A PRIESTLEY SUM OF FINITE TREES IS ACYCLIC

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Abstract. We show that the Priestley sum of finite trees contains no cyclic finite poset.

1. Introduction

A Priestley space is an ordered compact space satisfying a natural separation property, and Priestley maps are those which are continuous and preserve the order. The resulting category is dually equivalent to that of bounded distributive lattices (and 01-preserving lattice homomorphisms) by the famous Priestley duality ([10], [11], [8]). Since the latter category obviously admits arbitrary products, arbitrary coproducts (sums) of Priestley spaces exist. These are suitable compactifications of the disjoint union $\bigcup_J X_j$ of the summand spaces X_j, and this union appears as a dense subspace of the sum $X \equiv \coprod_J X_j$, which is, of course, bigger whenever the index set J is infinite.

Although the order structure of X is not yet fully understood, by now quite a few facts are known about the configurations (finite connected posets) that are present. Thus, for instance, no finite tree appears in X unless it also appears in some of the summands X_j ([2], [3], [5]). On the other hand, a configuration containing a cycle may be present in X without being present in any of the X_j's ([4], [6]). But, in all known constructions producing the latter phenomenon, the X_j's contain cycles, albeit not the one in question.

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The problem, then, naturally arises as to whether one can create a cycle in a sum without having a cycle in any of the summands. Using the forbidden tree result from [2] it is easy to see that this is impossible in the case of X_j's with tops, but the general case seems to be much harder. In this article we resolve the issue in the negative for arbitrary sums of finite trees, and hence, for arbitrary sums of finite acyclic Priestley spaces.

The major result appears here as Theorem 3.3.1, for which the preceding material is preparation: Section 2 contains the necessary terminology and background, and Section 3 contains the proof. The proof is organized into three cases analyzed in parallel. Although the key idea is already present in the motivational Lemma 3.1.1, the heart of the proof is a combinatorial analysis of trees and their convex subsets carried out in Subsection 3.2. The proof of the theorem itself is the content of Subsection 3.3.

2. Preliminaries

In this section we outline the background and notation necessary for what follows.

2.1. **Graphs, paths, and trees.** A *graph* is a couple $G = (X, E)$, where X is a set and E is a binary symmetric antireflexive relation on X. The elements of X are called the *vertices* or *nodes* of G, and the pairs $(x, y) \in E$ are called the *edges* of G. A *path of length n connecting x to y* is a sequence of the form

$$x = x_0, x_1, \ldots, x_n = y,$$

where $(x_i, x_{i+1}) \in E$ for $0 \leq i < n$. If $x = y$, the path is called a *cycle*. If $y = x_n, x_{n+1}, \ldots, x_m = z$ is a path from y to z, then the *concatenation* of the two is the path

$$x = x_0, x_1, \ldots, x_n, x_{n+1}, \ldots x_m = z$$

from x to z.

The path $x_0, x_1, \ldots, x_n$ is called *simple* if the x_i's are distinct, save for the possibility that $x_0 = x_n$, in which case it is called a *simple cycle*. Note that any path $x_0, x_1, \ldots, x_n$ can be reduced to a simple path as follows. If, say, $x_i = x_j$ for $0 \leq i < j \leq n$, then the path may be replaced by the shorter path

$$x = x_0, \ldots, x_i, x_{j+1} \ldots x_n = y.$$

The graph G is said to be *connected* if any two of its vertices can be connected by a path, and *acyclic* if it admits no nontrivial simple cycles, i.e., no simple cycles of length $n \geq 3$. Finally, a *tree* is a connected acyclic graph, i.e., a graph such that any two of its vertices can be connected by precisely one simple path.

2.2. **Posets, diamonds, and crowns.** In a poset $(X, \leq)$, we say that *y is a successor to x*, or that *x is a predecessor to y*, and write $x \prec y$, to mean that $x < y$ and

$$\forall z\ (x \leq z \leq y \implies (x = z \text{ or } y = z)),$$

and we write $x \succ\!\prec y$ to mean that $x \prec y$ or $x \succ y$. Thus we have associated with the poset $(X, \leq)$ the graph $(X, \succ\!\prec)$, and the terminology of the associated graph is then applied to the poset. In particular, we say that the poset X is connected or a tree if $(X, \succ\!\prec)$ has the same property. We use the term *configuration* for a finite connected poset, and *acyclic configuration* as a synonym for tree configuration.

We abbreviate the path notation in a poset. Instead of writing a path S in the form $s_0, \ldots, s_n$, we write it as a word

$$S = s_0 \ldots s_n.$$

A *segment* of S is any path of the form $s_j \ldots s_k$, $0 \leq j \leq k \leq n$, and a *segmentation* of S is a sequence of segments of the form

$$(s_0 \ldots s_{j_1})\,(s_{j_1+1} \ldots s_{j_2}) \ldots (s_{j_k+1} \ldots s_n).$$

Note that the adjacent segments are not concatenated, for they do not share an endpoint. We abuse the path notation to the extent of using the same letter S for a path or segment $s_0 s_2 \ldots s_n$ as for the set $\{s_i : 0 \leq i \leq n\}$ of its nodes. Note that a segment is nonempty by definition, in the sense that it involves at least one node, even if the path is of length 0.

In a tree (connected acyclic poset) T, the reduction to the shortest path given in Subsection 2.1 can be simplified. A path $S = s_0 s_2 \ldots s_n$ can fail to be simple iff it contains a *redundancy*, i.e., $s_{i-1} = s_{i+1}$ for some i, $1 < i < n$. One step in the path reduction procedure is then just the replacement of S by $s_0 \ldots s_{i-1} s_{i+2} \ldots s_n$, and a path is *simple* iff no redundancy occurs. *In what follows, we make repeated use without comment of the important fact that, in a tree T, there is a unique simple path between any two nodes s and t.* We refer to this as *the shortest path from s to t*, and denote it $\langle s, t \rangle$.

Acyclic posets are characterized by the absence of the following three configurations as induced subposets.

- A *diamond* is a configuration of four distinct points

$$x_0 < x_1, x_3 < x_2,$$

with x_1 and x_3 incomparable. When dealing with the diamond, index arithmetic is assumed to be mod 4.

- A *proper* 2*-crown* is a configuration of four distinct points

$$x_0 < x_1 > x_2 < x_3 > x_0$$

such that there is no intermediate point x such that $x_0, x_2 < x < x_1, x_3$. When dealing with the 2-crown, index arithmetic is assumed to be mod 4.

- For $n > 2$, an n*-crown* is a configuration of $2n$ distinct points

$$x_0 < x_1 > x_2 < x_3 > \ldots < x_{2n-1} > x_0$$

such that no order relationships obtain beyond those displayed. When dealing with the n-crown, index arithmetic is assumed to be mod $2n$.

2.3. **Up-sets and down-sets.** In a poset $(X, \leq)$, for a subset $M \subseteq X$, we set

$$\downarrow M \equiv \{x : \exists m \in M \ (x \leq m)\} \text{ and } \uparrow M \equiv \{x : \exists m \in M \ (x \geq m)\},$$

and we abbreviate $\downarrow\{x\}$ to $\downarrow x$ and $\uparrow\{x\}$ to $\uparrow x$. The subset M is said to be a down-set if $M = \downarrow M$ and an up-set if $M = \uparrow M$. Obviously, unions and intersections of down-sets are down-sets, and similarly for up-sets. Note that an intersection $M = U \cap D$ of an up-set and a down-set is typically neither an up-set nor a down-set, but is *convex*, meaning that for all $x, y, z \in X$,

$$(x \leq y \leq z \text{ and } x, z \in M) \implies y \in M.$$

2.4. **Priestley duality.** A *Priestley space* is a compact ordered space $(X, \tau, \leq)$ such that whenever $x \nleq y$ there is a clopen up-set U with $y \notin U \ni x$. A Priestley map $f : (X, \tau, \leq) \to (X', \tau', \leq')$ is a continuous order-preserving function. The resulting category is designated **PSp**. Recall the *Priestley duality* ([8], [10], [11]) between **PSp** and the category **D** of bounded distributive lattices with bound-preserving lattice homomorphisms. It can be given by the pair of functors

$$\mathbf{Psp} \underset{\mathcal{P}}{\overset{\mathcal{U}}{\rightleftarrows}} \mathbf{D},$$

where

$$\mathcal{U}(X, \tau, \leq) = (\{U : U \text{ is a clopen up-set in } X\}, \cup, \cap, \emptyset, X),$$
$$\mathcal{U}(f)(U) = f^{-1}[U]$$

and

$$\mathcal{P}(L) = (\{F : F \text{ is a prime filter in } L\}, \tau, \subseteq)$$
$$\mathcal{P}(h)(F) = h^{-1}[F].$$

The topology τ is generated by basic sets of the form $\Sigma(a,b)$, $a, b \in L$, where

$$\Sigma(a,b) = \{F : a \notin F \ni b\}.$$

Due to the obvious existence of products in $\mathbf{D}$, we have the following fundamental observation.

Lemma 2.4.1. **PSp** *has coproducts.*

2.5. **Coproduct conventions.** Let $\{X_j : j \in J\}$ be a pairwise disjoint family of finite posets. If we denote by A_j the lattice $\mathcal{U}(X_j)$ of up-sets of X_j, then

$$\coprod_J X_j \cong \mathcal{P}\left(\prod_J A_j\right).$$

Now $\prod_J A_j$ is isomorphic to $A \equiv \{U : U \text{ is an up-set of } \bigcup_J X_j\}$ by means of the association

$$\prod_J A_j \ni a \longmapsto \bigcup_J a(j) \in A,$$

so that we have the coproduct represented as

$$X \equiv (\{F : F \text{ is a prime filter on } A\}, \tau, \subseteq),$$

with τ generated by basic sets of the form $\Sigma(a,b)$, $a, b \in A$, where

$$\Sigma(a,b) = \{F : a \notin F \ni b\}.$$

The canonical insertion $\rho_j : X_j \to X$ is the map

$$\rho_j(x) \equiv \{a : x \in a\} \in X, \ x \in X_j.$$

Lemma 2.5.1. *Each ρ_j is an order embedding, meaning that $\rho_j(x) \leq \rho_j(y)$ iff $x \leq y$ for $x, y \in X_j$. And each $\rho_j[X_j]$ is order independent in X, meaning that no point of $\rho_j[X_j]$ is related to any point of $X \smallsetminus \rho_j[X_j]$. And $\bigcup_J \rho_j[X_j]$ is both a subposet and a dense subspace of X.*

Proof. See [9] for a penetrating analysis of the structure of X, of which this lemma is only a small part. □

From now on we identify each X_j with its image under ρ_j, and use letters like x, y, and z to designate the elements of X, whether they lie in $\bigcup_J X_j$ or in the remainder $X \smallsetminus \bigcup_J X_j$.

We record a consequence of Lemma 2.5.1 for our subsequent use. Let $T \equiv \bigcup_J X_j$.

Lemma 2.5.2. *For any subset $U \subseteq X$, $(\uparrow_X U) \cap T = \uparrow_T (U \cap T)$, meaning*

$$\{t \in T : \exists x \in U \ (t \geq x)\} = \{t \in T : \exists t' \in (U \cap T) \ (t \geq t')\},$$

and likewise $(\downarrow_X U) \cap T = \downarrow_T (U \cap T)$.

3. The proof of the acyclicity of a sum of trees

The proof of the main result, Theorem 3.3.1 below, will proceed by analyzing the three cases set out in the trichotomy of Subsection 2.2. That is, we will show that a sum of finite trees houses no diamond, no proper 2-crown, and no n-crown with $n > 2$. We develop all three cases in parallel in order to emphasize the common aspects of the arguments.

3.1. **A motivational lemma.** We begin with the observation that the presence of a cycle in a Priestley space is signaled by the existence of a certain finite collection of convex clopen subsets in a specific relationship to one another.

Lemma 3.1.1. *Let X be a Priestley space.*

(1) *Four points $x_0 < x_1, x_3 < x_2$ form a diamond iff x_1 and x_3 have clopen down-set neighborhoods U_1 and U_3, respectively, such that $x_1 \notin U_3$ and $x_3 \notin U_1$.*

(2) *Four points $x_0, x_2 < x_1, x_3$ form a 2-crown iff each x_i has a clopen neighborhood U_i such that U_0 and U_2 are up-sets, U_1 and U_3 are down-sets, and*

$$\bigcap_{0 \leq i \leq 3} U_i = \emptyset.$$

(3) *The following are equivalent for a collection of $2n$ points, $n > 2$, related as follows:*

$$x_0 < x_1 > x_2 < \ldots > x_{2n-2} < x_{2n-1} > x_0.$$

(a) *The collection forms an n-crown, i.e., no order relationships hold among the x_i's other than those displayed.*

(b) *Each x_i has a clopen neighborhood U_i such that the U_i's are up-sets for even indices and down-sets for odd indices, and such that $U_i \cap U_j = \emptyset$ for even i and odd j such that $|i - j| \neq 1$. (Index arithmetic is* mod $2n$.*)*

(c) *Each x_i, i even, is contained in a clopen upset U_i which contains only those x_j for which $|i - j| \leq 1$. (Index arithmetic is* mod $2n$.*)*

Proof. (1) If x_1 is unrelated to x_3 then the existence of the U_i's is a consequence of the total order disconnectedness of X. The converse is obvious. (2) The condition certainly implies the absence of an intermediate point, since such a point would lie in the displayed intersection. So suppose the crown is proper. Viewing the points of X as prime filters on some bounded distributive lattice L, the lack of an intermediate point translates into the condition that the filter generated by $x_0 \cup x_2$ meets the ideal generated by $(T \smallsetminus x_1) \cup (T \smallsetminus x_3)$. That condition, in turn, produces lattice elements $a_0 \in x_0$, $a_1 \notin x_1$, $a_2 \in x_2$, and $a_3 \notin x_3$ such that

$$a_1 \vee a_3 \geq a_0 \wedge a_2.$$

The sets

$$\begin{aligned} U_0 &\equiv \{x \in X : a_0 \in x\}, \\ U_1 &\equiv \{x \in X : a_1 \notin x\}, \\ U_2 &\equiv \{x \in X : a_2 \in x\}, \\ U_3 &\equiv \{x \in X : a_3 \notin x\}, \end{aligned}$$

clearly have the required properties.

(3) If (b) holds then it is clear that x_i can be related to x_j only if $|i - j| \leq 1$, from which (a) follows. The implication from (c) to (b) goes by setting

$$U_{2i-1} \equiv X \smallsetminus \bigcup_{|2i-1-2k| \neq 1} U_{2k}.$$

So assume (a) to prove (c). We view the x_i's as prime filters on some bounded distributive lattice. If i is an even index and j is an odd index such that $|i - j| \neq 1$, the fact that $x_i \not\leq x_j$ produces a lattice element $a_{ij} \in x_i \smallsetminus x_j$, so we define

$$a_i \equiv \bigwedge_{\substack{\text{odd } j \\ |i-j| \neq 1}} a_{ij} \in x_i \smallsetminus \bigcup_{\substack{\text{odd } j \\ |i-j| \neq 1}} x_j.$$

The sets $U_i \equiv \{x : a_i \in x\}$ clearly have the required properties. □

3.2. **In a finite tree.** In all three cases, the proof proceeds by analyzing the combinatorial properties of a finite collection of subsets U_i of the type that arise in Lemma 3.1.1. *But we emphasize that the analysis takes place, not in an arbitrary Priestley space X, but instead in a given finite tree T. Therefore, for the remainder of this section, T will represent a fixed finite tree and $\{U_i : 0 \leq i \leq n\}$ will represent subsets of T which are either up-sets or down-sets.*

The sets U_i and their complements partition T into subsets of the form $\bigcap_{0\leq i\leq n} U_i'$, where U_i' stands for either U_i or $T \smallsetminus U_i$. We term such subsets *basic*, and remind the reader that we pointed out in Subsection 2.3 that they are convex. What is most important for our purposes is that any path S in T has a unique segmentation

$$S = S_1 S_2 \ldots S_k$$

such that the nodes of each S_i lie within the same basic subset, and such that nodes of adjacent segments lie in different basic subsets. We refer to this as the *basic segmentation* of S.

We single out certain basic sets for special consideration, and denote these sets V_i. (Roughly speaking, these are the basic subsets corresponding to the points of the diamond or the crown under consideration.) In the case of the diamond, we are given clopen down-sets U_1 and U_3, from which we define

$$\begin{aligned} V_0 &\equiv U_1 \cap U_3, \\ V_1 &\equiv U_1 \cap (T \smallsetminus U_3), \\ V_2 &\equiv (T \smallsetminus U_1) \cap (T \smallsetminus U_3), \\ V_3 &\equiv (T \smallsetminus U_1) \cap U_3. \end{aligned}$$

In the case of the 2-crown, we are given clopen up-sets U_0 and U_2 and clopen down-sets U_1 and U_3 such that $\bigcap U_i = \emptyset$, from which we define

$$\begin{aligned} V_0 &\equiv U_0 \cap U_1 \cap (T \smallsetminus U_2) \cap U_3, \\ V_1 &\equiv U_0 \cap U_1 \cap U_2 \cap (T \smallsetminus U_3), \\ V_2 &\equiv (T \smallsetminus U_0) \cap U_1 \cap U_2 \cap U_3, \\ V_3 &\equiv U_0 \cap (T \smallsetminus U_1) \cap U_2 \cap U_3. \end{aligned}$$

In the case of the n-crown, $n > 2$, we are given n clopen up-sets U_{2i}, $0 \leq i < n$, from which we define

$$V_{2i} \equiv U_{2i} \cap \bigcap_{0\leq i\neq j<n} (T \smallsetminus U_{2j}), \ 0 \leq i < n.$$

$$V_{2i+1} \equiv U_{2i} \cap U_{2i+2} \cap \bigcap_{0\leq i\neq j\neq i+1<n} (T \smallsetminus U_{2j}), \ 0 \leq i < n.$$

We come now to the central definition, namely that of a *tour*. We begin by defining a *fundamental tour*. (As usual, index arithmetic is $\bmod 4$ in the case of the diamond and 2-crown, and $\bmod 2n$ in the case of the n-crown.) In the case of the diamond, a fundamental tour from s to t is a path R from s to t whose basic segmentation is of the form

$$R_0 R_1 R_2 R_3 R_4,$$

where either $R_i \subseteq V_i$ for $0 \leq i \leq 4$ or $R_i \subseteq V_{4-i}$ for $0 \leq i \leq 4$. In the case of the n-crown, $n > 2$, a fundamental tour from s to t is a path R from s to t whose basic segmentation is of the form

$$R_0 R_1 \ldots R_{2n},$$

where either $R_i \subseteq V_i$ for $0 \leq i \leq 2n$ or $R_i \subseteq V_{2n-i}$ for $0 \leq i \leq 2n$. In the case of the 2-crown, a fundamental tour from s to t is a path from s to t whose basic segmentation can be expressed in the form

$$R_0 S_0 R_1 S_1 R_2 S_2 R_3 S_3 R_4,$$

such that either $R_i \subseteq V_i$ for $0 \leq i \leq 4$ or $R_i \subseteq V_{4-i}$ for $0 \leq i \leq 4$. Any S_i may be void, meaning that no segment appears in that position, but if it represents an actual segment, then its nodes must occupy a basic subset different from any of the V_j's, and different from those of S_{i+1}. In addition, if S_3 and S_0 are nonvoid then their nodes must occupy different basic subsets.

Lemma 3.2.1. *Let R be a fundamental tour with basic segmentation $W_0 \ldots W_k$. Then the nodes of W_{i-1} fall into a different basic subset than do those of W_{i+1}, $0 < i < k$, and likewise for the nodes of W_{k-1} and W_1.*

Lemma 3.2.2. *When applied to a fundamental tour, the reduction to the shortest path (see Subsection 2.1) produces a fundamental tour.*

Proof. Let $R = s_0 \ldots s_n$ be a fundamental tour with basic segmentation $W_0 \ldots W_k$, and suppose a redundancy occurs at s_i, i.e., $s_{i-1} = s_{i+1}$, for some point s_i in segment W_j. But s_{i-1} must then also fall into W_j, since the only alternative is for s_{i-1} to fall into W_{j-1} and s_{i+1} to fall into W_{j+1}, and this cannot happen by Lemma 3.2.1. □

Finally, a *tour* from s to t is a path R from s to t which results from concatenating an odd number of fundamental tours. That is to say that, in the basic segmentation $W_0 W_1 \ldots W_n$ of R, there is an odd integer k, and there are indices

$$0 = i_0 < i_1 \ldots < i_k = n,$$

such that each segment of the form $W_{i_j} W_{i_j+1} \ldots W_{i_{j+1}}$, $0 \leq j < k$, is a fundamental tour.

Lemma 3.2.3. *No tour is closed. That is, there is no tour from a point to itself.*

Proof. Suppose, on the contrary, that R is a tour from s to s with basic segmentation $W_0 \ldots W_n$ and indices $0 = i_0 < i_1 \ldots < i_k = n$ such that each segment $W_{i_j} \ldots W_{i_{j+1}}$ is fundamental. Let k be the least integer for which such a tour exists. Lemma 3.2.2 implies that $k > 1$, of course, but more to the point, it allows us to assume without loss of generality that each segment $W_{i_j} \dots W_{i_{j+1}}$ is simple.

We claim that, for $l < m$, the nodes of W_{i_l} are unrelated to those of W_{i_m}. For if, say,

$$W_{i_l} \ni r > t \in W_{i_m}$$

then there is a path P from t to r lying entirely within V_0, and this gives rise to two closed paths. One is the segment of R from r to t concatenated with P, and the other is the segment of R from t to s concatenated with the segment of R from s to r concatenated with the reversal of P. Each of these paths is a concatenation of fundamental tours, and the total number of these fundamental tours is k. It follows that one of these paths is a closed tour composed of fewer than k fundamental tours, contrary to hypothesis.

Let us refer to a fundamental tour as *positive* if $R_i \subseteq V_i$ for all i, and as *negative* if $R_i \subseteq V_{4-i}$ or $R_i \subseteq V_{2n-i}$. Because k is odd, there must be two adjacent fundamental tours with the same parity, i.e., some index i_j such that both $W_{i_{j-1}} \dots W_{i_j}$ and $W_{i_j} \dots W_{i_{j+1}}$ are, say, positive. (This includes the possibility that $j = k$, meaning that both $W_{i_{k-1}} \dots W_n$ and $W_0 \dots W_{i_1}$ are positive.)

Let q be the last node of $W_{i_j - 1}$ and r the first node of $W_{i_j + 1}$. (In case $j = k$, q is chosen to be the last element of W_{n-1} and r is chosen to be the first element of W_1.) We now have two paths from q to r. The first is the segment of R from q to r. (If $j = k$, this is interpreted to mean the segment of R from q to s concatenated with the segment of R from s to r.) The second is the reversal of the segment of R from s to q concatenated with the reversal of the segment of R from r to s. (If $j = k$ this is interpreted to mean the reversal of the segment of R from r to q.)

The contradiction arises from observing that the two paths share only their endpoints, a situation which clearly cannot arise in a tree. For, other than the endpoints, the nodes of the first path lie entirely within V_0, and, by the claim above, are unrelated to the V_0 nodes of the second path. □

For our purposes, the important way in which tours arise is in the approximation of crowns. In the case of the diamond, we consider five points of T, arranged so that

$$s_0 < s_1 < s_2 > s_3 > s_4,$$

with $s_i \in V_i$. We fill in all points intermediate between the s_i's to get the path

$$\begin{aligned} &s_0 = t_0 \prec \ldots \prec t_{m_1} = s_1 \prec t_{m_0+1} \prec \ldots \prec t_{m_2} = s_2 \\ &s_2 = t_{m_2} \succ \ldots \succ t_{m_3} = s_3 \succ t_{m_2+1} \succ \ldots \succ t_{m_4} = s_4. \end{aligned}$$

In the case of the 2-crown, we again consider five points of T, this time arranged so that

$$s_0 < s_1 > s_2 < s_3 > s_4,$$

with $s_i \in V_i$. When the intermediate points are filled in, we get the path

$$\begin{aligned} &s_0 = t_0 \prec \ldots \prec t_{m_1} = s_1 \succ t_{m_0+1} \succ \ldots \succ t_{m_2} = s_2 \\ &s_2 = t_{m_2} \prec \ldots \prec t_{m_3} = s_3 \succ t_{m_2+1} \succ \ldots \succ t_{m_4} = s_4. \end{aligned}$$

In the case of the n-crown, $n > 2$, we consider $2n + 1$ points of T, arranged so that

$$s_0 < s_1 > s_2 < \ldots s_{2n-1} > s_{2n},$$

with $s_i \in V_i$. When the intermediate points are filled in, we get the path

$$\begin{aligned} &s_0 = t_0 \prec \ldots \prec t_{m_1} = s_1 \succ t_{m_0+1} \succ \ldots \succ t_{m_2} = s_2 \\ &\ldots \\ &s_{2n-1} = t_{m_{2n-1}} \succ \ldots \succ t_{m_{2n}} = s_{2n}. \end{aligned}$$

Lemma 3.2.4. *The path $\langle t_i \rangle$ described above is a tour.*

Proof. Consider first the case of the diamond. Since U_1 and U_3 are down-sets, and since $s_0 \in V_0 = U_1 \cap U_3$ and $s_2 \notin U_1 \cup U_3$, there exist integers k and l such that, for $0 \leq i \leq m_2$,

$$t_i \in U_3 \text{ iff } i \leq k, \text{ and } t_i \in U_1 \text{ iff } i \leq l.$$

And since $t_{m_1} = s_1 \in V_1 = U_1 \smallsetminus U_3$,

$$0 \leq k < m_1 \leq l < m_2.$$

Likewise there exist integers n and p, $m_2 \leq n < m_3 \leq p < m_4$, such that, for $m_2 \leq i \leq m_4$,

$$t_i \notin U_3 \text{ iff } i \leq n, \text{ and } t_i \notin U_1 \text{ iff } i \leq p.$$

Thus the basic segmentation of $\langle t_i \rangle$ is

$$(t_0 \cdots t_k)\,(t_{k+1} \cdots t_l)\,(t_{l+1} \cdots t_n)\,(t_{n+1} \cdots t_{m_4})\,,$$

and this segmentation clearly satisfies the requirements to be a tour.

Next consider the case of the 2-crown. Since $t_0 \in U_0$ and U_0 is an upset, $t_i \in U_0$ for $0 \leq i \leq m_1$, and since $t_{m_1} \in U_1$ and U_1 is a down-set, $t_i \in U_1$ for $0 \leq i \leq m_1$. Reasoning in similar fashion, we get that

$$t_i \in \begin{cases} U_0 \cap U_1, & 0 \leq i \leq m_1, \\ U_1 \cap U_2, & m_1 \leq i \leq m_2, \\ U_2 \cap U_3, & m_2 \leq i \leq m_3, \\ U_3 \cap U_4, & m_3 \leq i \leq m_4. \end{cases}$$

Since $t_{m_1} \notin U_3 \ni t_0$, there is an integer k, $0 \leq k < m_1$, such that $t_i \in U_3$ iff $i \leq k$. Likewise there is an integer l, $0 \leq l < m_1$, such that $t_i \notin U_2$ iff $i \leq l$. And $k \leq l$ lest $\cap U_i \neq \emptyset$. Reasoning in similar fashion, we get integers m, n, p, q, u, and v,

$$m_2 \leq m \leq n < m_2 \leq p \leq q < m_3 \leq u \leq v < m_4,$$

such that

$$t_i \in U_0 \text{ iff } i \leq m, \text{ and } t_i \notin U_3 \text{ iff } i \leq n,\ m_1 \leq i \leq m_2,$$
$$t_i \in U_1 \text{ iff } i \leq p, \text{ and } t_i \notin U_0 \text{ iff } i \leq q,\ m_2 \leq i \leq m_3,$$
$$t_i \in U_2 \text{ iff } i \leq u, \text{ and } t_i \notin U_1 \text{ iff } i \leq v,\ m_3 \leq i \leq m_4.$$

These indices allow us to give the basic segmentation of $\langle t_i \rangle$ in the required form $R_0S_0R_1S_1R_2S_2R_3S_3R_4$, as follows.

$$R_0 \equiv (t_0 \cdots t_k) \subseteq U_0 \cap U_1 \cap (T \smallsetminus U_2) \cap U_3 = V_0$$
$$S_0 \equiv (t_{k+1} \cdots t_l) \subseteq U_0 \cap U_1 \cap (T \smallsetminus U_2) \cap (T \smallsetminus U_3)$$
$$R_1 \equiv (t_{l+1} \cdots t_m) \subseteq U_0 \cap U_1 \cap U_2 \cap (T \smallsetminus U_3) = V_1$$
$$S_1 \equiv (t_{m+1} \cdots t_n) \subseteq (T \smallsetminus U_0) \cap U_1 \cap U_2 \cap (T \smallsetminus U_3)$$
$$R_2 \equiv (t_{n+1} \cdots t_p) \subseteq (T \smallsetminus U_0) \cap U_1 \cap U_2 \cap U_3 = V_2$$
$$S_2 \equiv (t_{p+1} \cdots t_q) \subseteq (T \smallsetminus U_0) \cap (T \smallsetminus U_1) \cap U_2 \cap U_3$$
$$R_3 \equiv (t_{q+1} \cdots t_u) \subseteq U_0 \cap (T \smallsetminus U_1) \cap U_2 \cap U_3 = V_3$$
$$S_3 \equiv (t_{u+1} \cdots t_v) \subseteq U_0 \cap (T \smallsetminus U_1) \cap (T \smallsetminus U_2) \cap U_3$$
$$R_4 \equiv (t_{v+1} \cdots t_{m_4}) \subseteq U_0 \cap U_1 \cap (T \smallsetminus U_2) \cap U_3 = V_0$$

(We interpret S_0 to be void if $k = l$, and similarly for S_1, S_2, and S_3.) Cursory inspection reveals that this segmentation meets the requirements to be a tour in the case of the 2-crown. In the case of the n-crown, $n > 2$, no new ideas are involved in the proof and for that reason we leave it to the reader. □

We arrive finally at the main point of this section, Proposition 3.2.5, for which we need one more definition. It is the definition of a *small* subset of V_0, which is slightly different in each case. In the case of the diamond, $S \subseteq V_0$ is small provided that

$$\downarrow(\downarrow(\uparrow(\uparrow S \cap V_1) \cap V_2) \cap V_3) \cap S = \emptyset.$$

In the case of the 2-crown, $S \subseteq V_0$ is small provided that

$$\downarrow(\uparrow(\downarrow(\uparrow S \cap V_1) \cap V_2) \cap V_3) \cap S = \emptyset.$$

In the case of the n-crown, $n > 2$, $S \subseteq V_0$ is small provided that

$$\downarrow(\uparrow(\ldots(\downarrow(\uparrow S \cap V_1) \cap V_2)\ldots) \cap V_{2n-1}) \cap S = \emptyset.$$

Proposition 3.2.5. *V_0 can be partitioned into disjoint small subsets S_1 and S_2 such that each S_i satisfies $\uparrow S_i \cap V_0 = S_i$.*

Proof. We write $s\Omega t$ to indicate that there is a tour from s to t. Note that $s\Omega t$ iff $t\Omega s$, and that

$$(\dagger) \qquad \exists\, r_i\ (s\Omega r_1 \Omega r_2 \Omega t) \implies s\Omega t.$$

Let S_1 be maximal among subsets of V_0 with respect to the property

$$(*) \qquad (s \in S \text{ and } s\Omega t) \implies t \notin S.$$

We claim that $S_2 \equiv V_0 \setminus S_1$ satisfies $(*)$. For suppose $r_1\Omega r_2$ for $r_i \in S_2$. By the maximality of S_1, there would then have to exist $s_i \in S_1$ such that $s_i\Omega r_i$, which would imply by $(\dagger)$ the contradiction $s_1\Omega s_2$. We next claim that any two nodes r and s for which $\langle r, s\rangle \subseteq V_0$ must lie in the same S_i, for otherwise $r\Omega s$ and the tour from r to s could be concatenated with $\langle r, s\rangle$ to produce a closed tour, contrary to Lemma 3.2.3. Therefore, though S_2 may be empty, S_1 is not, since for any $r \in V_0$, the nonempty set of nodes which can be reached from r by a path lying entirely within V_0 enjoys property $(*)$. Finally, it follows, also from the second claim, that $\uparrow S_i \cap V_0 = S_i$.

It remains only to show that the S_i's are small. We argue in the case of the 2-crown, the other cases being quite similar. If S_1, for example, is not small then there must be an element

$$\begin{aligned}
&s_5 \in \downarrow(\uparrow(\downarrow(\uparrow S_1 \cap V_2) \cap V_3) \cap V_4) \cap S_1, \text{ say}\\
&s_5 < s_4 \in \uparrow(\downarrow(\uparrow R_1 \cap V_2) \cap V_3) \cap V_4, \text{ hence}\\
&s_4 > s_3 \in \downarrow(\uparrow R_1 \cap V_2) \cap V_3, \text{ hence}\\
&s_3 < s_2 \in \uparrow R_1 \cap V_2, \text{ hence}\\
&s_2 > s_1 \in R_1.
\end{aligned}$$

But, exactly as in the discussion above, these five points give rise to a tour from s_1 to s_5, in violation of the defining condition $(*)$ for S_1. $\square$

3.3. **Acyclicity of a sum of trees.** Suppose $\{T_j : j \in J\}$ is a family of trees. Following the coproduct conventions of Subsection 2.5, we denote by A_j the lattice of up-sets of T_j, and by A the lattice of up-sets of $T \equiv \bigcup_J T_j$. (A is isomorphic to $\prod_J A_j$.) We take the elements of the sum $X \equiv \coprod_J T_j$ to be the prime filters of A, and we regard T as a subset of X.

Theorem 3.3.1. *A sum of finite trees is acyclic.*

Proof. It is sufficient to show that X contains no diamond, no proper 2-crown, and no n-crown, $n > 2$. We show that X contains no proper 2-crown; the other cases can be handled in similar fashion. So suppose that $x_1, x_2 < x_3, x_4$ is a proper 2-crown in X, and let U_i, $0 \leq i \leq 3$, be clopen neighborhoods of the points satisfying Lemma 3.1.1(2). Let

$$\begin{aligned} V_0' &\equiv U_0 \cap U_1 \cap (X \smallsetminus U_2) \cap U_3, \\ V_1' &\equiv U_0 \cap U_1 \cap U_2 \cap (X \smallsetminus U_3), \\ V_2' &\equiv (X \smallsetminus U_0) \cap U_1 \cap U_2 \cap U_3, \\ V_3' &\equiv U_0 \cap (X \smallsetminus U_1) \cap U_2 \cap U_3, \end{aligned}$$

and note that each V_i' is a convex clopen subset of X containing x_i. For every $j \in J$, let S_1^j and S_2^j be the result of applying Proposition 3.2.5 to the subsets $U_i \cap T_j$, let $S_i \equiv \bigcup_J S_i^j$, let $a_i \equiv \uparrow S_i$, let $R_i \equiv \{x \in X : S_i \in x\}$, and let $W_i \equiv R_i \cap V_0'$. Since each a_i lies in A, each R_i is a clopen up-set and each W_i is a convex clopen subset of V_0'.

We claim that the W_i''s partition V_0'. This is basically because $W_i \cap T = S_i$ and the S_i's partition $V_0' \cap T$, but we will elaborate. First of all, $W_1 \cap W_2$ is empty, since this open set meets the dense subset T in $S_1 \cap S_2 = \emptyset$. And secondly, $V_0' \subseteq W_1 \cup W_2$, since otherwise the nonempty open set $V_0' \smallsetminus (W_1 \cup W_2)$ would meet T nontrivially, a state of affairs ruled out by the fact that $V_0' \cap T \subseteq S_1 \cap S_2$ by construction.

It follows that one of the W_i's must contain x_0. But x_0 cannot lie in W_1, for if it did then

$$\begin{aligned} x_1 &\in \uparrow W_1 \cap V_1', \\ x_2 &\in \downarrow(\uparrow W_1 \cap V_1') \cap V_2', \\ x_3 &\in \uparrow(\downarrow(\uparrow W_1 \cap V_1') \cap V_2') \cap V_3', \\ x_0 &\in \downarrow(\uparrow(\downarrow(\uparrow W_1 \cap V_1') \cap V_2') \cap V_3') \cap W_1, \end{aligned}$$

meaning that the set displayed last, call it W_1', is nonempty. But this is not the case, since, by Lemma 2.5.2, that open set meets T in

$$\downarrow(\uparrow(\downarrow(\uparrow S_1 \cap V_1) \cap V_2) \cap V_3) \cap S_1,$$

where $V_i \equiv T \cap V_i'$, and the latter set is empty by virtue of the smallness of each S_1^j. Likewise $x_1 \notin S_2'$, and this is a contradiction. We conclude that X contains no proper 2-crown. □

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