Short Cycle Covers of Graphs with Minimum Degree Three

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Abstract

The Shortest Cycle Cover Conjecture asserts that the edges of every bridgeless graph with m edges can be covered by cycles of total length at most $7m/5 = 1.4m$. We show that every bridgeless graph with minimum degree three that contains m edges has a cycle cover comprised of three cycles of total length at most $44m/27 \approx 1.6296m$; this extends a bound of Fan [J. Graph Theory 18 (1994), 131–141] for cubic graphs to the class of all graphs with minimum degree three.

1 Introduction

Cycle covers of graphs are closely related to several deep and open problems in graph theory. A *cycle* in a graph is a subgraph with all degrees even. A *cycle cover* is a collection of cycles such that each edge is contained in at least one of the cycles; we say that each edge is *covered*. The Cycle Double Cover

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Conjecture of Seymour [24] and Szekeres [26] asserts that every bridgeless graph G has a collection of cycles containing each edge of G exactly twice which is called a *cycle double cover*. In fact, it was conjectured by Celmins [5] and Preissmann [22] that every graph has such a collection of five cycles.

The Cycle Double Cover Conjecture is known to be implied by several other conjectures, e.g., the Berge-Fulkerson Conjecture [10] asserting that every cubic bridgeless graph G has 6 perfect matchings covering each edge of G twice. Another conjecture that implies the Cycle Double Cover Conjecture is the Shortest Cycle Cover Conjecture of Alon and Tarsi [1] asserting that every bridgeless graph with m edges has a cycle cover of total length at most $7m/5$. Recall that the *length* of a cycle is the number of edges contained in it and the length of the cycle cover is the sum of the lengths of its cycles. The reduction of the Cycle Double Cover Conjecture to the Shortest Cycle Cover Conjecture can be found in the paper of Jamshy and Tarsi [15].

The best known general result on short cycle covers is due to Alon and Tarsi [1] and Bermond, Jackson and Jaeger [3]: every bridgeless graph with m edges has a cycle cover of total length at most $5m/3 \approx 1.667m$. As it is the case with most conjectures in this area, there are numerous results on short cycle covers for special classes of graphs, e.g., graphs with no short cycles, well connected graphs or graphs admitting a nowhere-zero 4-/5-flow, see e.g. [7, 8, 12, 13, 16, 23]. The reader is referred to the monograph of Zhang [27] for further exposition of such results where an entire chapter is devoted to results on the Shortest Cycle Cover Conjecture.

The least restrictive of such refinements of the general bound of Alon and Tarsi [1] and Bermond, Jackson and Jaeger [3] is the result of Fan [7] that every *cubic* bridgeless with m edges has a cycle cover of total length at most $44m/27 \approx 1.630m$. This result has recently been improved in [17] where it is shown that every cubic bridgeless graph with m edges has a cycle cover of total length at most $34m/21 \approx 1.619m$. In this paper, we strengthen the result of Fan [7] in another direction: we show that every m-edge bridgeless graph *with minimum degree three* has a cycle cover of total length at most $44m/27 \approx 1.630m$, i.e., we extend the result from [7] on cubic graphs to all graphs with minimum degree three. As in [7], the cycle cover that we construct consists of at most three cycles.

Though the improvements of the original bound of $5m/3 = 1.667m$ on the length of a shortest cycle cover of an m-edge bridgeless graph can seem to be rather minor, obtaining a bound below $8m/5 = 1.600m$ for a significant class of graphs might be quite challenging since the bound of $8m/5$ is implied by Tutte's 5-Flow Conjecture [16].

2 Notation

Let us briefly introduce notation used throughout this paper. We only focus on those terms where confusion could arise and refer the reader to standard graph theory textbooks, e.g. [6], for exposition of other notions.

Graphs considered in this paper can have loops and multiple (parallel) edges. If E is a set of edges of a graph G, $G \setminus E$ denotes the graph with the same vertex set with the edges of E removed. If $E = \{e\}$, we simply write $G \setminus e$ instead of $G \setminus \{e\}$. For an edge e of G, G/e is the graph obtained by contracting the edge e, i.e., G/e is the graph with the end-vertices of e identified, the edge e removed and all the other edges, including new loops and parallel edges, preserved. Note that if e is a loop, then $G/e = G \setminus e$. Finally, for a set E of edges of a graph G, G/E denotes the graph obtained by contracting all edges contained in E. If G is a graph and v is a vertex of G of degree two, then the graph obtained from G by *suppressing* the vertex v is the graph obtained from G by contracting one of the edges incident with v, i.e., the graph obtained by replacing the two-edge path with the inner vertex v by a single edge.

An *edge-cut* in a graph G is a set E of edges such that the vertices of G can be partitioned into two sets A and B such that E contains exactly edges with one end-vertex in A and the other in B. Such an edge-cut is also denoted by $E(A, B)$. Note that edge-cuts need not be minimal sets of edges whose removal increases the number of components of G. An edge forming an edge-cut of size one is called a *bridge* and graphs with no edge-cuts of size one are said to be *bridgeless*. Note that we do not require bridgeless graphs to be connected. Also observe that if G has no edge-cuts of size k, then G/F also has no edge-cuts of size k for every set F of edges of G.

As said before, a *cycle* of a graph G is a subgraph of G with all vertices of even degree. A *circuit* is a connected subgraph with all vertices of degree two and a 2-*factor* is a spanning subgraph with all vertices of degree two.

3 Rainbow Lemma

In this section, we state and prove a variant of the following folklore lemma, referred to as the Rainbow Lemma. The Rainbow Lemma has been implicitly used in some of previous work, e.g. [7, 18, 20], and is closely related to the notion of parity 3-edge-colorings from the Ph.D. thesis of Goddyn [11].

Lemma 1 (Rainbow Lemma)**.** *Let G be a bridgeless cubic graph. G contains a 2-factor F such that the edges of G not contained in F can be colored with three colors, red, green and blue in the following way:*

- *every even circuit of F contains an even number of vertices incident with red edges, an even number of vertices incident with green edges and an even number number of vertices incident with blue edges, and*
- *every odd circuit of F contains an odd number of vertices incident with red edges, an odd number of vertices incident with green edges and an odd number number of vertices incident with blue edges.*

In the rest of this paper, a 2-factor F with an edge-coloring satisfying the constraints given in Lemma 1 will be called a *rainbow 2-factor*.

In this section, we prove a weighted variant of the Rainbow Lemma which is needed in our further considerations. Later, in Section 7, we further generalize the argument to exclude certain edge-colorings of the edges not contained in the 2-factor F. However, we think that presenting a less general version of the lemma first will help the reader to follow our arguments later.

A key ingredient of the proof of Lemma 1 is the following classical result of Jaeger:

Theorem 2 (Jaeger [14])**.** *If G is a graph that contains no edge-cuts of size one or three, then G has a nowhere-zero 4-flow.*

Another ingredient for the proof of our modifications of the Rainbow Lemma is the notion of fractional perfect matchings. Let us briefly survey some classical results from this area. The reader is referred to a recent monograph of Schrijver [25] for a more detailed exposition.

A *perfect matching* M of a graph G is the set of edges such that every vertex of G is incident with exactly one edge of M. A perfect matching M can also be viewed as a zero-one vector $u_M \in \{0,1\}^{E(G)}$ such that for each vertex v, the entries of u corresponding to the edges incident v sum to one. A fractional perfect matching is a generalization of this notion: a non-negative vector $u \in \mathbb{R}^{E(G)}$ is said to be a *fractional perfect matching* of the graph G if it can be expressed as a convex combination of vectors u_M corresponding to perfect matchings M of G. The convex polytope formed by all vectors corresponding to fractional perfect matching is called the *perfect matching polytope* of the graph G.

A natural question is whether it is possible to explicitly find the inequalities describing the perfect matching polytope for a graph G. Clearly, all vectors u of the perfect matching polytope have non-negative entries between 0 and 1 (inclusively) and satisfy that the sum of the entries of u corresponding to the edges incident a vertex v sum to one for every vertex v. These two constraints turn out to fully describe the perfect matching polytope if the graph is bipartite [2], however, they are not sufficient for a full description of the perfect matching polytope of non-bipartite graphs. In the general case, the description of the perfect matching polytope is given as follows:

Theorem 3 (Edmonds [4]). *Let G be a graph. A vector $u \in \mathbb{R}^{E(G)}$ is contained in the perfect matching polytope of G if and only if:*

- *all the entries of u are between 0 and 1 (inclusively),*
- *the sum of the entries corresponding to the edges incident with a vertex v is equal to one for every vertex v of G, and*
- *the sum of the entries corresponding to the edges with one end-vertex in a subset $V' \subseteq V(G)$ and with the other end-vertex not in V' is at least one for every subset $V' \subseteq V(G)$ of odd cardinality.*

Note that the last condition of Theorem 3 applied for $V' = V(G)$ implies that the perfect matching polytope is empty if the number of the vertices of G is odd.

We are now ready to prove a weighted variant of the Rainbow Lemma.

Lemma 4. *Let G be a bridgeless cubic graph with edges assigned weights and let w_0 be the total weight of all the edges of G. The graph G contains a rainbow 2-factor F such that the total weight of the edges of F is at least $2w_0/3$ and the 2-factor F contains no circuits of length three.*

Proof. Observe first that Theorem 3 implies that the vector $u \in \mathbb{R}^{E(G)}$ with all entries equal to $1/3$ is contained in the perfect matching polytope of

G. Hence, there exist perfect matchings $M_1, \ldots, M_k$ of G and coefficients $\alpha_i \in (0,1]$, $i = 1, \ldots, k$, such that

$$u = \sum_{i=1}^{k} \alpha_i u_{M_i} \qquad \text{and} \qquad \sum_{i=1}^{k} \alpha_i = 1 \ .$$

Let w_i be the sum of the weights of the edges contained in the perfect matching M_i. Since $u = \sum_{i=1}^{k} \alpha_i u_{M_i}$, we conclude that

$$w_0/3 = \sum_{i=1}^{k} \alpha_i w_i \ .$$

Since $w_0/3$ is a convex combination of the weights w_i, there exists an index $i_0 \in \{1, \ldots, k\}$ such that $w_{i_0} \leq w_0/3$. Let F be the complement of M_{i_0}.

Let us now focus on the graph $H = G/F$. Every edge-cut of H corresponds to an edge-cut of G of the same size. In particular, H has no edge-cuts of size one. Assume that H has an edge-cut of size three and let V_1 and V_2 be the vertices of G corresponding to the two parts of H. Since the graph G is cubic and the size of the edge-cut $E(V_1, V_2)$ is odd, both the parts V_1 and V_2 must contain an odd number of vertices of G.

Let $E(V_1, V_2) = \{e_1, e_2, e_3\}$. The sum of the entries of each of the vectors $u_{M_1}, \ldots, u_{M_k}$ corresponding to the edges e_1, e_2 and e_3 is at least one since V_1 contains an odd number of vertices. On the other hand, the sum of the entries of the vector u, which is a convex combination of the vectors $u_{M_1}, \ldots, u_{M_k}$, is equal to one. Hence, the sum of the three entries of each of the vectors $u_{M_1}, \ldots, u_{M_k}$ corresponding to the edges e_1, e_2 and e_3 must also be equal to one. In particular, M_{i_0} contains exactly one of the edges e_1, e_2 and e_3 which is impossible since $\{e_1, e_2, e_3\} \subseteq M_{i_0}$. We conclude that H has no edge-cuts of size one or three. This also implies that F has no circuits of length three.

Theorem 2 yields that H has a nowhere-zero 4-flow. Fix a nowhere-zero flow $\varphi : E(H) \to \mathbb{Z}_2^2$. The edges of $\varphi^{-1}(01)$ are colored with red, the edges of $\varphi^{-1}(10)$ with green and the edges of $\varphi^{-1}(11)$ with blue. Since φ is a $\mathbb{Z}_2^2$-flow of H, a vertex of H of odd degree is incident with an odd number of red edges, an odd number of green edges and an odd number of blue edges (counting loops twice). Similarly, the vertices of H of even degree are incident with an even number of red edges, green edges and blue edges. Since the weight of the edges of M_{i_0} is at most $w_0/3$, the statement of the lemma follows. $\square$

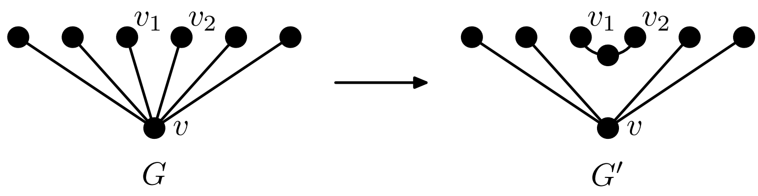

Figure 1: Splitting the pair v_1 and v_2 from the vertex v.

4 Intermezzo

In order to help the reader to follow our arguments, we present another proof of the classical result of Alon and Tarsi [1] and Bermond, Jackson and Jaeger [3] that every bridgeless graph with m edges has a cycle cover of length at most $5m/3$. In the rest of the paper, we refine the arguments presented above to obtain an improved bound for graphs with minimum degree three.

The core of our proof is the Rainbow Lemma. In order to apply the lemma, we first reduce vertices of degrees four or more. This will be achieved through vertex splitting which we now introduce. Consider a graph G, a vertex v and two neighbors v_1 and v_2 of v. The graph $G.v_1vv_2$ that is obtained by removing the edges vv_1 and vv_2 from G and adding a two-edge path v_1v_2 (see Figure 1) is said to be obtained by *splitting the pair v_1 and v_2 from the vertex v*. Note that if $v_1 = v \neq v_2$, i.e., the edge vv_1 is a loop, the graph $G.v_1vv_2$ is the graph obtained from G by removing the loop vv_1 and subdividing the edge vv_2. Similarly, if $v_1 \neq v = v_2$, $G.v_1vv_2$ is obtained by removing the loop vv_2 and subdividing the edge vv_1. Finally, if $v_1 = v = v_2$, then the graph $G.v_1vv_2$ is obtained from G by removing the loops vv_1 and vv_2 and introducing a new vertex joined by two parallel edges to v.

There are several deep results on splitting vertices in graphs preserving edge-connectivity, see the classical works of Fleischner [9], Mader [21] and Lovász [19]. Let us now formulate one of the simplest possible corollaries of results in this area.

Lemma 5. *Let G be a bridgeless graph. For every vertex v of G of degree four or more, there exist two neighbors v_1 and v_2 of the vertex v such that the graph $G.v_1vv_2$ is also bridgeless.*

Let us now reprove the upper bound of $5m/3$ on the length of the shortest cycle cover of an m-edge bridgeless graph. The proof that we present differs both from the proof of Alon and Tarsi [1] which is based on 6-flows and the proof of Bermond, Jackson and Jaeger [3] based on 8-flows; on the other hand, its main idea resembles the proof of Fan [7] for cubic graphs.

Theorem 6. *Let G be a bridgeless graph with m edges. G has a cycle cover of length at most* $5m/3$*.*

Proof. If G has a vertex v of degree four or more, then, by Lemma 5, v has two neighbors v_1 and v_2 such that the graph $G.v_1vv_2$ is also bridgeless. Let G' be the graph $G.v_1vv_2$. The number of edges of G' is the same as the number of edges of G and every cycle of G' corresponds to a cycle of G of the same length. Hence, a cycle cover of G' corresponds to a cycle cover of G of the same length. Through this process we can reduce any bridgeless graph to a bridgeless graph with maximum degree three. In particular, we can assume without loss of generality that the graph G has maximum degree three and G is connected (otherwise, cover each component separately).

If G is a circuit, the statement is trivial. Otherwise, we proceed as described in the rest. First, we suppress all vertices of degree two in G. Let G_0 be the resulting cubic (bridgeless) graph. We next assign each edge e of G_0 the weight equal to the number of edges in the path corresponding to e in G. In particular, the total weight of the edges of G is equal to m. Let F_0 be a rainbow 2-factor with the properties described in Lemma 4.

The 2-factor F_0 corresponds to a set F of disjoint circuits of the graph G which do not necessarily cover all the vertices of G. Let w_F be the weight of the edges contained in the 2-factor F_0, and r, g and b the weight of red, green and blue edges, respectively. By symmetry, we can assume that $r \leq g \leq b$. Since the weight w_F of the edges contained in the 2-factor F_0 is at least $2m/3$, the sum $r + g + b$ is at most $m/3$. Finally, let $\mathcal{R}$ be the set of edges of G corresponding to red edges of G_0, $\mathcal{G}$ the set of edges corresponding to green edges, and $\mathcal{B}$ the set of edges corresponding to blue edges. By the choice of edge-weights, the cardinality of $\mathcal{R}$ is r, the cardinality of $\mathcal{G}$ is g and the cardinality of $\mathcal{B}$ is b.

For a circuit C contained in F and for a set of edges of E such that $C \cap E = \emptyset$, we define $C(E)$ to be the set of vertices of C incident with the edges of E. If $C(E)$ has even cardinality, it is possible to partition the edges of C into two sets $C(E)^A$ and $C(E)^B$ such that

- each vertex of $C(E)$ is incident with one edge of $C(E)^A$ and one edge of $C(E)^B$, and

- each vertex of C not contained in $C(E)$ is incident with either two edges of $C(E)^A$ or two edges of $C(E)^B$.

Note that if $C(E) = \emptyset$, then $C(E)^A$ contains no edges of C and $C(E)^B$ contains all the edges of C (or vice versa). We will always assume that the number of edges of $C(E)^A$ does not exceed the number of edges of $C(E)^B$, i.e., $|C(E)^A| \leq |C(E)^B|$.

The desired cycle cover of G which is comprised of three cycles can now be defined. The first cycle $\mathcal{C}_1$ contains all the red and green edges and the edges of $C(\mathcal{R} \cup \mathcal{G})^A$ for all circuits C of the 2-factor F. The second cycle $\mathcal{C}_2$ contains all the red and green edges and the edges of $C(\mathcal{R} \cup \mathcal{G})^B$ for all circuits C of F. Finally, the third cycle $\mathcal{C}_3$ contains all the red and blue edges and the edges of $C(\mathcal{R} \cup \mathcal{B})^A$ for all circuits C of F.

Let us first verify that the cycles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ cover the edges of G. Clearly, every edge not contained in F, i.e., a red, green or blue edge, is covered by at least one of the cycles. On the other hand, every edge of F is contained either in the cycle $\mathcal{C}_1$ or the cycle $\mathcal{C}_2$. Hence, the cycles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ form a cycle cover of G.

It remains to estimate the lengths of the cycles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$. Each edge of F is covered once by the cycles $\mathcal{C}_1$ and $\mathcal{C}_2$; since $|C(E)^A| \leq |C(E)^B|$ for every circuit C of F, at most half of the edges of F is also covered by the cycle $\mathcal{C}_3$. We conclude that the total length of the constructed cycle cover is at most:

$$3r + 2g + b + |F| + |F|/2 \leq 2(r + g + b) + 3w_F/2 =$$

$$3(r + g + b + w_F)/2 + (r + g + b)/2 \leq 3m/2 + m/6 = 5m/3\ .$$

This finishes the proof of the theorem. $\square$

5 Splitting and expanding vertices

In Section 9, we will apply the Rainbow Lemma in a way analogous to that in the proof of Theorem 6. However, not every edge-coloring is suitable for our further needs. In order to exclude some "bad" edge-colorings, we will first modify the graph $H = G/F_0$ from the proof of the Rainbow Lemma to assure that some of its edges must get the same color. This modification will be done through splitting some of the vertices of $H = G/F_0$ without introducing edge-cuts of size one or three.

Another corollary of the classical results on vertex splittings is that it is always possible to split off a pair of neighbors of every vertex without introducing edge-cuts of size one or three:

Lemma 7. *Let G be a graph with no edge-cuts of size one or three. For every vertex v of G of degree four, six or more, there exist two neighbors v_1 and v_2 of the vertex v such that the graph $G.v_1vv_2$ also contains no edge-cuts of size one or three.*

Not even this lemma is sufficient for our purposes and we will need some corollaries of results on vertex splitting established in [17].

Lemma 8. *Let G be a graph with no edge-cuts of size one or three, and let v be a vertex of degree four and v_1, v_2, v_3 and v_4 its four neighbors. The graph $G.v_1vv_2$ or the graph $G.v_2vv_3$ contains no edge-cuts of size one or three.*

Lemma 9. *Let G be a graph with no edge-cuts of size one or three, and let v be a vertex of degree six and $v_1, \ldots, v_6$ its neighbors. At least one of the graphs $G.v_1vv_2$, $G.v_2vv_3$ and $G.v_3vv_4$ contains no edge-cuts of size one or three.*

Lemma 10. *Let G be a graph with no edge-cuts of size one or three, and let v be a vertex of degree six or more and $v_1, \ldots, v_k$ its neighbors ($k \geq 6$). At least one of the graphs $G.v_1vv_2$, $G.v_2vv_3$, $G.v_3vv_4$, $G.v_4vv_5$ and $G.v_5vv_6$ contains no edge-cuts of size one or three.*

In [17], these lemmas are stated and proven for simple graphs and for another variant of vertex splitting in which the newly created vertices of degree two are suppressed. Since the two notions of vertex splitting differ only by subdividing some of the edges, and every graph can be made simple by subdividing all its edges, and subdividing edges cannot create edge-cuts of size one or three if they did not exist before, the proofs presented in [17] readily translate to our scenario.

We need one more vertex operation in our arguments in Section 9—vertex expansions. If G is a graph, v a vertex of G and V_1 and V_2 a partition of the neighbors of v into two sets, then the graph $G : v : V_1$ is the graph obtained from G by removing the vertex v and introducing two new vertices v_1 and v_2, joining v_1 to the vertices of V_1, v_2 to the neighbors of v not contained in V_1, and adding an edge v_1v_2. We say that $G : v : V_1$ is obtained by *expanding the vertex v with respect to the set V_1*. See Figure 2 for an example. Let us

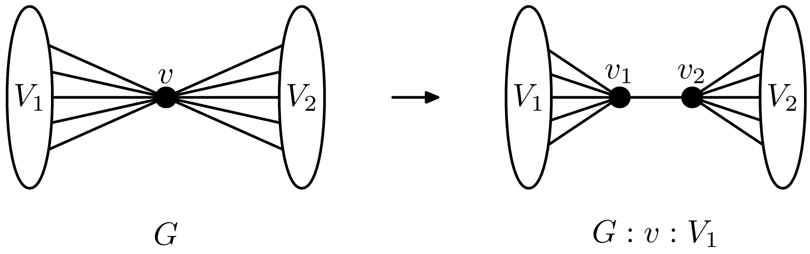

Figure 2: An example of the expansion a vertex v with respect to set V_1.

remark that this operation will be applied only to vertices v incident with no parallel edges.

In Section 9, we use the following auxiliary lemma which directly follows from results of Fleischner [9]:

Lemma 11. *Let G be a bridgeless graph and v a vertex of degree four in G incident with no parallel edges. Further, let v_1, v_2, v_3 and v_4 be the four neighbors of v. The graph $G : v : \{v_1, v_2\}$ or the graph $G : v : \{v_2, v_3\}$ is also bridgeless.*

6 Special types of $\mathbb{Z}_2^2$-flows

As mentioned before, we need a modification of the Rainbow Lemma excluding certain edge-colorings of the graph $H = G/F_0$. Some of the "bad" edge-colorings will be excluded by vertex splitting introduced in Section 5. However, vertex splitting itself is not sufficient to exclude all bad edge-colorings. In this section, we establish an auxiliary lemma that guarantees the existence of a special nowhere-zero $\mathbb{Z}_2^2$-flow.

Lemma 12. *Let G be a bridgeless graph admitting a nowhere-zero $\mathbb{Z}_2^2$-flow. Assume that*

- *for every vertex v of degree five, there are given two multisets A_v and B_v of three edges incident with v such that $|A_v \cap B_v| = 2$ (loops can appear twice in the same set), and*

- *for every vertex v of degree six, the incident edges are partitioned into three multisets A_v, B_v and C_v of size two each (loops appear twice in these sets).*

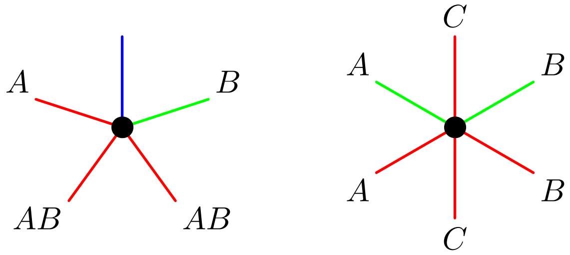

Figure 3: Bad vertices of degree five and six (symmetric cases are omitted). The letters indicate edges contained in the sets A, B and C.

The graph G has a nowhere-zero $\mathbb{Z}_2^2$-flow φ such that

- *for every vertex v of degree five, the flow φ is constant on neither of the sets A_v and B_v, and*
- *for every vertex v of degree six, the edges incident with v have all the three possible flow values, or the flow φ is constant on A_v, or it is constant on B_v, or it is not constant on C_v.*

Proof. By Theorem 2, G has a nowhere-zero $\mathbb{Z}_2^2$-flow φ. For simplicity, we refer to edges with the flow value 01 red, 10 green and 11 blue. Note that each vertex of odd degree is incident with odd numbers of red, green and blue edges and each vertex of even degree is incident with even numbers of red, green and blue edges (counting loops twice). We say that a vertex v of degree five is *bad* if φ is constant on A_v or on B_v, and it is *good*, otherwise. Similarly, a vertex v of degree six is *bad* if φ has only two possible flow values at v and it is not constant on A_v and on B_v and is constant on C_v; otherwise, v is good. Choose a $\mathbb{Z}_2^2$-flow φ of H with the least number of bad vertices. If there are no bad vertices, then there is nothing to prove. Assume that there is a bad vertex v.

Let us first analyze the case that the degree of v is five. Let $e_1, \dots, e_5$ be the edges incident with v. By symmetry, we can assume that $A_v = \{e_1, e_2, e_3\}$, $B_v = \{e_2, e_3, e_4\}$, the edges e_1, e_2 and e_3 are red, the edge e_4 is green and the edge e_5 is blue (see Figure 3). We now define a closed trail W in H formed by red and blue edges. The first edge of W is e_1.

Let $f = ww'$ be the last edge of W defined so far. If $w' = v$, then f is one of the edges e_2, e_3 and e_5 and the definition of W is finished. Assume that $w' \neq v$. If w' is not a vertex of degree five or six or w' is a bad vertex,

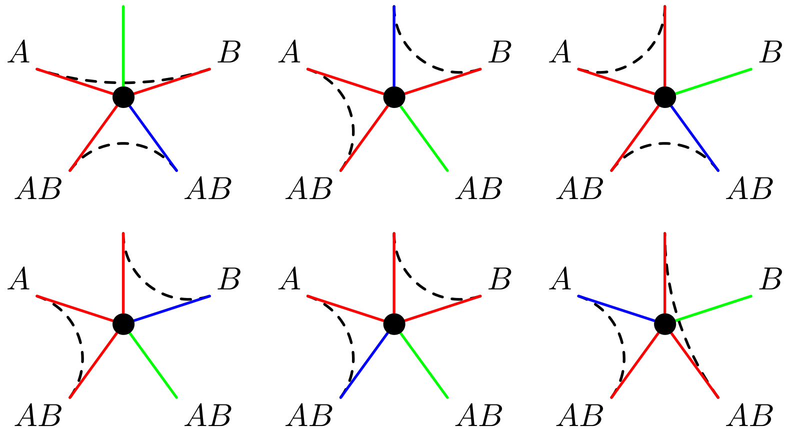

Figure 4: Routing the trail W (indicated by dashed edges) through a good vertex of degree five with three red edges. The letters indicate edges contained in the sets A and B. Symmetric cases are omitted.

add to the trail W any red or blue edge incident with w' that is not already contained in W.

If w' is a good vertex of degree five, let $f_1, \ldots, f_5$ be the edges incident with w', $A_{w'} = \{f_1, f_2, f_3\}$ and $B_{w'} = \{f_2, f_3, f_4\}$. If w' is incident with a single red and a single blue edge, leave w' through the other edge that is red or blue. Otherwise, there are three red edges and one blue edge or vice versa. The next edge f' of the trail W is determined as follows (note that the role of red and blue can be swapped):

Red edges	Blue edge	$f = f_1$	$f = f_2$	$f = f_3$	$f = f_4$	$f = f_5$
f_1, f_2, f_4	f_3	$f' = f_4$	$f' = f_3$	$f' = f_2$	$f' = f_1$	N/A
f_1, f_2, f_4	f_5	$f' = f_2$	$f' = f_1$	N/A	$f' = f_5$	$f' = f_4$
f_1, f_2, f_5	f_3	$f' = f_5$	$f' = f_3$	$f' = f_2$	N/A	$f' = f_1$
f_1, f_2, f_5	f_4	$f' = f_2$	$f' = f_1$	N/A	$f' = f_5$	$f' = f_4$
f_1, f_4, f_5	f_2	$f' = f_2$	$f' = f_1$	N/A	$f' = f_5$	$f' = f_4$
f_2, f_3, f_5	f_1	$f' = f_2$	$f' = f_1$	$f' = f_5$	N/A	$f' = f_3$

See Figure 4 for an illustration of these rules.

If w' is a good vertex of degree six, proceed as follows. If φ is constant on $A_{w'}$ and $f \in A_{w'}$, let the next edge f' of W be the other edge contained in $A_{w'}$; if φ is constant on $A_{w'}$ and $f \notin A_{w'}$, let f' be any red or blue edge not contained in $A_{w'}$ or in W. A symmetric rule applies if φ is constant on $B_{w'}$, i.e., f' is the other edge of $B_{w'}$ if $f \in B_{w'}$ and f' is a red or blue edge not contained in $B_{w'}$ or W, otherwise.

If φ is not constant on $C_{w'}$ and $f \in C_{w'}$ and the other edge of $C_{w'}$ is red or blue, set f' to be the other edge of $C_{w'}$; if $f \in C_{w'}$ and the other edge of $C_{w'}$ is green, choose f' to be any red or blue edge incident with w' that is not contained in W. If $f \notin C_{w'}$ (and φ is not constant on $C_{w'}$), choose f' to be a red or blue edge incident with w' not contained in W that is also not contained in $C_{w'}$. If such an edge does not exist, choose f' to be the red or blue edge contained in $C_{w'}$ (note that the other edge of $C_{w'}$ is green since w' is incident with an even number of red, green and blue edges).

It remains to consider the case that w' is incident with two edges of each color and is constant on $C_{w'}$ and neither of $A_{w'}$ and $B_{w'}$. If f is blue, set f' to be any red edge incident with w' not contained in W and if f is red, set f to be any such blue edge. See Figure 5 for an illustration of these rules.

The definition of the trail W is now finished. Let us swap the red and blue colors on W. It is straightforward to verify that all good vertices remain good and the vertex v become good (see Figures 3–5). In particular, the number of bad vertices is decreased which contradicts the choice of φ.

Let us now analyze the case that there is a bad vertex v of degree six, i.e., the colors of the edges of A_v are distinct, the colors of the edges of B_v are distinct and the colors of the edges of C_v are the same and not all the flow values are present at the vertex v (see Figure 3). By symmetry, we can assume that the two edges of A_v are red and green, the two edges of B_v are also red and green, and the two edges of C_v are both red (recall that the vertex v is incident with even numbers of red, green and blue edges). As in the case of vertices of degree five, we find a trail formed by red and blue edges and swap the colors of the edges on the trail. The first edge of the trail is any red edge incident with v and the trail W is finished when it reaches again the vertex v. After swapping red and blue colors on the trail W, the vertex v is incident with two edges of each of the three colors. Again, the number of bad vertices has been decreased which contradicts our choice of the flow φ. $\square$

7 Rainbow Lemma revisited

In this section, we establish a modification of the Rainbow Lemma from Section 3. In addition to the statement of Lemma 4, we exclude certain edge-colorings of edges incident with short circuits of the chosen 2-factor. Let us be more precise. If $C = v_1 \dots v_k$ is a circuit of a cubic graph and e_i

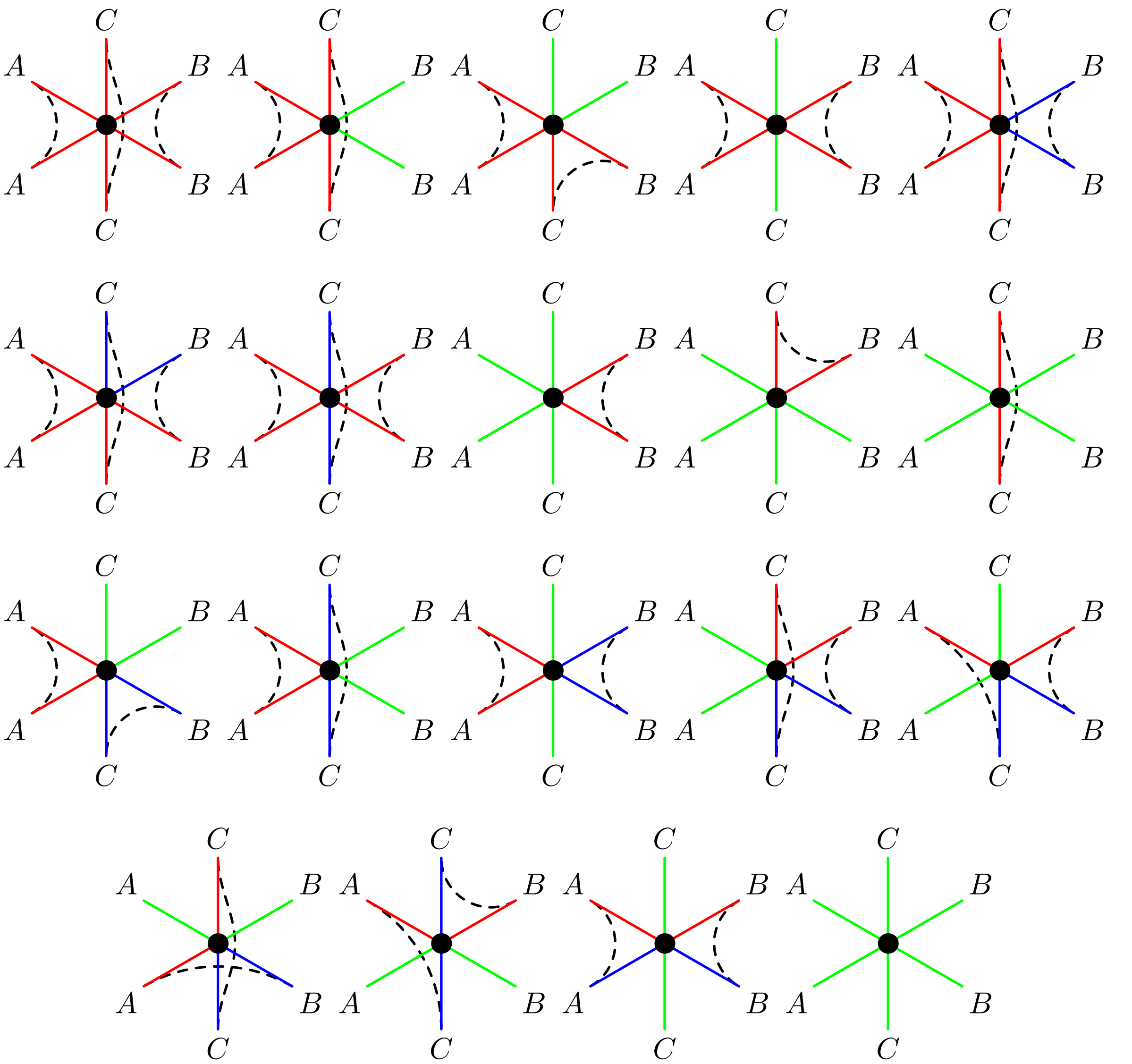

Figure 5: Routing the trail W (indicated by dashed edges) through a good vertex of degree six. The letters indicate edges contained in the sets A, B and C. Symmetric cases are omitted.

the edge incident with v_i not contained in C, then the pattern of C is a k-tuple $X_1 \ldots X_k$ where X_i is R if the color of e_i is red, G if it is green, and B if it is blue. A pattern P is *compatible* with a pattern P' if P' can be obtained from P by a permutation of the red, green and blue colors followed by replacement of some of the colors with the letter x (which represents a wild-card). For example, the pattern RGRGBBGG is compatible with RBRxxxBx.

We can now state and prove the modification of the Rainbow Lemma.

Lemma 13. *Let G be a bridgeless cubic graph with edges assigned non-negative integer weights and w_0 be the total weight of the edges. In addition, suppose that no two edges with weight zero have a vertex in common. The graph G contains a rainbow 2-factor F such that the total weight of the edges of F is at most $2w_0/3$. Moreover, the patterns of circuits with four edges of weight one are restricted as follows. Every circuit $C = v_1 \ldots v_k$ of F that consists of four edges of weight one and at most four edges of weight zero (and no other edges) has a pattern:*

- *compatible with RRxx or xRRx if C has no edges of weight zero (and thus $k = 4$),*
- *compatible with RxGxx or RRRGB if the only edge of C of weight zero is v_4v_5 (and thus $k = 5$),*
- *compatible with xxRRxx, xxxxRR, xxRGGR or xRxGGR if the only edges of C of weight zero are v_3v_4 and v_5v_6 (and thus $k = 6$),*
- *not compatible with RRGRRG, RRGRGR, RGRRRG or RGRRGR if the only edges of C of weight zero are v_2v_3 and v_5v_6 (and thus $k = 6$),*
- *compatible with xRRxxxx, xxxRRxx, xxxxxRR, xRGxxRB, xRGxxBR, xRGxxGB, xRGxxBG, xxxRGRG or xxxRGGR if the only edges of C of weight zero are v_2v_3, v_4v_5 and v_6v_7 (and thus $k = 7$), and*
- *compatible with RRxxxxxx, xxRRxxxx, xxxxRRxx, xxxxxxRR, RGGRxxxx, xxRGGRxx, xxxxRGGR or GRxxxxRG if the edges v_1v_2, v_3v_4, v_5v_6 and v_7v_8 of C have weight zero (and thus $k = 8$).*

Proof. As in the proof of Lemma 4, we first find a perfect matching M with weight at least $w_0/3$ such that the graph $H = G/F$ has no edge-cuts of size one or three where F is the 2-factor of G complementary to M. Note that in the proof of Lemma 4, we found a matching M with weight at most $w_0/3$ but the same argument also yields the existence of a matching M with weight at least $w_0/3$.

Next, we modify the graph $H = G/F$ in such a way that an application of Lemma 12 will yield a $\mathbb{Z}_2^2$-flow that yields an edge-coloring satisfying the conditions from the statement of the lemma. Let w be a vertex of H corresponding to a circuit $v_1 \dots v_k$ of F consisting of four edges with weight one and some edges with weight zero, and let e_i be the edge of M incident with v_i. Finally, let w_i be the neighbor of w in H that corresponds to the circuit containing the other end-vertex of the edge e_i. The graph H is modified as follows (see Figure 6):

- if $k = 4$, split the pair w_1 and w_2 or the pair w_2 and w_3 from w in such a way that the resulting graph has no edge-cuts of size one or three (at least one of the two splittings works by Lemma 8).

- if $k = 5$ and the weight of the edge v_4v_5 is zero, set $A_w = \{e_1, e_3, e_5\}$ and $B_w = \{e_1, e_3, e_4\}$.

- if $k = 6$ and the weights of the edges v_3v_4 and v_5v_6 are zero, split the pair w_3 and w_4, w_4 and w_5, or w_5 and w_6 from w without creating edge-cuts of size one or three (one of the splitting works by Lemma 9). If the pair w_4 and w_5 is split off, split further the pair w_2 and w_6, or the pair w_3 and w_6 from w again without creating edge-cuts of size one or three (one of the splitting works by Lemma 8).

- if $k = 6$ and the weights of the edges v_2v_3 and v_5v_6 are zero, set $A_w = \{e_2, e_3\}$, $B_w = \{e_5, e_6\}$ and $C_w = \{e_1, e_4\}$.

- if $k = 7$ and the weights of the edges v_2v_3, v_4v_5 and v_6v_7 are zero, split one of the pairs w_i and w_{i+1} from w for $i \in \{2, 3, 4, 5, 6\}$ (the existence of such a splitting is guaranteed by Lemma 10). If w_3 and w_4 is split off, set $A_w = \{e_1, e_2, e_6\}$ and $B_w = \{e_1, e_2, e_7\}$. If w_5 and w_6 is split off, set $A_w = \{e_1, e_2, e_7\}$ and $B_w = \{e_1, e_3, e_7\}$.

- if $k = 8$ and the weights of the edges v_1v_2, v_3v_4, v_5v_6 and v_7v_8 are equal to zero, split one of the pairs w_i and w_{i+1} from w for some $i \in \{1, \dots, 8\}$ (indices taken modulo eight) without creating edge-cuts of size one or three. This is possible by Lemma 10. If i is odd, then there are no further modifications to be performed. If i is even, one of the pairs

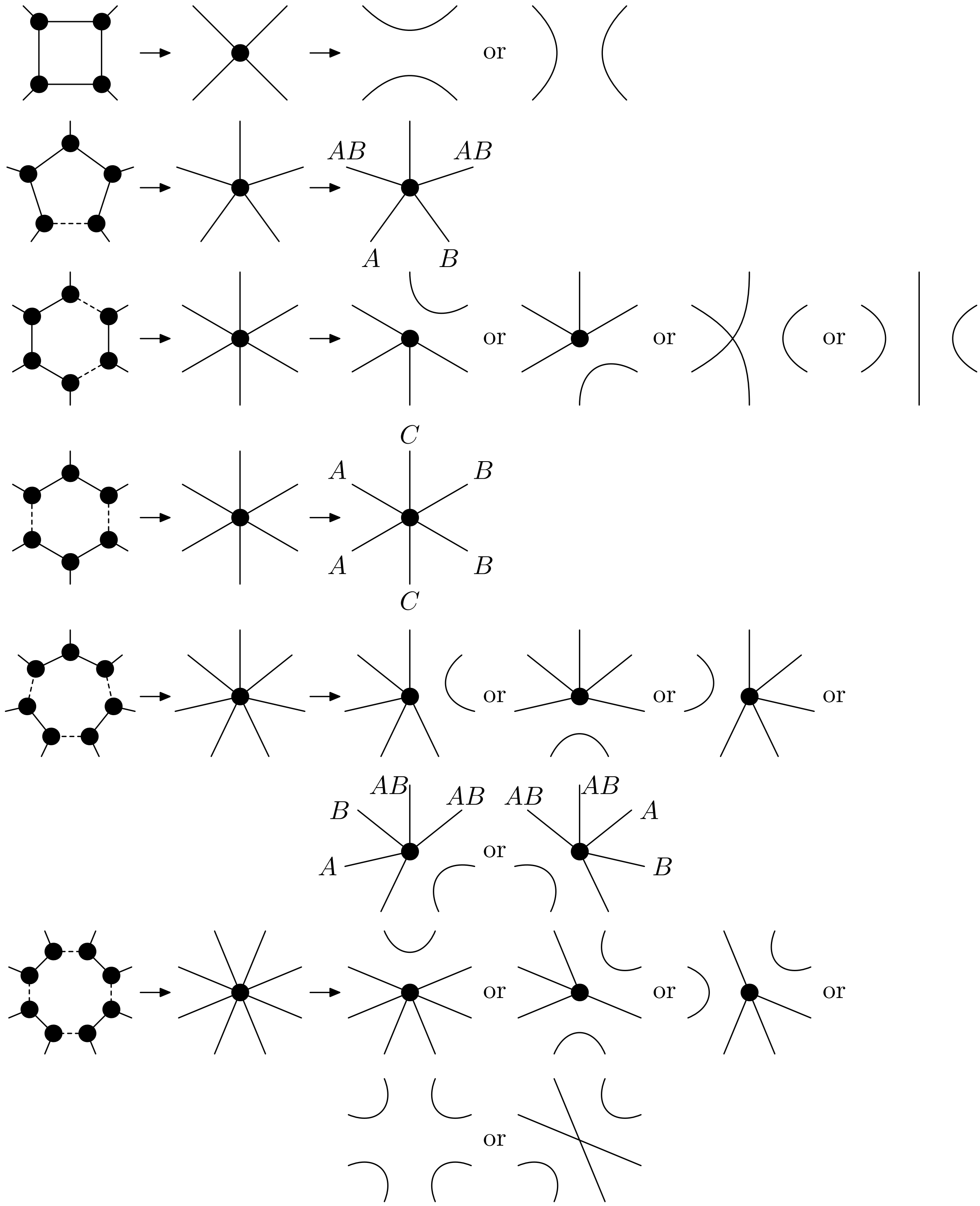

Figure 6: Modifications of the graph H performed in the proof of Lemma 6. The edges of weight one are solid and the edges of weight zero are dashed. The sets A_w, B_w and C_w are indicated by letters near the edges. Vertices of degree two obtained through splittings are not depicted and some symmetric cases are omitted in the case of a circuit of length eight.

w_{i+3} and w_{i+4}, w_{i+4} and w_{i+5}, and w_{i+5} and w_{i+6} is further split off from the vertex w in such a way that no edge-cuts of size one or three are created (one of the splittings has this property by Lemma 9). In case that the vertices w_{i+4} and w_{i+5} are split off, split further the pair of vertices w_{i+2} and w_{i+3} or the pair of vertices w_{i+3} and w_{i+6}, again, without creating edge-cuts of size one or three (and do not split off other pairs of vertices in the other cases). Lemma 8 guarantees that one of the two splittings work.

Fix a nowhere-zero $\mathbb{Z}_2^2$-flow φ with the properties described in Lemma 12 with respect to the sets A_w, B_w and C_w as defined before (and where the sets A_w, B_w and C_w are undefined, choose them arbitrarily). The edges of $\varphi^{-1}(01)$ are colored with red, the edges of $\varphi^{-1}(10)$ with green and the edges of $\varphi^{-1}(11)$ with blue as in the proof of Lemma 4. This defines the coloring of the edges of G not contained in F.

Clearly, F is a rainbow 2-factor. It remains to verify that the patterns of circuits with four edges of weight one are as described in the statement of the lemma. Let $C = v_1 \ldots v_k$ be a circuit of F consisting of four edges with weight one and some edges with weight zero, and let c_i be the color of the edge of M incident with v_i. We distinguish six cases based on the value of k and the position of zero-weight edges (symmetric cases are omitted):

- if $k = 4$, then all the edges of C have weight one. By the modification of H, it holds that $c_1 = c_2$ or $c_2 = c_3$. Hence, the pattern of C is compatible with RRxx or xRRx.

- if $k = 5$ and the weight of v_4v_5 is zero, then either $c_1 \neq c_3$, or $c_1 = c_3 \notin \{c_4, c_5\}$. Since C is incident with an odd number of edges of each color, its pattern is compatible with RxGxx or RRRGB.

- if $k = 6$ and the weights of v_3v_4 and v_5v_6 are zero, then $c_3 = c_4$, or $c_4 = c_5$ and $c_2 = c_6$, or $c_4 = c_5$ and $c_3 = c_6$, or $c_5 = c_6$. Hence, the pattern of C is compatible with xxRRxx, xRxRRR or xRxGGR, xxRRRR or xxRGGR, or xxxxRR.

- if $k = 6$ and the weights of v_2v_3 and v_5v_6 are zero, then the pattern of C contains all three possible colors or it is compatible with xRRxxx, xxxxRR or RxxGxx. In particular, it is not compatible with any of the patterns listed in the statement of the lemma.

- if $k = 7$ and the weights of v_2v_3, v_4v_5 and v_6v_7 are zero, then $c_i = c_{i+1}$ for some $i \in \{2, 3, 4, 5, 6\}$ by the modification of H. If i is even, then the pattern of C is compatible with xRRxxxx, xxxRRxx or xxxxxRR. If $i = 3$, then $c_1 \neq c_2$ or $c_1 = c_2 \notin \{c_6, c_7\}$. Hence, the pattern of C is compatible with RGRRxxx, RGBBxxx, RRGGxGB or RRGGxBG (unless $c_2 = c_3$). Since C is incident with an odd number of edges of each colors, its pattern is compatible with one of the patterns listed in the statement of the lemma. A symmetric argument applies if $i = 5$ and either $c_1 \neq c_7$ or $c_1 = c_7 \notin \{c_2, c_3\}$.
- if $k = 8$ and the weights v_iv_{i+1}, $i = 1, 3, 5, 7$, then $c_i = c_{i+1}$ for $i \in \{1, \ldots, 8\}$ by the modification of H. If there is such odd i, the pattern of C is compatible with RRxxxxxx, xxRRxxxx, xxxxRRxx or xxxxxxRR. Otherwise, at least one of the following holds for some even i: $c_{i+3} = c_{i+4}$, $c_{i+4} = c_{i+5}$ or $c_{i+5} = c_{i+6}$. In the first and the last case, the pattern is again compatible with RRxxxxxx, xxRRxxxx, xxxxR-Rxx or xxxxxxRR. If $c_{i+4} = c_{i+5}$, then $c_{i+2} = c_{i+3}$ or $c_{i+3} = c_{i+6}$. Hence, the pattern of C is compatible with xRRGGRRx, xRRGGBBx, xRRxRGGR, xRRxGRRG, xRRxGBBG or one of the patterns rotated by two, four or six positions. All these patterns are listed in the statement of the lemma.

□

8 Reducing parallel edges

In this section, we show that it is enough to prove our main theorem for graphs that do not contain parallel edges of certain type. We state and prove four auxiliary lemmas that simplify our arguments presented in Section 9. The first two lemmas deal with the cases when there is a vertex incident only with parallel edges leading to the same vertex.

Lemma 14. *Let G be an m-edge bridgeless graph with vertices v_1 and v_2 joined by $k \geq 3$ parallel edges. If the degree of v_1 is k, the degree of v_2 is at least $k + 3$ and the graph $G' = G \setminus v_1$ has a cycle cover with three cycles of length at most $44(m - k)/27$, then G has a cycle cover with three cycles of length at most $44m/27$.*

Proof. Let $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ be the cycles of total length at most $44(m-k)/27$ covering the edges of G' and $e_1, \ldots, e_k$ the k parallel edges between the vertices v_1 and v_2. If k is even, add the edges $e_1, \ldots, e_k$ to $\mathcal{C}_1$. If k is odd, add the edges $e_1, \ldots, e_{k-1}$ to $\mathcal{C}_1$ and the edges e_{k-1} and e_k to $\mathcal{C}_2$. Clearly, we have obtained a cycle cover of G with three cycles. The length of the cycles is increased at most by $k+1$ and thus it is at most

$$\frac{44m-44k}{27}+k+1=\frac{44m-17k+27}{27}\leq\frac{44m}{27}\ .$$

□

Lemma 15. *Let G be an m-edge bridgeless graph with vertices v_1 and v_2 joined by $k \geq 4$ parallel edges. If the degree of v_1 is k, the degree of v_2 is $k+2$ and the graph G' obtained from G by removing all the edges between v_1 and v_2 and suppressing the vertex v_2 has a cycle cover with three cycles of length at most $44(m-k-1)/27$, then G has a cycle cover with three cycles of length at most $44m/27$.*

Proof. Let $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ be the cycles of total length at most $44(m-k-1)/27$ covering the edges of G' and $e_1, \ldots, e_k$ the k parallel edges between the vertices v_1 and v_2. Let v' and v'' be the two neighbors of v_2 distinct from v_1. Note that it can hold that $v' = v''$. The edges v_2v' and v_2v'' are included in those cycles $\mathcal{C}_i$ that contain the edge $v'v''$. The edges $e_1, \ldots, e_{k-1}$ are included to $\mathcal{C}_1$. In addition, the edge e_k is included to $\mathcal{C}_1$ if k is even. Otherwise, the edges e_{k-1} and e_k are included to $\mathcal{C}_2$.

The length of all the cycles is increased by at most $3+k+1=k+4$. Hence, the total length of the cycle cover is at most

$$\frac{44m-44k-44}{27}+k+4=\frac{44m-17k+64}{27}\leq\frac{44m}{27}\ .$$

□

In the next two lemmas, we deal with the case that each of the two vertices joined by several parallel edges is also incident with another vertex.

Lemma 16. *Let G be an m-edge bridgeless graph with vertices v_1 and v_2 joined by $k \geq 2$ parallel edges. If the degree of v_1 is at least $k+1$, the degree of v_2 is at least $k+2$ and the graph G' obtained by contracting all the edges between v_1 and v_2 has a cycle cover with three cycles of length at most $44(m-k)/27$, then G has a cycle cover with three cycles of length at most $44m/27$.*

Proof. Let $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ be the cycles of total length at most $44(m-k)/27$ covering the edges of G' and $e_1, \ldots, e_k$ the k parallel edges between the vertices v_1 and v_2. By symmetry, we can assume that the cycles $\mathcal{C}_1, \ldots, \mathcal{C}_{i_0}$ contain an odd number of edges incident with v_1 and the cycles $\mathcal{C}_{i_0+1}, \ldots, \mathcal{C}_3$ contain an even number of such edges for some $i_0 \in \{0, 1, 2, 3\}$. Since $\mathcal{C}_1, \ldots, \mathcal{C}_3$ form a cycle cover of G', if v_1 is incident with an odd number of edges of $\mathcal{C}_i$, $i = 1, 2, 3$, then v_2 is incident with an odd number of edges of $\mathcal{C}_i$ and vice versa.

The edges are added to the cycles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ as follows based on the value of i_0 and the parity of k:

i_0	k	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$
0	odd	$e_1, \ldots, e_{k-1}$	e_{k-1}, e_k	
0	even	$e_1, \ldots, e_k$		
1	odd	$e_1, \ldots, e_k$		
1	even	$e_1, \ldots, e_{k-1}$	e_{k-1}, e_k	
2	odd	$e_1, \ldots, e_k$	e_k	
2	even	$e_1, \ldots, e_{k-1}$	e_k	
3	odd	$e_1, \ldots, e_{k-2}$	e_{k-1}	e_k
3	even	$e_1, \ldots, e_{k-1}$	e_{k-1}	e_k

Clearly, we have obtained a cycle cover of G with three cycles. The length of the cycles is increased at most by $k+1$ and thus it is at most

$$\frac{44m - 44k}{27} + k + 1 = \frac{44m - 17k + 27}{27} \leq \frac{44m}{27} .$$

□

Lemma 17. *Let G be an m-edge bridgeless graph with vertices v_1 and v_2 joined by $k \geq 3$ parallel edges. If the degrees of v_1 and v_2 are $k+1$ and the graph G' obtained by contracting all the edges between v_1 and v_2 and suppressing the resulting vertex of degree two has a cycle cover with three cycles of length at most $44(m-k-1)/27$, then G has a cycle cover with three cycles of length at most $44m/27$.*

Proof. Let $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ be the cycles of total length at most $44(m-k-1)/27$ covering the edges of G', let $e_1, \ldots, e_k$ be the k parallel edges between the vertices v_1 and v_2, and let v'_i be the other neighbor of v_i, $i = 1, 2$. Add the edges incident with $v_1v'_1$ and $v_2v'_2$ to those cycles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ that contain the edge $v_1'v_2'$ and then proceed as in the proof of Lemma 16. The length of the cycles is increased by at most $3+k+1=k+4$ and thus it is at most

$$\frac{44m-44k-44}{27}+k+4=\frac{44m-17k+64}{27}\leq\frac{44m}{27}$$

where the last inequality holds unless $k=3$. If $k=3$ and the edge $v_1'v_2'$ is contained in at most two of the cycles, the length is increased by at most $2+k+1=k+3=6$. If $k=3$ and the edge $v_1'v_2'$ is contained in three of the cycles, each of the parallel edges is added to exactly one of the cycles and thus the length is increased by at most $3+k=6$. In both cases, the length of the new cycle cover can be estimated as follows:

$$\frac{44m-44\cdot 3-44}{27}+6=\frac{44m-14}{27}\leq\frac{44m}{27}\ .$$

□

9 Main result

We are now ready to prove the main result of this paper.

Theorem 18. *Let G be a bridgeless graph with m edges and with minimum degree three or more. The graph G has a cycle cover of total length at most $44m/27$ that is comprised of at most three cycles.*

Proof. By Lemmas 14–17, we can assume without loss of generality that if vertices v_1 and v_2 of G are joined by k parallel edges, then either $k=2$ and the degrees of both v_1 and v_2 are equal to $k+1=3$, or $k=3$, the degree of v_1 is $k=3$ and the degree of v_2 is $k+2=5$ (in particular, both v_1 and v_2 have odd degrees). Note that the graphs G' from the statement of Lemmas 14–17 are also bridgeless graphs with minimum degree three and have fewer edges than G which implies that the reduction process described in Lemmas 14–17 eventually finishes.

Let us now proceed with the proof under the assumption that the only parallel edges contained in G are pairs of edges between two vertices of degree three and triples of edges between a vertex of degree three and a vertex of degree five. As the first step, we modify the graph G into bridgeless graphs $G_1, G_2, \ldots$ eventually obtaining a bridgeless graph G' with vertices of degree two, three and four. Set $G_1=G$. If G_i has no vertices of degree five or more, let $G' = G_i$. If G_i has a vertex v of degree five or more, then Lemma 5 yields that there are two neighbors v_1 and v_2 of v such that the graph $G_i.v_1vv_2$ is also bridgeless. We set G_{i+1} to be the graph $G_i.v_1vv_2$. We continue while the graph G_i has vertices of degree five or more. Clearly, the final graph G' has the same number of edges as the graph G and every cycle of G' corresponds to a cycle of G.

Next, each edge of G' is assigned weight one, each vertex of degree four is expanded to two vertices of degree three as described in Lemma 11 and the edge between the two new vertices of degree three is assigned weight zero (note that the vertex splitting preserves the parity of the degree of the split vertex and thus no vertex of degree four is incident with parallel edges). The resulting graph is denoted by G_0. Note that every cycle C of G_0 corresponds to a cycle C' of G and the length of C' in G is equal to the sum of the weights of the edges of C. Next, the vertices of degree two in G_0 are suppressed and each edge e is assigned the weight equal to the sum of the weights of edges of the path of G_0 corresponding to e. The resulting graph is denoted by G'_0. Clearly, G'_0 is a cubic bridgeless graph. Also note that all the edges of weight zero in G_0 are also contained in G'_0 and no vertex of G'_0 is incident with two edges of weight zero. Finally, observe that the total weight of the edges of G'_0 is equal to m.

We apply Lemma 13 to the cubic graph G'_0. Let F'_0 be the rainbow 2-factor of G'_0 and let F_0 be the cycle of G_0 corresponding to the 2-factor of F'_0. Note that F_0 is a union of disjoint circuits. Let $\mathcal{R}_0$, $\mathcal{G}_0$ and $\mathcal{B}_0$ be the sets of edges of G_0 contained in paths corresponding to red, green and blue edges in G'_0. Let r_0 be the weight of the red edges in G_0, g_0 the weight of green edges and b_0 the weight of blue edges. Lemma 13 yields $r_0 + g_0 + b_0 \geq m/3$.

We construct two different cycle covers, each comprised of three cycles, and eventually combine the bounds on their lengths to obtain the bound claimed in the statement of the theorem.

The first cycle cover. The first cycle cover that we construct is a cycle cover of the graph G_0 (which yields a cycle cover of G of the same length as explained earlier). Let d_ℓ be the number of circuits of F_0 of weight ℓ. Note that d_3 can be non-zero since a circuit of weight three need not have length three in G_0. The cycle $\mathcal{C}_1$ contains all the red and green edges, i.e., the edges contained in $\mathcal{R}_0 \cup \mathcal{G}_0$, the cycle $\mathcal{C}_2$ contains the red and blue edges and the cycle $\mathcal{C}_3$ contains the green and blue edges. Recall now the notation $C(E)^A$ and $C(E)^B$ used in the proof of Theorem 6 for circuits C and set E of edges that are incident with even number of vertices of C. In addition, $C(E)^A_*$

denotes the edges of $C(E)^A$ with weight one and $C(E)^B_*$ denotes such edges of $C(E)^B$. In the rest of the construction of the first cycle cover, we always assume that $|C(E)^A_*| \le |C(E)^B_*|$. The sets $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ are completed to cycles in a way similar to that used in the proof of Theorem 6.

For a circuit C of F_0, the edges of $C^1 = C(\mathcal{R}_0 \cup \mathcal{G}_0)^A$ are added to the cycle $\mathcal{C}_1$. The edges C^2 added to $\mathcal{C}_2$ are either the edges of $C(\mathcal{R} \cup \mathcal{B})^A$ or $C(\mathcal{R} \cup \mathcal{B})^B$—we choose the set with fewer edges with weight one in common with $C^1 = C(\mathcal{R} \cup \mathcal{G})^A$. Finally, the edges added to $\mathcal{C}_3$ are chosen so that every edge of C is covered an odd number of times; explicitly, the edges $C^3 = C^1 \bigtriangleup C^2 \bigtriangleup C$ are added to $\mathcal{C}_3$. Note that C^3 is either $C(\mathcal{G} \cup \mathcal{B})^A$ or $C(\mathcal{G} \cup \mathcal{B})^B$. In particular, the sets $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ form cycles.

We now estimate the number of the edges of C of weight one contained in $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$. Let C_* be the edges of weight one contained in the circuit C, $\ell = |C_*|$ and $C^i_* = C^i \cap C_*$ for $i = 1, 2, 3$. By the choice of C^2, the number of edges of weight one in $C^1 \cap C^2$ is $|C^1_* \cap C^2_*| \le |C^1_*|/2$. Consequently, the number of edges of C of weight one contained in the cycles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ is:

$$\begin{aligned} |C^1_*| + |C^2_*| + |C^1_* \bigtriangleup C^2_* \bigtriangleup C_*| &= \\ |C^1_* \cup C^2_*| + |C^1_* \cap C^2_*| + |C_* \setminus (C^1_* \cup C^2_*)| + |C^1_* \cap C^2_*| &= \\ |C_*| + 2|C^1_* \cap C^2_*| \, . \end{aligned}$$

Since $|C(\mathcal{R}_0 \cup \mathcal{G}_0)^A_*| \le |C(\mathcal{R}_0 \cup \mathcal{G}_0)^B_*|$, the number of edges contained in the set $C^1_* = C(\mathcal{R}_0 \cup \mathcal{G}_0)^A_*$ is at most $\ell/2$. By the choice of C^2, $|C^1_* \cap C^2_*| \le |C^1_*|/2$. Consequently, it holds that

$$|C^1_* \cap C^2_*| \le |C^1_*|/2 \le \ell/4 \tag{1}$$

and eventually conclude that the sets $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ contain at most $\ell + 2\lfloor \ell/4 \rfloor$ edges of the circuit C with weight one.

If $\ell = 4$, the estimate given in (1) can be further refined. Let C' be the circuit of G' corresponding to C. Clearly, C' is a circuit of length four. Color a vertex v of the circuit C'

red if v has degree three and is incident with a red edge, or v has degree four and is incident with green and blue edges,

green if v has degree three and is incident with a green edge, or v has degree four and is incident with red and blue edges,

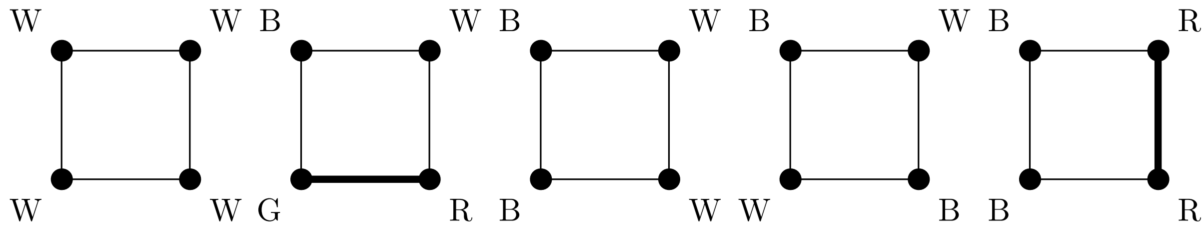

Figure 7: An improvement for circuits of length four considered in the proof of Theorem 18. The letters R, G, B and W stand for red, green, blue and white colors. Note that it is possible to freely permute the red, green and blue colors. The edges included to $C(\mathcal{R}_0 \cup \mathcal{G}_0)^A$ are bold. Symmetric cases are omitted.

blue if v has degree three and is incident with a blue edge, or v has degree four and is incident with red and green edges, and

white otherwise.

Observe that either C' contains a white vertex or it contains an even number of red vertices, an even number of green vertices and an even number of blue vertices. If C' contains a white vertex, it is easy to verify that

$$|C^1_*| = |C(\mathcal{R}_0 \cup \mathcal{G}_0)^A_*| \leq 1 \tag{2}$$

for a suitable permutation of red, green and blue colors. The same holds if C' contains two adjacent vertices of the same color (see Figure 7).

If the circuit of F'_0 corresponding to C contains an edge of weight two or more, then C contains a white vertex and the estimate (2) holds. Otherwise, all vertices of C have degree three in G_0 and thus the circuit C is also contained in F'_0. Since the edges of C have weight zero and one only, the pattern of C is one of the patterns listed in Lemma 13. A close inspection of possible patterns of C' yields that the cycle C' contains a white vertex or it contains two adjacent vertices with the same color. We conclude that the estimate (2) applies. Hence, if $\ell = 4$, the estimate (1) can be improved to 0.

We now estimate the length of the cycle cover of G_0 formed by the cycles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$. Since each red, green and blue edge is covered by exactly two of the cycles, we conclude that:

$$2(r_0 + g_0 + b_0) + 2d_2 + 3d_3 + 4d_4 + 7d_5 + 8d_6 + 9d_7 + \sum_{\ell=8}^{\infty} \frac{3\ell}{2} d_\ell \;\; =$$

$$2(r_0+g_0+b_0)+\frac{3}{2}\sum_{\ell=2}^{\infty}\ell d_\ell-d_2-3d_3/2-2d_4-d_5/2-d_6-3d_7/2 \quad = \\ \frac{3m}{2}+\frac{r_0+g_0+b_0}{2}-d_2-3d_3/2-2d_4-d_5/2-d_6-3d_7/2\,. \qquad (3)$$

Note that we have used the fact that the sum $r_0+g_0+b_0+\sum_{\ell=2}^{\infty}\ell d_\ell$ is equal to the number of the edges of G.

The second cycle cover. The second cycle cover is constructed in an auxiliary graph G'' which we now describe. Every vertex v of G is eventually split to a vertex of degree three or four in G'. The vertex of degree four is then expanded. Let $r(v)$ be the vertex of degree three obtained from v or one of the two vertices obtained by the expansion of the vertex of degree four obtained from v. By the construction of F_0, each $r(v)$ is contained in a circuit of F_0. The graph G'' is constructed from the graph G_0 as follows: every vertex of G_0 of degree two not contained in F_0 that is obtained by splitting from a vertex v is identified with the vertex $r(v)$. The edges of weight zero contained in the cycle F_0 are then contracted. Let F be the cycle of G'' corresponding to the cycle F_0 of G_0. Note that F is formed by disjoint circuits and it contains d_ℓ circuits of weight/length ℓ.

Observe that G'' can be obtained from G by splitting some of its vertices (perform exactly those splittings yielding vertices of degree two contained in the circuits of F_0) and then expanding some vertices. In particular, every cycle of G'' is also a cycle of G. Edges of weight one of G'' one-to-one correspond to edges of weight one of G_0, and edges of weight zero of G'' correspond to edges of weight zero of G_0 not contained in F_0. Hence, the weight of a cycle in G'' is the length of the corresponding cycle in G.

The edges not contained in F are red, green and blue (as in G_0). Each circuit of F is incident either with an odd number of red edges, an odd number of green edges and an odd number of blue edges, or with an even number of red edges, an even number of green edges and an even number of blue edges (chords are counted twice). Let $H=G''/F$. If H contains a red circuit (which can be a loop), recolor such a circuit to blue. Similarly, recolor green circuits to blue. Let $\mathcal{R}$, $\mathcal{G}$ and $\mathcal{B}$ be the resulting sets of red, green and blue edges and r, g and b their weights. Clearly, $r+g+b=r_0+g_0+b_0$. Also note that each circuit of F is still incident either with an odd number of red edges, an odd number of green edges and an odd number of blue edges, or with an even number of red edges, an even number of green edges and an even number of blue edges. Since the red edges form an acyclic subgraph of

$H = G''/F$, there are at most $\sum_{\ell=2}^{\infty} d_\ell - 1$ red edges and thus the total weight r of red edges is at most $\sum_{\ell=2}^{\infty} d_\ell$ (we forget "-1" since it is not important for our further estimates). A symmetric argument yields that $g \leq \sum_{\ell=2}^{\infty} d_\ell$.

Let us have a closer look at circuits of F with weight two. Such circuits correspond to pairs of parallel edges of G'' (and thus of G). By our assumption, the only parallel edges contained in G are pairs of edges between two vertices v_1 and v_2 of degree three and triples of edges between vertices v_1 and v_2 of degree three and five.

In the former case, both v_1 and v_2 have degree three in G''. Consequently, each of them is incident with a single colored edge. By the assumption on the edge-coloring, the two edges have the same color.

In the latter case, the third edge v_1v_2 which corresponds to a loop in G''/F is blue. Hence, the other two edges incident with v_2 must have the same color, which is red, green or blue.

In both cases, the vertex of H corresponding to the circuit v_1v_2 is an isolated vertex in the subgraph of H formed by red edges or in the subgraph formed by green edges (or both). It follows we can improve the estimate on r and g:

$$r + g \leq 2\sum_{\ell=2}^{\infty} d_\ell - d_2 = d_2 + 2\sum_{\ell=3}^{\infty} d_\ell \tag{4}$$

We are now ready to construct the cycle cover of the graph G''. Its construction closely follows the one presented in the proof of Theorem 6. The cycle cover is formed by three cycles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$. The cycles $\mathcal{C}_1$ and $\mathcal{C}_2$ contain all red and green edges and the cycle $\mathcal{C}_3$ contains all red and blue edges. We now explain how to alter the definition of the sets $C(E)^A$ and $C(E)^B$ to the setting needed in the construction of these three cycles. Let C be a circuit of F. Consider a set E of edges disjoint from C with an even number of end-vertices on the circuit C. The set $C(E)$ is defined to be the set of the vertices of C incident with an *odd* number of edges of E. Clearly, $|C(E)|$ is even. As before, it is possible to partition the edges of C into two sets $C(E)^A$ and $C(E)^B$ such that

- each vertex of $C(E)$ is incident with one edge of $C(E)^A$ and one edge of $C(E)^B$, and
- each vertex of C not contained in $C(E)$ is incident with either two edges of $C(E)^A$ or two edges of $C(E)^B$.

As before, we always assume that $|C(E)^A| \leq |C(E)^B|$. Note that if all the vertices of C have degree three, the new definition coincides with the earlier one.

For every circuit C of F, the edges of $C(\mathcal{R} \cup \mathcal{G})^A$ are added to the cycle $\mathcal{C}_1$, the edges of $C(\mathcal{R} \cup \mathcal{G})^B$ to the cycle $\mathcal{C}_2$, and the edges of $C(\mathcal{R} \cup \mathcal{B})^A$ to the cycle $\mathcal{C}_3$. Clearly, the sets $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ are cycles of G'' and correspond to cycles of G whose length is equal to the the weight of the cycles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ in G''.

We now estimate the total weight of the cycles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$. Each red edge is covered three times, each green edge twice and each blue edge once. Each edge of F is contained in either $\mathcal{C}_1$ or $\mathcal{C}_2$ and for every circuit C of F at most half of its edges are also contained in $\mathcal{C}_3$. We conclude that the total length of the cycles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ can be bounded as follows (note that we apply (4) to estimate the sum $r+g$ and we also use the fact that the number of the edges of F is at most $2m/3$ by Lemma 13):

$$\begin{aligned}
3r + 2g + b + \sum_{\ell=2}^{\infty} \left\lfloor \frac{3\ell}{2} \right\rfloor d_\ell &= \\
m + 2r + g + \sum_{\ell=2}^{\infty} \left\lfloor \frac{\ell}{2} \right\rfloor d_\ell &\leq \\
m - d_2 + \sum_{\ell=2}^{\infty} \left(\left\lfloor \frac{\ell}{2} \right\rfloor + 3 \right) d_\ell &= \\
\frac{13m}{8} - \frac{5(r_0+g_0+b_0)}{8} - d_2 + \sum_{\ell=2}^{\infty} \left(\left\lfloor \frac{\ell}{2} \right\rfloor + 3 - \frac{5\ell}{8} \right) d_\ell &\leq \\
\frac{43m}{24} - \frac{5(r_0+g_0+b_0)}{8} - d_2 + \sum_{\ell=2}^{\infty} \left(\left\lfloor \frac{\ell}{2} \right\rfloor + 3 - \frac{7\ell}{8} \right) d_\ell &\leq \\
\frac{43m}{24} - \frac{5(r_0+g_0+b_0)}{8} - d_2 + \sum_{\ell=2}^{6} \left(\left\lfloor \frac{\ell}{2} \right\rfloor + 3 - \frac{7\ell}{8} \right) d_\ell &= \\
\frac{43m}{24} - \frac{5(r_0+g_0+b_0)}{8} + 5d_2/4 + 11d_3/8 + 3d_4/2 + 5d_5/8 + 3d_6/4\ . & \qquad (5)
\end{aligned}$$

The last inequality follows from the fact that $\left\lfloor \frac{\ell}{2} \right\rfloor + 3 - \frac{7\ell}{8} \leq 0$ for $\ell \geq 7$.

The length of the shortest cycle cover of G with three cycles exceeds neither the bound given in (3) nor the bound given in (5). Hence, the length of such a cycle cover of G is bounded by any convex combination of the two bounds, in particular, by the following:

$$\frac{5}{9}\cdot\left(\frac{3m}{2}+\frac{r_0+g_0+b_0}{2}-d_2-3d_3/2-2d_4-d_5/2-d_6-3d_7/2\right)+$$

$$\frac{4}{9}\cdot\left(\frac{43m}{24}-\frac{5(r_0+g_0+b_0)}{8}+5d_2/4+11d_3/8+3d_4/2+5d_5/8+3d_6/4\right)=$$

$$\frac{44m}{27}-2d_3/9-4d_4/9-2d_6/9-5d_7/6\leq\frac{44m}{27}\ .$$

The proof of Theorem 18 is now completed. □

Acknowledgement

This research was initiated while the second author attended PIMS Workshop on Cycle Double Cover Conjecture held at University of British Columbia, Vancouver, Canada. Support of Pacific Institute for Mathematical Sciences (PIMS) during the workshop is greatly acknowledged. The authors are also indebted to Robert Šámal for fruitful discussions on the subject of this paper.

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