The edge-closure of a claw-free graph is the line graph of a multigraph

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Abstract.
Ryjáček introduced a closure concept in claw-free graphs based on local completion at a locally connected vertex. He showed that the closure of a graph is the line graph of a triangle-free graph. Brousek and Holub gave an analogous closure concept of claw-free graphs, called the edge-closure, based on local completion at a locally connected edge. In this paper, it is shown that the edge-closure is the line graph of a multigraph.

Keywords: Claw-free, Closure concept, Edge-closure concept, Hamiltonicity, Stable property

AMS Subject Classification (2000): 05C45, 05C35

1 Introduction

In this paper, by a graph we mean a simple undirected graph (without loops and multiple edges). By a multigraph we mean a graph in which multiple edges and loops are allowed. We use [3] for terminology and notations not defined here. The circumference, i.e., the length of a longest cycle in G, is denoted by $c(G)$. For a nonempty set $A \subseteq V(G)$, the induced subgraph on A in G is denoted by $\langle A \rangle_G$. For any $A \subset V(G)$, G–A stands for the graph $\langle V(G) \setminus A \rangle_G$. An edge xy is *pendant* if $d_G(x) = 1$ or $d_G(y) = 1$.
For a connected graph H, a graph G is said to be H*-free*, if G does not contain a copy of H as an induced subgraph; the graph H will be also

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[2]Research Supported by Grant No. 1M0545 of the Czech Ministry of Education.

referred to in this context as a *forbidden subgraph*. The graph $K_{1,3}$ will be called the *claw* and in the special case $H = K_{1,3}$ we say that G is *claw-free*.
Let $x \in V(G)$. The *neighbourhood* of x, denoted by $N_G(x)$, is the set of all vertices adjacent to x. For a nonempty set $A \subset V(G)$, $N_G(A)$ denotes the set of all vertices of G–A adjacent to at least one vertex of A, and $N_G[A] = N_G(A) \cup A$. For an edge $xy \in E(G)$ we set $N_G(xy) = N_G(\{x, y\})$ and $N_G[xy] = N_G[\{x, y\}]$. A vertex $x \in V(G)$ is said to be *locally connected* if $\langle N_G(x) \rangle$ is connected. A graph G is *locally connected* if every vertex of G is locally connected. Analogously, an edge $xy \in E(G)$ is *locally connected* if $\langle N_G(xy) \rangle$ is connected and a graph G is *edge-locally connected* if every edge of G is locally connected.
For an arbitrary vertex $x \in V(G)$, let $B_x = \{uv|\, u, v \in N_G(x),\ uv \notin E(G)\}$ and $G_x = (V(G),\ E(G) \cup B_x)$. The graph G_x is called the *local completion* of G at x. A locally connected vertex x with $B_x \neq \emptyset$ is called *eligible* (in G). We say that a graph F is the *closure* of G, denoted by $F = \mathrm{cl}(G)$, if there is no eligible vertex in F and there is a sequence of graphs $G_1, \ldots, G_t$ and vertices $x_1, \ldots, x_{t-1}$ such that $G_1 = G$, $G_t = F$, x_i is an eligible vertex of G_i and $G_{i+1} = (G_i)_{x_i}$, $i = 1, \ldots t-1$ (equivalently, $\mathrm{cl}(G)$) is obtained from G by a series of local completions at eligible vertices, as long as this is possible). The following basic result was proved by Ryjáček.

Theorem A [5]. *Let G be a claw-free graph. Then*

(i) $\mathrm{cl}(G)$ *is well-defined (i.e., uniquelly determined),*

(ii) there is a triangle-free graph H such that $\mathrm{cl}(G) = L(H)$,

(iii) c(G)=c($\mathrm{cl}(G)$*).*

Consequently, if G is claw-free, then so is $\mathrm{cl}(G)$. A claw-free graph G, for which $G = \mathrm{cl}(G)$, will be called *closed*.
For an edge $xy \in E(G)$, let $B_{xy} = \{u, v|\, u, v \in N_G[xy],\ uv \notin E(G)\}$ and let $G_{xy} = (V(G), E(G) \cup B_{xy})$. The graph G_{xy} is called the *local completion* of G at xy. A locally connected edge xy is called *eligible* (in G), if $B_{xy} \neq \emptyset$ and xy is not a pendant edge in G. We say that a graph F is the *edge-closure* of G, denoted by $F = \mathrm{cl}'(G)$, if there is no eligible edge in F and there is a sequence of graphs $G_1, \ldots,\ G_t$ and edges $e_1, \ldots,\ e_{t-1}$ such that $G_1 = G$,

$G_t = F$, e_i is an eligible edge of G_i and $G_{i+1} = (G_i)_{e_i}$, $i = 1, \ldots, t-1$. A claw-free graph G, for which $G = \operatorname{cl}'(G)$, will be called *edge-closed.* In [4], there are examples showing the independence between the closure introduced by Ryjáček in [5] and the edge-closure (i.e., none of the closures can be obtained by using the other one).
The following theorem shows the basic properties of the edge-closure of a graph.

Theorem B [4]. *Let G be a claw-free graph. Then*

(i) *the closure* $\operatorname{cl}'(G)$ *is well defined,*

(ii) *the graph* $\operatorname{cl}'(G)$ *is claw-free,*

(iii) $c(G) = c(\operatorname{cl}'(G))$.

Beineke in [1] characterized line graphs of graphs in terms of forbidden induced sugraphs. He showed that a graph G is the line graph of a graph H if and only if G does not contain any of nine forbidden subgraphs. One of them, given in Fig. 1, is edge-closed, implying that the edge-closure of a graph is not a line graph of a graph. In this paper we show that the edge-closure of a claw-free graph is the line graph of a multigraph.

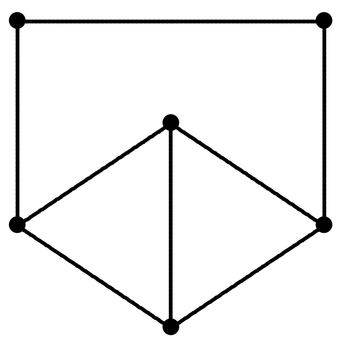


Fig. 1

Bermond and Meyer [2] characterized line graphs of multigraphs:

Theorem C [2]. *Let G be a multigraph. A graph H is the line graph of a multigraph G if and only if H contains none of the seven graphs given in Fig. 2.*

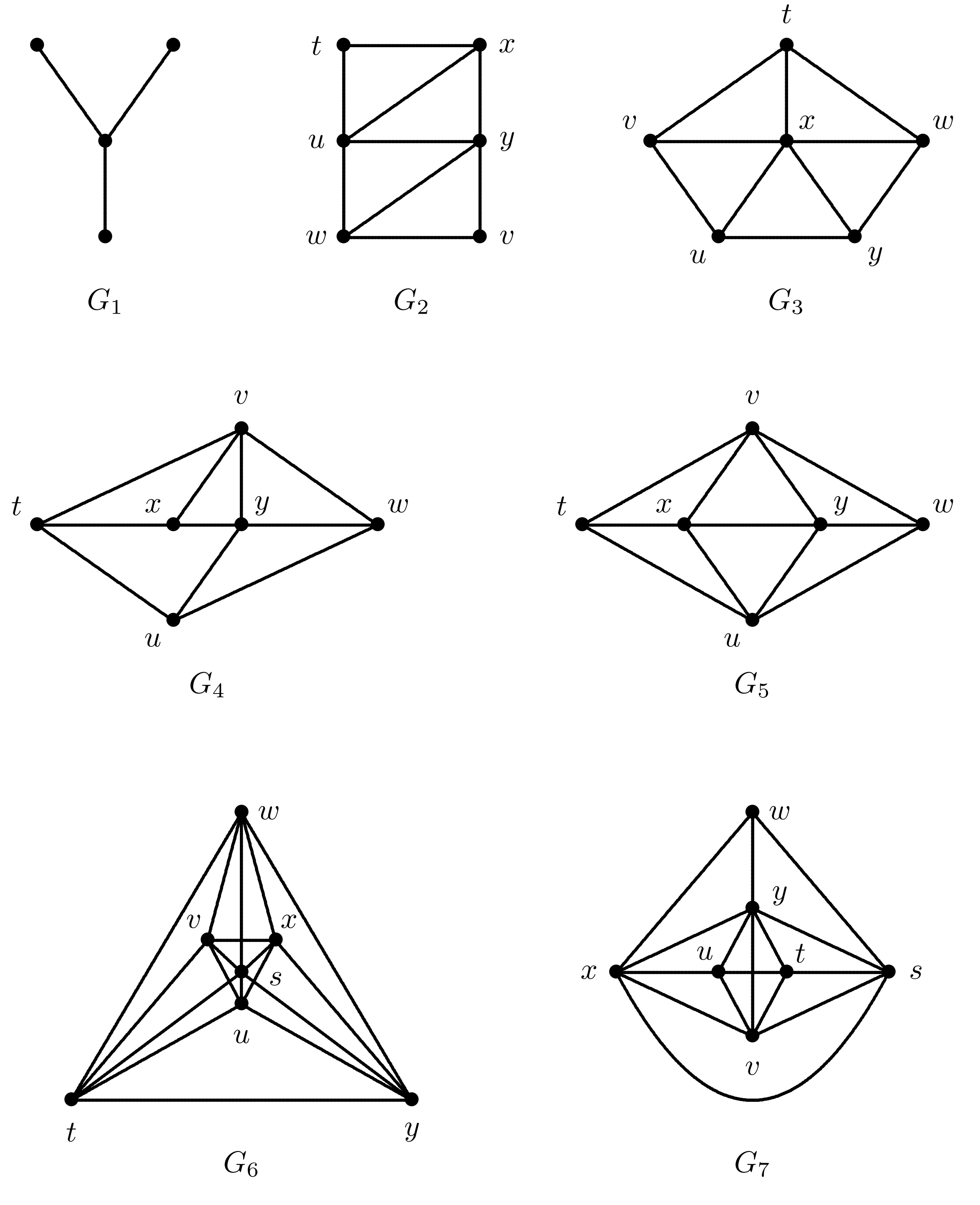


Fig. 2

Note that we will always keep the labeling of the vertices of the graphs $G_2, G_3, \ldots, G_7$ as shown in Fig. 2.

2 Main result

Theorem 1. *Let G be a claw-free graph. Then $\mathrm{cl}'(G)$ is the line graph of a multigraph.*

Proof. We will show that every edge-closed graph contains none of the seven forbidden subgraphs depicted in Fig. 2. Let G be a claw-free graph, let H be the edge-closure of G. By Theorem B, H is claw-free.
Now, to the contrary, we suppose that H contains an induced subgraph F isomorphic to one of the graphs $G_2, G_3, \ldots, G_7$. Since H is edge-closed, none of the edges of F is eligible in H.
Consider a pair of vertices x, y of the graph F as shown in Fig. 2. Since xy is not eligible in H and $\langle N_H(xy)\rangle$ is not complete, there is at least one neighbouring vertex z of xy in $\langle N_H(xy)\rangle$ such that z belongs to a different component of $\langle N_H(xy)\rangle$ than the vertices t, u, v, w. Choose an arbitrary vertex z with this property and note that H is claw-free by Theorem B.

Case 1: $F \simeq G_2$ or $F \simeq G_4$ or $F \simeq G_5$. Suppose that $xz \in E(H)$. Since H is claw-free, the subgraph $\langle x, t, y, z\rangle$ is not an induced claw in H. Clearly $ty \notin E(H)$, since otherwise F is not induced in H, a contradiction. If $zt \in E(G)$, then z belongs to the same component of $\langle N_H(xy)\rangle$ as the vertices t, u, v, w implying that the edge xy is eligible in H, a contradiction again.

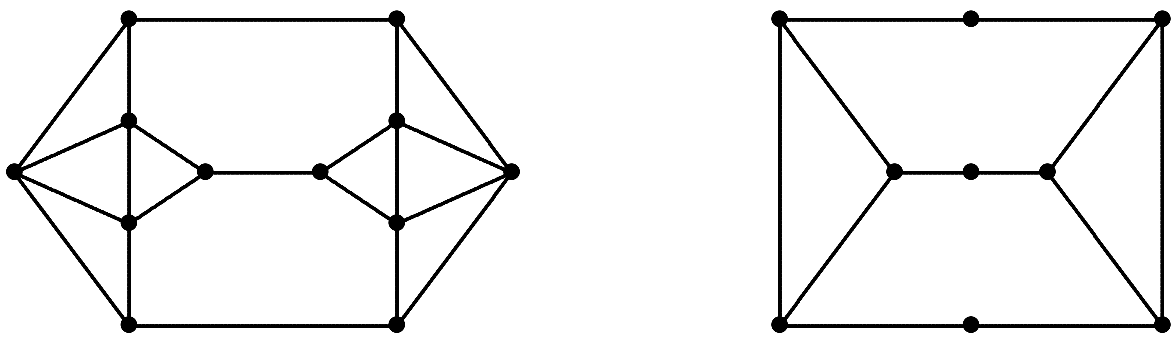
Hence $yz \in E(H)$. Consider the subgraph induced by the vertices y, u, v, z. Since H is claw-free, there is at least one of the edges uv, uz, vz. For the edge uv the subgraph F is not induced in H, a contradiction. Thus at least one of the edges zu, zv belongs to H implying that z belongs to the same component of $\langle N_H(xy)\rangle$ as t, u, v, w. This yields that the edge xy is eligible in H, a contradiction. Hence H is G_2, G_4, G_5-free.

Case 2: $F \simeq G_3$ or $F \simeq G_6$ or $F \simeq G_7$. Suppose that $yz \in E(H)$. Since H is claw-free, at least one of the edges wu, wz, uz belongs to H. For the edge uw the subgraph F is not induced in H, a contradiction. If $uz \in E(H)$ or $wz \in E(H)$, then z belongs to the same component of $\langle N_H(xy)\rangle$ as t, u, v, w. This yields that the edge xy eligible in H, a contradiction.

Now suppose that $xz \in E(H)$. Since H is claw-free, at least one of the edges wu, wz, uz belongs to H. For the edge uw the subgraph F is not induced in H, a contradiction. If $uz \in E(H)$ or $wz \in E(H)$, then the vertex z belongs to the same component of $\langle N_H(xy)\rangle$ as the vertices t, u, v, w. This implies that xy is eligible in H, a contradiction again. Hence we have shown that H is G_3, G_6, G_7-free.

Thus we have shown that $\mathrm{cl}'(G)$ contains none of the forbidden subgraphs given in Theorem C. Hence the edge-closure of a claw-free graph G is the line graph of a multigraph. ■

We have shown that the edge-closure of a claw-free graph G is not necessarily a line graph of a graph. The following example shows that there is an edge-closed graph H such that H has no triangle-free line graph original.



$L(G) = H$ $\quad\quad$ G

Fig. 3

References

[1] L.W. Beineke: Characterizations of derived graphs.
J. Combin. Theory Ser. B 9 (1970), 129-135.

[2] J.C. Bermond, J.C. Meyer: Graphe representatif des arretes d'un multigraphe.
J. Math. Pures et Appl. 52 (1973), 299-308

[3] J.A. Bondy, U.S.R. Murty.: Graph Theory with Applications.
Macmillan, London and Elsevier (1976).

[4] J. Brousek, P. Holub: Edge-closure concept in claw-free graphs and stability of forbidden subgraphs.
Journal of Graph Theory, submitted

[5] Z. Ryjáček: On a Closure Concept in Claw-Free Graphs.
Journal of Combinatorial Theory Ser. B. 70 (1997), 217-224.