal{M}(Q_d)$ of the d-dimensional hypercube is bipartite and connected for $d \geq 4$. This proves Kreweras' conjecture [2] that the graph M_d is connected, where M_d is obtained from $\mathcal{M}(Q_d)$ by contracting all vertices of $\mathcal{M}(Q_d)$ which correspond to isomorphic perfect matchings.

1 Introduction

A set of edges $P \subseteq E$ of a graph $G = (V, E)$ is *matching* if every vertex of G is incident with at most one edge of P. If a vertex v of G is incident with an edge of P, then v is *covered* by P, otherwise v is *uncovered* by P. A matching P is *perfect* if every vertex of G is covered by P.

The *d-dimensional hypercube* (shortly *d-cube*) Q_d is a graph whose vertex set consists of all binary vectors of length d, with two vertices being adjacent whenever the corresponding vectors differ at exactly one coordinate. The binary vectors are labelled by the set $[d] := \{1, 2, \ldots, d\}$.

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It is well known that Q_d is Hamiltonian for every $d \geq 2$. This statement can be traced back to 1872 [4]. Since then the research on Hamiltonian cycles in d-cubes satisfying certain additional properties has received considerable attention. An interested reader can find more details about this topic in the survey of Savage [3]. Dvořák [5] showed that every set of at most $2d-3$ edges of Q_d ($d \geq 2$) that induces vertex-disjoint paths is contained in a Hamiltonian cycle. Dimitrov et al. [6] proved that for every perfect matching P of Q_d ($d \geq 3$) there exists some Hamiltonian cycle that faults P, if and only if P is not a set of all edges of one dimension of Q_d.

The *matching graph* $\mathcal{M}(G)$ of a graph G on even number of vertices has a vertex set of all perfect matchings of G, with two vertices being adjacent whenever the union of the corresponding perfect matchings forms a Hamiltonian cycle of G. There is a natural one-to-one correspondence between Hamiltonian cycles of G and edges of $\mathcal{M}(G)$. The problem of determining $h(d)$, the number of Hamiltonian cycles of d-cube, is a well-known open problem. Douglas [7] presents upper and lower bounds

$$\left(\prod_{i=5}^{d-1} i^{2^{d-i-1}}\right) d(1344)^{2^{d-4}} 2^{2^{d-2}-1-d} \leq h(d) \leq \left(\frac{d(d-1)}{2}\right)^{2^{d-1}-2^{d-1-\log_2(d)}}.$$

We are interested in structural properties of $\mathcal{M}(Q_d)$.

We say that two perfect matchings P and R are isomorphic if there exists an isomorphism $f : V(Q_d) \to V(Q_d)$ such that $f(u)f(v) \in R$ for every edge $uv \in P$. This relation of isomorphism is an equivalence and it factors the set of all perfect matchings. Kreweras [2] considered a graph M_d which is obtained from $\mathcal{M}(Q_d)$ by contracting all vertices of each class of this equivalence.

Kreweras [2] proved by inspection of all perfect matchings that the graphs M_3 and M_4 are connected and he conjectured that the graph M_d is connected for every $d \geq 3$. It is more general to also ask whether the graph $\mathcal{M}(Q_d)$ is connected since the connectivity of $\mathcal{M}(Q_d)$ implies the connectivity of M_d. The answer is negative for $d = 3$ (see Figure 1). However, we prove that this is the only counter-example.

We also prove that the matching graph $\mathcal{M}(K_{n,n})$ of the complete bipartite graph $K_{n,n}$ is bipartite for even n, which implies that $\mathcal{M}(Q_d)$ is bipartite. This is an interesting property which helps us to find a walk in $\mathcal{M}(Q_d)$ of even length.

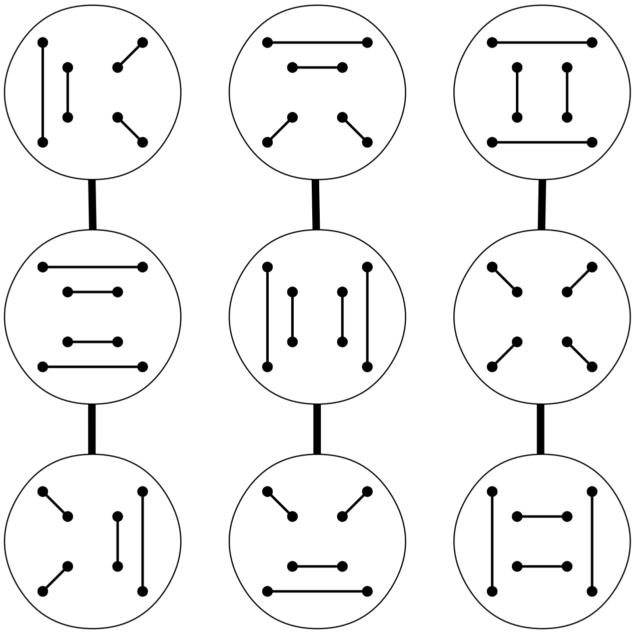

Figure 1: The matching graph $\mathcal{M}(Q_3)$. The circles and bold lines are vertices and edges of $\mathcal{M}(Q_3)$.

2 Perfect matchings extend to Hamiltonian cycles

Let $K(G)$ be the complete graph on the vertices of a graph G. If G is bipartite and connected, then let $B(G)$ be complete bipartite graph with same color classes as G. Let P be a perfect matching of $K(Q_d)$. Let $\Gamma(P)$ be the set of all perfect matchings R of Q_d such that $P \cup R$ is a Hamiltonian cycle of $K(Q_d)$. Note that if P is a perfect matching of Q_d and $R \in \Gamma(P)$, then $P \cup R$ is a Hamiltonian cycle of Q_d, so PR is an edge of $\mathcal{M}(Q_d)$.

Kreweras conjectured [2] that every perfect matching in the d-cube with $d \geq 2$ extends to a Hamiltonian cycle. We [1] proved following stronger form of this conjecture.

Theorem 1 ([1]). *For every perfect matching P of $K(Q_d)$ the set $\Gamma(P)$ is non-empty where $d \geq 2$.*

We say that an edge uv of $K(Q_d)$ *crosses* a dimension $\alpha \in [d]$ if vertices u and v differ in dimension α, otherwise uv *avoids* α. A perfect matching P of $K(Q_d)$ crosses α if P contains an edge crossing α, otherwise P avoids α. Let I_d^α be the perfect matching of Q_d that contains all edges in dimension $\alpha \in [d]$. Observe that a perfect matching P of Q_d crosses α if and only if $P \cap I_d^\alpha \neq \emptyset$.

Proposition 2. *Let P be a perfect matching of $K(Q_d)$ avoiding $\beta \in [d]$ and $e \in I_d^\beta$. There exists $R \in \Gamma(P)$ containing e.*

Proof. The proof proceeds by induction on d. The statement holds for $d = 2$. Let us assume that the statement is true for every k-cube Q_k with $2 \leq k \leq d-1$ and let us prove it for d.

Clearly, P crosses some $\alpha \in [d] \setminus \{\beta\}$. We divide the d-cube Q_d by dimension α into two $(d-1)$-subcubes Q^1 and Q^2 so that $e \in E(Q^1)$. Let $K^i := K(Q^i)$ and $P^i := P \cap E(K^i)$ for $i \in \{1, 2\}$.

The set of edges P^1 is a matching of K^1 which is not perfect since P crosses α. Let M be the set of vertices of K^1 that are uncovered by P^1. The size of M is even. If we divide Q^1 by dimension β, then numbers of vertices of M on both subcubes of Q^1 are even because P^1 avoids β. We choose an arbitrary perfect matching S^1 on vertices of M such that S^1 avoids β. The perfect matching $P^1 \cup S^1$ of K^1 avoids β. By induction there exists a perfect matching $R^1 \in \Gamma(P^1 \cup S^1)$ of Q^1 containing e. Let

$$S^2 := \left\{ xy \in E(K^2) \;\middle|\; \begin{array}{c} \exists x', y' \in V(Q^1) \text{ such that } xx', yy' \in P \text{ and} \\ \text{there exists a path between } x' \text{ and } y' \text{ of } P^1 \cup R^1 \end{array} \right\}. \tag{1}$$

Observe that $P^1 \cup R^1$ is a partition of Q^1 into vertex-disjoint paths between vertices uncovered by P^1. For every path between x' and y' of this partition there exist vertices x and y of Q^2 such that $xx', yy' \in P$. Thus, the set of edges S^2 is a matching of K^2. Moreover, the set of edges $P^2 \cup S^2$ is a perfect matching of K^2 because S^2 covers each vertex covered by P but not by P^2. Hence, there exists a perfect matching $R^2 \in \Gamma(P^2 \cup S^2)$ of Q^2 by Theorem 1. Clearly, $R := R^1 \cup R^2$ is a perfect matching of Q_d containing e. Finally, $R \in \Gamma(P)$ by Lemma 3. $\square$

Lemma 3. *Let P be a perfect matching of $K(Q_d)$ crossing $\alpha \in D$. Let the d-cube Q_d be divided into two $(d-1)$-subcubes Q^1 and Q^2 by dimension α. Let $K^i := K(Q^i)$ and $P^i := P \cap E(K^i)$ for $i \in \{1, 2\}$. Let S^1 be a perfect matching on vertices of $K(Q^1)$ uncovered by P^1. Let $R^1 \in \Gamma(P^1 \cup S^1)$. Let S^2 be given by Equation (1). Let $R^2 \in \Gamma(P^2 \cup S^2)$ and $R := R^1 \cup R^2$. Then $R \in \Gamma(P)$.*

Proof. We prove that $P \cup R$ is a Hamiltonian cycle of $K(Q_d)$. Suppose on the contrary that C is a cycle of $P \cup R$ which is not Hamiltonian. Since P crosses α, both S^1 and S^2 are non-empty sets. Because $P^i \cup S^i \cup R^i$ is a Hamiltonian cycle of K^i, whole cycle C cannot belong to K^i, for $i \in \{1, 2\}$. So C has edges in both K^1 and K^2. Now, we shorten every path $xx' \cdots y'y$ such that $x, y \in V(Q^2)$; $x', y' \in V(Q^1)$; $xx', yy' \in P$ and $x' \cdots y'$ is a path of

$P^1 \cup R^1$ by the edge $xy \in S^2$. Hence, we obtain a cycle C' of $(P^2 \cup S^2) \cup R^2$. We prove that C' does not contain a vertex of K^2 which is a contradiction because $(P^2 \cup S^2) \cup R^2$ is a Hamiltonian cycle of K^2.

If C does not contain a vertex u of K^2, then C' also does not contain u. Suppose that C does not contain a vertex v of K^1. Let x' and y' be the end vertices of the path of $P^1 \cup R^1$ that contains v. Let $xx', yy' \in P$. Observe that $x, y \in V(K^2)$ and $xy \in S^2$. Hence, C' does not contain x and y. $\square$

Observe that the perfect matching R obtained in Lemma 3 avoids dimension α. Interested reader may ask whether there exists a perfect matching R in Theorem 1 that avoids given set of dimension $A \subset [d]$. Clearly, the graph on edges of P and allowed edges of Q_d (i.e. edges of Q_d that avoid every dimension of A) must be connected. Gregor [8] proved that this is also a sufficient condition which implies following lemma.

Lemma 4. *For every perfect matching P of $K(Q_d)$ and $\alpha \in [d]$ there exists $R \in \Gamma(P)$ avoiding α if and only if P crosses α where $d \geq 2$.*

3 Bipartitness of $\mathcal{M}(K_{n,n})$

There is a natural one-to-one correspondence between perfect matchings of the complete bipartite graph $K_{n,n}$ and permutations on a set of size n. A permutation π is *even* if $n - k$ is even where k is a number of cycles of π, otherwise π is *odd*. It is well-known that $\pi_1 \circ \pi_2$ is even if and only if permutations π_1 and π_2 have same parity. Hence, the inverse permutation π_2^{-1} has same parity as π_2.

Let $c(P)$ be the number of components of the graph on a set of edges P. Recall that $B(G)$ is the complete bipartite graph with same color classes as a bipartite and connected graph G.

Let P_1 and P_2 be perfect matchings of $K_{n,n}$ and π_1 and π_2 be their corresponding permutations. Observe that $c(P_1 \cup P_2)$ is equal to the number of cycles of $\pi_1 \circ \pi_2^{-1}$. If n is even and $P_1 \cup P_2$ is a Hamiltonian cycle of $K_{n,n}$, then π_1 and π_2 have different parity. Hence, $\mathcal{M}(K_{n,n})$ is bipartite for n even. The matching graph $\mathcal{M}(Q_d)$ is also bipartite because $\mathcal{M}(Q_d)$ is a subgraph of $\mathcal{M}(B(Q_d))$ which is isomorphic to $\mathcal{M}(K_{2^{d-1},2^{d-1}})$.

Above discussion proves following theorem.

Theorem 5. *The matching graphs $\mathcal{M}(Q_d)$ and $\mathcal{M}(B(Q_d))$ are bipartite.*

We did not define which perfect matchings of $B(Q_d)$ are even and odd. But we know that perfect matchings P_1 and P_2 of $B(Q_d)$ belong to same color class of $\mathcal{M}(B(Q_d))$ if and only if $c(P_1 \cup P_2)$ is even. Hence, we fix one perfect matching of $B(Q_d)$ to be even.

Let us recall that I_d^α is the perfect matching of Q_d that contains all edges in dimension $\alpha \in [d]$. We simply count that $c(I_d^\alpha \cup I_d^\beta) = 2^{d-2}$ for every two different dimensions $\alpha, \beta \in [d]$ because the graph on edges $I_d^\alpha \cup I_d^\beta$ consists of 2^{d-2} independent cycles of size 4. Hence, perfect matchings I_d^α and I_d^β belong to same color class of $\mathcal{M}(B(Q_d))$ for $d \geq 3$. We call a perfect matching P of $B(Q_d)$ *even* if $c(P \cup I_d^1)$ is even, otherwise *odd* where $d \geq 3$.

4 Walks in $\mathcal{M}(Q_d)$

We will prove that $\mathcal{M}(Q_d)$ is connected by induction on d. Therefore, we need to know how we can make a walk in $\mathcal{M}(Q_d)$ from a walk in $\mathcal{M}(Q_{d-1})$. In this section we present two lemmas which help us.

Let P^0 and P^1 be perfect matchings of Q_{d-1}. We denote by $\langle P^0|P^1\rangle$ the perfect matching of Q_d containing P^i in the $(d-1)$-subcube of vertices having i in coordinate d for $i \in \{0,1\}$.

Lemma 6. *Let $P_1, P_2, P_3, R_1, R_2,$ and R_3 be perfect matchings of Q_{d-1} such that $P_1 \cup P_2$, $P_2 \cup P_3$, $R_1 \cup R_2$, and $R_2 \cup R_3$ are Hamiltonian cycles of Q_{d-1}. If $P_2 \cap R_2 \neq \emptyset$, then there exists a perfect matching S of Q_d such that $\langle P_1|R_1\rangle \cup S$ and $S \cup \langle P_3|R_3\rangle$ are Hamiltonian cycles of Q_d. Moreover, S crosses the dimension d and every dimension that is crossed by P_2 or R_2.*

Proof. Let $uv \in P_2 \cap R_2$. Let u_i be the vertex of Q_d obtained from u by appending i into dimension d where $i \in \{0,1\}$. Vertices v_0 and v_1 are defined similarly.

Let $S := (\langle P_2|R_2\rangle \setminus \{u_0v_0, u_1v_1\}) \cup \{u_0u_1, v_0v_1\}$. The graph on edges $\langle P_1|R_1\rangle \cup \langle P_2|R_2\rangle$ consists of two cycles covering all vertices of Q_d. These cycles are joined together in $\langle P_1|R_1\rangle \cup S$. Hence, $\langle P_1|R_1\rangle \cup S$ is a Hamiltonian cycle of Q_d. Similarly, $S \cup \langle P_3|R_3\rangle$ is a Hamiltonian cycle of Q_d.

The edge u_0u_1 crosses dimension d so S also crosses d. Let us consider $\beta \in [d-1]$ which is crossed by P_2 or R_2. Without lost of generality we suppose that P_2 crosses β. There exist at least 2 edges crossing β in P_2. It can happen that the edge u_0v_0 is one of them so at least one edge crossing β remains in S. ☐

Let P be a perfect matching of $K(Q_d)$ and $A \subseteq [d]$. We say that P *crosses* A if P crosses α for every $\alpha \in A$.

Lemma 7. *Let $P_1, P_2, P_3,$ and R_1 be perfect matchings of Q_{d-1} such that $P_1 \cup P_2$ and $P_2 \cup P_3$ are Hamiltonian cycles of Q_{d-1}. Let $\alpha, \beta \in [d-1]$,*

$\alpha \neq \beta$. If P_2 crosses $[d-1] \setminus \{\alpha\}$ and R_1 avoids β, then there exists a perfect matching S of Q_d such that $\langle P_1|R_1\rangle \cup S$ and $S \cup \langle P_3|R_1\rangle$ are Hamiltonian cycles of Q_d and S crosses $[d] \setminus \{\alpha\}$.

Proof. Let $e \in P_2 \cap I^{\beta}_{d-1}$. There exists $R_2 \in \Gamma(R_1)$ containing e by Proposition 2. If we apply Lemma 6 on P_1, P_2, P_3, R_1, R_2, and R_1, then we obtain a perfect matching S which satisfies the requirements of this lemma. $\square$

5 Base of induction

Let us recall that M_d is obtained from $\mathcal{M}(Q_d)$ by contracting all vertices of $\mathcal{M}(Q_d)$ whose corresponding perfect matchings are isomorphic. Let P and R be perfect matchings of Q_d. If there exists a walk between vertices representing P and R in $\mathcal{M}(Q_d)$, then the length of the shortest one is $d(P, R)$, otherwise $d(P, R)$ is infinity. Hence, $d(P, R) < \infty$ means that P and R belong to same component of $\mathcal{M}(Q_d)$.

The proof, that $\mathcal{M}(Q_d)$ is connected, proceeds by induction on d. We present a base of this induction in this section. We showed that $\mathcal{M}(Q_3)$ has 3 components (see Figure 1) so the induction starts from $d = 4$. Kreweras [2] proved that M_4 is connected (see Figure 3). We prove that if M_d is connected and $d \geq 4$, then $\mathcal{M}(Q_d)$ is connected. Hence, $\mathcal{M}(Q_4)$ is connected.

First, we present a simple lemma.

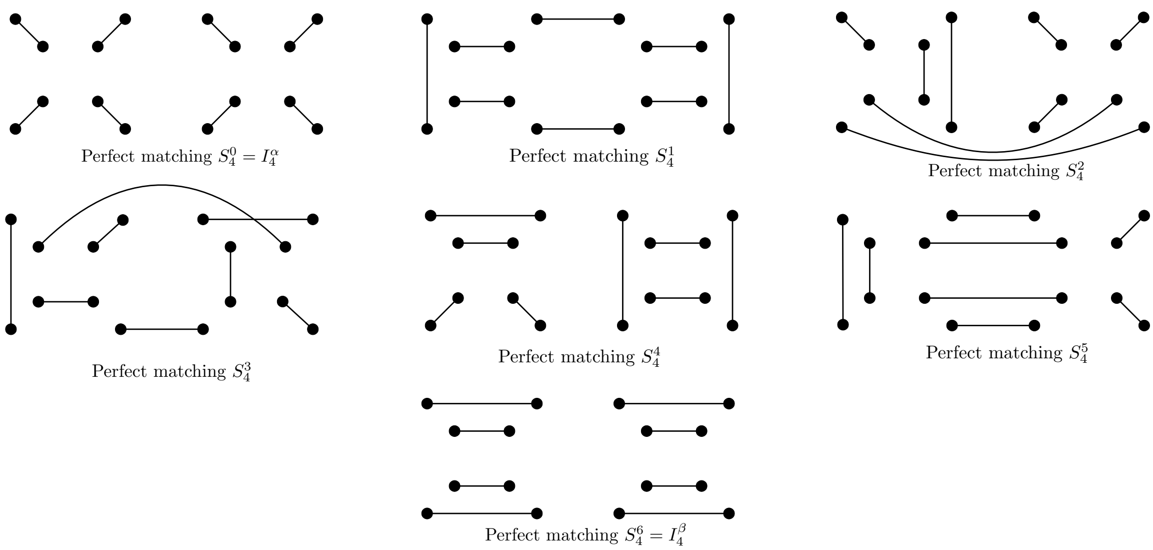

Figure 2: The walk between perfect matchings I_4^α and I_4^β in $\mathcal{M}(Q_4)$.

Lemma 8. *If $d \geq 4$, then $d(I^{\alpha}_d, I^{\beta}_d) \leq 6$ for every $\alpha, \beta \in [d]$, $\alpha \neq \beta$.*

Proof. The proof proceeds by induction on d. The walk between I_4^α and I_4^β is drawn on Figure 2.

Let $I_{d-1}^\alpha = S_{d-1}^0, S_{d-1}^1, S_{d-1}^2, S_{d-1}^3, S_{d-1}^4, S_{d-1}^5, S_{d-1}^6 = I_{d-1}^\beta$ be a walk in $\mathcal{M}(Q_{d-1})$. Let $S_d^i := \left\langle S_{d-1}^i | S_{d-1}^i \right\rangle$ for even i. For odd i let S_d^i be given by Lemma 6 where $P_1 = R_1 := S_{d-1}^{i-1}$, $P_2 = R_2 := S_{d-1}^i$, and $P_3 = R_3 := S_{d-1}^{i+1}$. Then $I_d^\alpha = S_d^0, S_d^1, S_d^2, S_d^3, S_d^4, S_d^5, S_d^6 = I_d^\beta$ is a walk in $\mathcal{M}(Q_d)$. $\square$

Let us recall that perfect matchings P and R are isomorphic if there exists an isomorphism $f : V(Q_d) \to V(Q_d)$ such that $f(u)f(v) \in R$ for edge $uv \in P$. This relation of isomorphism is an equivalence on the set of all perfect matching. Let $[P]$ be the equivalence class containing P. Observe that $[I_d] := \{I_d^\alpha \mid \alpha \in [d]\}$ is an equivalence class. If there exists a walk between $[P]$ and $[R]$ of M_d, then the length of the shortest one is $d([P], [R])$, otherwise $d([P], [R])$ is infinity.

Proposition 9. *If $d \geq 4$ and M_d is connected, then $\mathcal{M}(Q_d)$ is connected.*

Proof. We prove that vertices $\{P \in V(\mathcal{M}(Q_d)) \mid d([P], [I_d]) \leq k\}$ belong into one component of $\mathcal{M}(Q_d)$ by induction on k. This claim holds for $k = 0$ by Lemma 8.

Let P be a perfect matching of Q_d such that $d([P], [I_d]) = k$. There exists a perfect matching R of Q_d such that $d([R], [I_d]) = k - 1$ and $d([P], [R]) = 1$. Hence, there exists $R' \in \Gamma(P)$ isomorphic to R. By induction $d(I_d, R') < \infty$. Therefore, $d(I_d, R) < \infty$. $\square$

6 Induction step

We define a set of perfect matchings $\mathcal{Z}(d, k, \alpha)$ of Q_d by following induction on d, where $d \geq k \geq 3$ and $\alpha \in [d]$.

Definition 10. Let $\mathcal{Z}(d, d, \alpha)$ contains only I_d^α. The set $\mathcal{Z}(d, k, \alpha)$, where $d > k \geq 3$ and $\alpha \in [d]$, is the set of all perfect matchings of Q_d in the form $\langle P_1 | P_2 \rangle$, where $P_1 \in \mathcal{Z}(d - 1, k, \alpha)$ and P_2 is an even perfect matching of Q_{d-1} avoiding some $\beta \in [d - 1] \setminus \{\alpha\}$.

Observe that every $P \in \mathcal{Z}(d, k, \alpha)$ contains I_k^α in some k-subcube Q_k. We prove that the graph $\mathcal{M}(Q_d)$ is connected so we need to show that there exists a perfect matching I of Q_d such that for every perfect matching P of Q_d there exists a walk between P and I in $\mathcal{M}(Q_d)$. Lemma 8 says that perfect matchings $[I_d]$ belong to common component of $\mathcal{M}(Q_d)$ so it is sufficient to find a walk from P to an arbitrary one of $[I_d]$. Without lost of generality

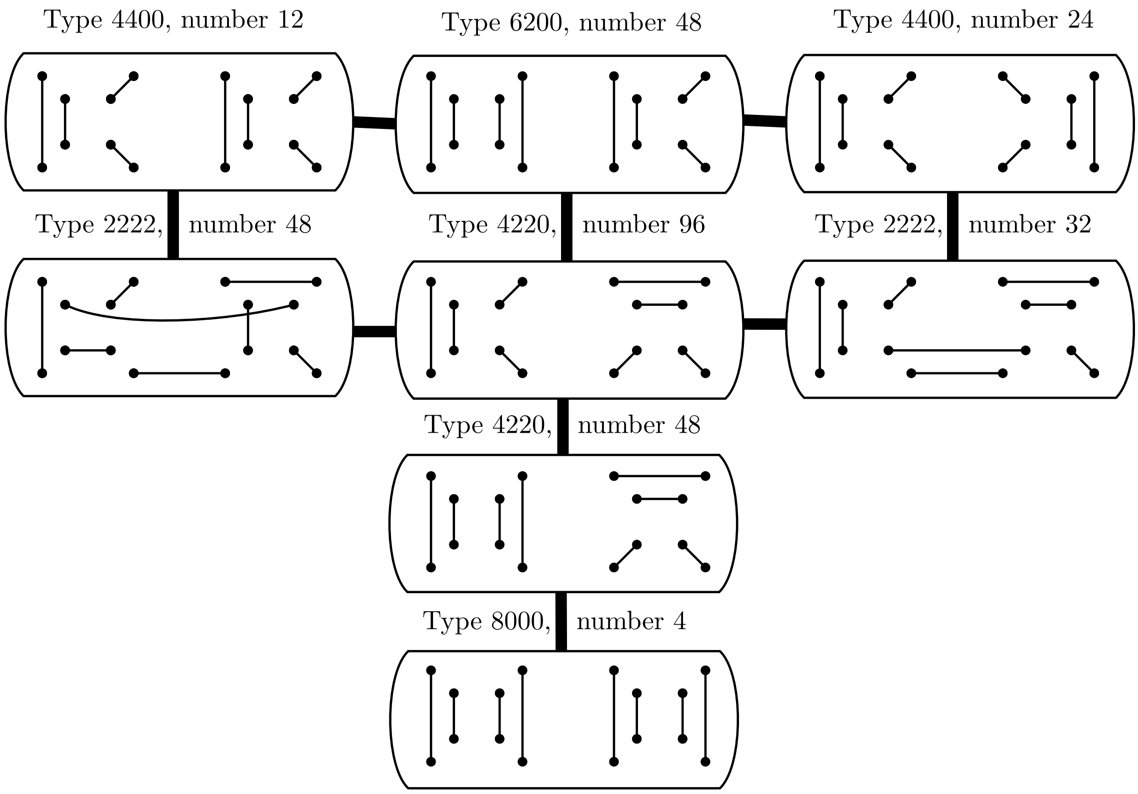

Figure 3: The graph M_4. For every equivalence class $[P]$ of isomorphism there is a frame which contains P. Four numbers of a type above frame are numbers of edges crossing each dimension. Above each frame there is also number of perfect matchings which are contracted to the equivalence class.

we assume that P is odd by Theorems 1 and 5. We find this walk in two steps: First, we find a walk from P to a perfect matching of $\mathcal{Z}(d,k,\alpha)$ for some $\alpha \in [d]$ and k, $d \geq k \geq 3$. Next, we find walks from $\mathcal{Z}(d,k,\alpha)$ to $\mathcal{Z}(d,k+1,\alpha)$ so by induction on k we obtain walks to $\mathcal{Z}(d,d,\alpha)$ which contains only I_d^α by definition.

Since Q_d is bipartite we call vertices of one color class *black* and the other *white*.

Lemma 11. *For every odd perfect matching P of $B(Q_d)$ there exists $Y \in \mathcal{Z}(d,k,\alpha)$ for some $\alpha \in [d]$ and k, $d \geq k \geq 3$, such that $d(P,Y) \leq 3$.*

Proof. We prove by induction on d that for every perfect matching P of $B(Q_d)$ there exist perfect matchings R, X and Y of Q_d such that $P \cup R$, $R \cup X$ and $X \cup Y$ are Hamiltonian cycles and X crosses $[d] \setminus \{\alpha\}$ and $Y \in \mathcal{Z}(d,k,\alpha)$.

First, we prove the statement for $d = 3$. Let P be an odd perfect matching of $B(Q_3)$. Therefore, $c(P \cup I_3^\alpha)$ is 1 or 3 for every $\alpha \in [3]$. If there exists $\alpha \in [3]$ such that $c(P \cup I_3^\alpha) = 1$, then we choose $R := Y := I_3^\alpha$ and $X \in \Gamma(R)$.

We prove that there exists $\alpha \in [3]$ such that $c(P \cup I_3^\alpha) = 1$. Suppose on the contrary that $c(P \cup I_3^\alpha) = 3$ for every $\alpha \in [3]$. The graph on edges $P \cup I_3^\alpha$ consists of two common edges and one cycle of size 4. Perfect matchings of $[I_3]$ are pairwise disjoint and P has two common edges with each of them. It is a contradiction because P has only 4 edges.

In the induction step we need to have at least 4 edges of P that cross a common dimension. Such dimension exists for every perfect matching P of $B(Q_d)$ if $d \geq 5$ by the Pigeonhole principle. Every perfect matching P of $B(Q_4)$ has 8 edges. If P contains an edge crossing at least two dimensions, then we use the Pigeonhole principle again.

A perfect matching P of Q_4 is *balanced* if it has 2 edges in every dimension. Luckily, Kreweras [2] proved that there are 8 perfect matchings of Q_4 up-to isomorphism and only two of them are balanced; see Figure 3. Check that the balanced perfect matchings S_4^3 drawn on Figure 2 and R^1 drawn of Figure 4 satisfy the requirements of this statement.

Now, we present the induction step. Let $\beta \in [d]$ such that P has at least 4 edges crossing β. Without lost of generality we assume that $\beta = d$. We divide Q_d into two $(d-1)$-subcubes Q^1 and Q^2 by dimension β. Let $B^i := B(Q^i)$ and $P^i := P \cap E(B^i)$ for $i \in \{1, 2\}$. Let M be the set of vertices of B^1 that are uncovered by P^1. We know that $|M| \geq 4$. Moreover, M has same number of black vertices as white ones.

Let b_1 and b_2 be two different black vertices of M and w_1 and w_2 be two different white vertices of M. Let S' be a matching of B^1 covering $M \setminus \{b_1, b_2, w_1, w_2\}$. We have two ways how to extend S' to be matching S^1 of B^1 covering M: We can insert edges $\{b_1w_1, b_2w_2\}$ or $\{b_1w_2, b_2w_1\}$. Those two ways give us two perfect matchings $P^1 \cup S^1$ of B^1 having different parity. Of course, we choose the way that gives us odd perfect matching $P^1 \cup S^1$.

Let R^1, X^1 and Y^1 be perfect matchings of Q^1 given by induction – $(P^1 \cup S^1) \cup R^1$, $R^1 \cup X^1$ and $X^1 \cup Y^1$ are Hamiltonian cycles of B^1 and X^1 crosses $[d-1] \setminus \{\alpha\}$ and $Y^1 \in \mathcal{Z}(d-1, k, \alpha)$. Hence, R^1 is even by Theorem 5. Let S^2 be given by Equation (1).

We prove that $P^2 \cup S^2$ is odd. Let $\bar{R}^2 \in \Gamma(P^2 \cup S^2)$ by Theorem 1. Let $\bar{R} := R^1 \cup \bar{R}^2$. By Lemma 3 holds $\bar{R} \in \Gamma(P)$ so $\bar{R}$ is even by Theorem 5. Also $\bar{R}^2$ is even because R^1 and $\bar{R}$ are even. Hence, $P^2 \cup S^2$ is odd by Theorem 5. Moreover, $P^2 \cup S^2 \neq I_{d-1}^\alpha$.

Hence, the perfect matching $P^2 \cup S^2$ crosses some $\gamma \in [d-1] \setminus \{\alpha\}$ and there exists $R^2 \in \Gamma(P^2 \cup S^2)$ avoiding γ by Lemma 4. Let $R := R^1 \cup R^2$. Therefore, $R \in \Gamma(P)$ by Lemma 3 and R is even by Theorem 5. Because R^1 is even so R^2 is even. We apply Lemma 7 on R^1, X^1, Y^1 and R^2 to obtain a perfect matching X such that $\langle R^1 | R^2 \rangle \cup X$ and $X \cup \langle Y^1 | R^2 \rangle$ are Hamiltonian cycles of Q_d and X crosses $[d] \setminus \{\alpha\}$. Finally, $Y := \langle Y^1 | R^2 \rangle \in \mathcal{Z}(d,k,\alpha)$ by definition. $\square$

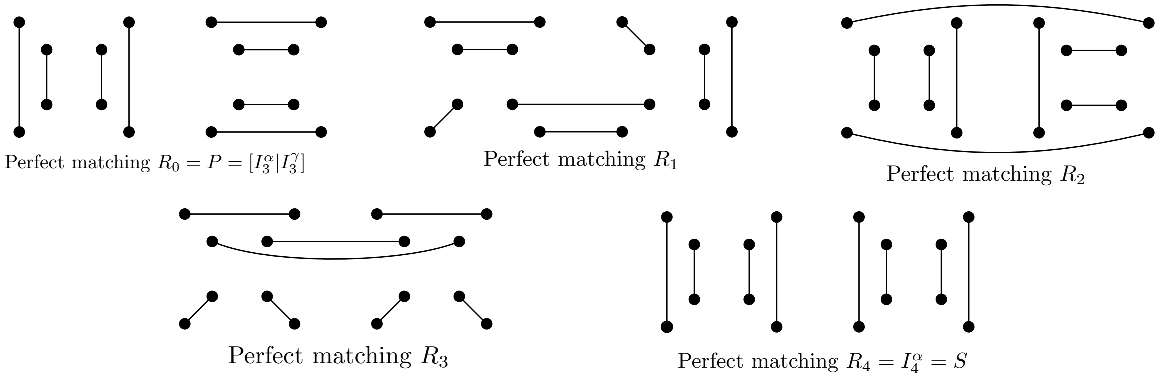

Figure 4: A walk between $P \in \mathcal{Z}(4,3,\alpha)$ and I_4^α.

Lemma 12. *Let $P \in \mathcal{Z}(d,k,\alpha)$ where $3 \leq k < d$ and $\alpha \in [d]$. If $\mathcal{M}(Q_k)$ is connected or $k = 3$, then there exists $S \in \mathcal{Z}(d,k+1,\alpha)$ such that $d(P,S) < \infty$.*

Proof. We prove by induction on d that for every $P \in \mathcal{Z}(d,k,\alpha)$ there exists a walk $P = R_0, R_1, \ldots, R_n = S$ in $\mathcal{M}(Q_d)$ of even length such that R_l crosses $[d] \setminus \{\alpha\}$ for every odd l and $S \in \mathcal{Z}(d,k+1,\alpha)$. The base of this induction is for $d = k+1$.

By definition of $\mathcal{Z}(d,k,\alpha)$ we divide P into perfect matchings P^1 and P^2 such that $P = \langle P^1 | P^2 \rangle$ and $P^1 \in \mathcal{Z}(d-1,k,\alpha)$ and P^2 is an even perfect matching of Q_{d-1} avoiding some $\beta \in [d-1] \setminus \{\alpha\}$.

First, we present the base of induction for $d = 4$, so $k = 3$. By definition $P^1 = I_3^\alpha$ and P^2 is even. There are two perfect matchings of Q_3 up-to isomorphism with different parity; see Figure 1. Hence, $P^2 = I_3^\gamma$ for some $\gamma \in [3]$. If $P^2 = I_3^\alpha$, then $P = I_4^\alpha$ which belongs to $\mathcal{Z}(4,4,\alpha)$ by definition. Otherwise, the walk on Figure 4 satisfies requirements of this lemma.

Now, we present the base of the induction for $k \geq 4$ and $k+1 = d$. In that case $P^1 = I_k^\alpha$. There exists a walk $P^2 = R_0, R_1, \ldots, R_n = I_k^\alpha$ on $\mathcal{M}(Q_k)$ of even length because $\mathcal{M}(Q_k)$ is connected and bipartite and P^2 is even. Let $R'_l := \langle P^1 | R_l \rangle$ for even l. Clearly, $R'_n \in \mathcal{Z}(d,k+1,\alpha)$ because $R'_n = I_{k+1}^\alpha$.

Let l be odd. Since R_l is odd, it holds $R_l \neq I_k^\alpha$. We choose an edge $e_l \in R_l \setminus I_k^\alpha$. By Proposition 2 there exists $Z_l \in \Gamma(I_k^\alpha)$ containing e_l. The perfect matching Z_l crosses $[k] \setminus \{\alpha\}$ by Lemma 4. We apply Lemma 6 on $R_{l-1}, R_l, R_{l+1}, I_k^\alpha, Z_l$, and I_k^α to obtain a perfect matching R'_l. The walk $P = R'_0, R'_1, \ldots, R'_n = I_{k+1}^\alpha$ satisfies the requirements.

Finally, we present the induction step for $k \geq 3$ and $k+1 < d$. By induction there exists a walk $P^1 = R_0, R_1, \ldots, R_n = S^1$ in $\mathcal{M}(Q_{d-1})$ of even length such that $S^1 \in \mathcal{Z}(d-1, k+1, \alpha)$ and R_l crosses $[d-1] \setminus \{\alpha\}$ for every odd l. Let $R'_l := \langle R_l | P^2 \rangle$ for even l. For odd l we apply Lemma 7 on R_{l-1}, R_l, R_{l+1} and P^2 to obtain a perfect matching R'_l of Q_d. Now, the walk $P = R'_0, R'_1, \ldots, R'_n = S$ satisfies the requirements and $S \in \mathcal{Z}(d, k+1, \alpha)$. $\square$

Corollary 13. *Let $P \in \mathcal{Z}(d, k, \alpha)$ where $3 \leq k \leq d$ and $\alpha \in [d]$. If $\mathcal{M}(Q_l)$ is connected for every $l \in \{4, 5, \ldots, d-1\}$, then $d(P, I_d^\alpha) < \infty$.*

Proof. The proof proceeds by induction on $d - k$. If $d = k$, then $P = I_d^\alpha$ by definition of $\mathcal{Z}(d, k, \alpha)$. Let $3 \leq k < d$. By Lemma 12 there exists $S \in \mathcal{Z}(d, k+1, \alpha)$ such that $d(P, S) < \infty$. By induction $d(S, I_d^\alpha) < \infty$. Hence, $d(P, I_d^\alpha) < \infty$. $\square$

Theorem 14. *The matching graph $\mathcal{M}(Q_d)$ is connected for $d \geq 4$.*

Proof. The proof proceeds by induction on d. Kreweras [2] proved that the graph M_4 is connected; see Figure 3. Hence, the graph $\mathcal{M}(Q_4)$ is connected by Proposition 9 and the statement holds for $d = 4$. Let us assume that the graph $\mathcal{M}(Q_l)$ is connected for every l with $4 \leq l \leq d-1$. Let us prove that for some $\beta \in [d]$ and for every perfect matching P of Q_d holds $d(P, I_d^\beta) < \infty$.

If P is even, then we choose $R \in \Gamma(P)$ by Theorem 1 which is odd by Theorem 5. Otherwise, we simply consider $R := P$. By Lemma 11 there exists $S \in \mathcal{Z}(d, k, \alpha)$ such that $d(R, S) \leq 3$. By Corollary 13 it holds $d(R, I_d^\alpha) < \infty$ and $d(I_d^\alpha, I_d^\beta) \leq 6$ by Lemma 8. $\square$

Corollary 15. *The graph M_d is connected for $d \geq 3$.*

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References

[1] J. Fink: *Perfect Matchings Extend to Hamilton Cycles in Hypercubes*, Submitted.

[2] G. Kreweras: *Matchings and Hamiltonian cycles on hypercubes*, Bull. Inst. Combin. Appl. 16 (1996), 87–91.

[3] C. Savage: *A survey of combinatorial Gray codes*, SIAM Rev. 39 (1997), 605–629.

[4] L. Gros: *Théorie du Baguenodier*, Aimé Vingtrinier, Lyon, 1872.

[5] T. Dvořák: *Hamiltonian cycles with prescribed edges in hypercubes*, SIAM J. Discrete Math. 19(2005), 135–144.

[6] D. Dimitrov, T. Dvořák, P. Gregor, R. Škrekovski: *Gray Codes Faulting Matchings*, submitted.

[7] R. J. Douglas: *Bounds on the number of Hamiltonian circuits in the n-cube*, Discrete Math. 17(1977), 143–146.

[8] P. Gregor: *Perfect matchings extending on subcubes to Hamiltonian cycles of hypercubes*, manuscript.