Choosability of Squares of K_4-minor Free Graphs

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Abstract

Lih, Wang and Zhu [Discrete Math. 269 (2003), 303–309] proved that the chromatic number of the square of a K_4-minor free graph with maximum degree Δ is bounded by $\lfloor 3\Delta/2 \rfloor + 1$ if $\Delta \geq 4$, and is at most $\Delta + 3$ for $\Delta \in \{2, 3\}$. We show that the same bounds hold for the list chromatic number of squares of K_4-minor free graphs. The same result was also proved independently by Hetherington and Woodall.

1 Introduction

Graph coloring is a very active area of graph theory. A particular attention is focused on colorings of planar graphs. Wegner's conjecture asserts that the square of every planar graph with maximum degree Δ has a coloring with approximately $3\Delta/2$ colors. Recall that the *square* G^2 of a graph G is the graph on the same vertex set with any two vertices at distance at most two joined by an edge.

Conjecture 1 (Wegner 1977, [17]). *Let G be a planar graph with maximum degree Δ. The chromatic number of G^2 is at most 7, if $\Delta = 3$, at most $\Delta + 5$, if $4 \leq \Delta \leq 7$, and at most $\left\lfloor \frac{3\Delta}{2} \right\rfloor + 1$, otherwise.*

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Conjecture 1 has been recently verified by Thomassen [15] for graphs with maximum degree three and it remains open for $\Delta \geq 4$. For larger values of Δ, there is a series [9, 18, 8, 7, 1, 2, 3] of improvements of the upper bound with the currently best known upper bound $\lfloor 5\Delta/3 \rfloor + 78$ due to Molloy and Salavatipour [13, 14]. A significant progress on the conjecture has been recently achieved by by Havet, van den Heuvel, McDiarmid and Reed [5] who established the upper bound $(3/2 + o(1))\Delta$, i.e., verified the conjecture to be asymptotically true.

Planar graphs are known to be precisely the graphs that do not contain the graphs K_5 and $K_{3,3}$ as minors. Recall that a graph H is a minor of a graph G if H can be obtained from G by deleting vertices and edges and contracting edges. Conjecture 1 is known to be true for outerplanar graphs [11], the class of graphs that do not contain $K_{2,3}$ or K_4 as a minor. The validity of the conjectured bound was later extended to K_4-minor free graphs:

Theorem 1 (Lih, Wang and Zhu 2003, [12])**.** *The chromatic number of the square of a K_4-minor free graph G of maximum degree Δ is at most $\lfloor 3\Delta/2 \rfloor + 1$ if $\Delta \geq 4$ and at most $\Delta + 3$ if $2 \leq \Delta \leq 3$.*

In this paper, we consider a list variant of the problem. Recall that the *list chromatic number* of a graph G is the smallest integer k such that the vertices of G can be colored for an arbitrary assignment of lists of k colors to each vertex in such a way that every vertex receives a color from its list. Clearly, the list chromatic number is always at least the chromatic number of G. The notion of list colorings was introduced by Erdős, Rubin and Taylor [4] and Vizing [16] and it forms a very rich area of graph colorings nowadays. It is well-known that the list chromatic number of a complete bipartite graph $K_{n,n}$ is $\Theta(\log n)$ and thus the difference between the chromatic number and the list chromatic number of a graph can be arbitrarily large.

We show that the bounds established in [12] for the chromatic number of a K_4-minor free graph also hold for list coloring (see Theorem 16). Since the bounds are the best possible for the ordinary coloring, they are also the best possible for list colorings. We build on a technique used in [10] to establish tight upper bounds of $L(p, q)$-labelings of K_4-minor free graphs which exploits a close relation of K_4-minor free graphs and series-parallel graphs. More details are given in the next section.

After finishing our work on this manuscript, we have learnt that the same bounds on the list chromatic number of K_4-minor free graphs were earlier established by Hetherington and Woodall [6]. In addition to the results on list coloring, the paper [6] also contains optimal bounds on the coloring number (degeneracy of K_4-minor free graphs). In particular, it is shown that a K_4-minor free graph with maximum degree Δ is $(\Delta + 2)$-degenerate if $\Delta \in \{2, 3\}$ and it is $\lceil 3\Delta/2 \rceil$-degenerate, otherwise. Both the bounds are the best possible.

2 Series-Parallel Graphs

In this section, we introduce notation related to K_4-minor free graphs and series-parallel graphs in particular. Before introducing the notion of series-parallel graphs, let us remark that the definition of series-parallel graphs slightly varies throughout the literature. *Series-parallel graphs* can be obtained by the following recursive construction based on graphs with two distinguished vertices called *poles*. The simplest series-parallel graph is an edge uv with the two poles being its end-vertices. If G_1 and G_2 are series-parallel graphs with poles u_1 and v_1, and u_2 and v_2, respectively, then the graph G obtained by identifying the vertices v_1 and u_2 is also a series-parallel graph and its two poles are the vertices u_1 and v_2. The graph G obtained in this way is called the *serial join* of G_1 and G_2. The *parallel join* of G_1 and G_2 is the graph obtained by identifying the pairs of vertices u_1 and u_2 and v_1 and v_2 with the poles being the identified vertices. The series-parallel graphs are precisely those that can be obtained from edges by a series of serial and parallel joins.

It is well-known that every 2-edge-connected K_4-minor free graph is a series-parallel graph (but the converse is not true).

Lemma 2. *Every block of a K_4-minor free graph is a series-parallel graph.*

The construction of a particular series-parallel graph G can be encoded by a rooted tree referred to as an *SP-decomposition tree* of G. Each node of the tree corresponds to a subgraph of G obtained at a step of the recursive construction of G. The leaves correspond to simple paths with their end-vertices being poles (such graphs are obtained by successive serial joins of edges) and each inner node of the tree corresponds to either a serial or a parallel join, i.e., there are two types of inner nodes: *S-nodes* and *P-nodes*. The inner nodes have at least two children and the subgraphs corresponding to their children were joined together by a sequence of serial or parallel joins depending on the type of the node. Since the result of a sequence of serial joins depends on the order in which the serial joins are applied, the children of each inner node have a fixed order. Without loss of generality, we can assume that the children of a P-node are S-nodes and leaves only, and the children of an S-node are P-nodes and leaves only (otherwise, the nodes can be merged together). We can also assume that no two consecutive children of an S-node are leaves.

Figure 1: A $S(P, \ell)$-subgraph and the corresponding subtree. The subgraph is obtained by a serial join (using the vertex v) of an edge vw and a P-subgraph A with poles u and v. The P-subgraph A itself is a parallel join of several paths with endvertices u and v.

An SP-decomposition tree corresponding to a series-parallel graph G is not unique and there is quite a lof of freedom in its choice [10]:

Lemma 3. *Let G be a series-parallel graph and v a vertex of G. There exists an SP-decomposition tree such that v is one of the poles of the graph corresponding to the root of the SP-decomposition tree.*

We now adopt the notation for describing series-parallel subgraphs based on their SP-decomposition, which was used in [10]. A subgraph of G corresponding to a leaf of the tree, i.e., a path consisting of 2-vertices, is called an *ℓ-subgraph* of G (ℓ stands for leaf). A subgraph obtained by a parallel join of A_1-subgraph, A_2-subgraph, ..., A_k-subgraph, is a $P(A_1, \ldots, A_k)$-subgraph and a subgraph obtained by a serial join of such subgraphs is an $S(A_1, \ldots, A_k)$-subgraph. For instance, a $P(\ell, \ell, \ell)$-subgraph is a subgraph of G that corresponds to a P-node with three leaves. Since the result of a serial join depends on the order in which the subgraphs are joined, we require the sequence $A_1, \ldots, A_k$ to respect this order. Subgraphs obtained by a parallel join of several A-subgraphs are called $P(A*)$-subgraphs and those obtained by a serial join $S(A*)$-subgraphs. $P(\ell*)$-subgraphs are called P-subgraphs for short. An example of this notation can be found in Figure 1.

Finally, we introduce a special name for particular P-subgraphs of a series-parallel graph G. A P-subgraph of G obtained by a parallel join of several two-edge paths and possibly an edge is called a *crystal*. Its vertices distinct from its poles are said to be its *inner* vertices. A crystal whose poles are not adjacent is a *diamond*. The *size* of a crystal is the number of edges incident with each of its poles, i.e., for diamonds, it is the number of the inner vertices, and it is the number of the inner vertices increased by one otherwise. Notice that size of any crystal is at least 2.

3 The Result

In this section, we state and prove the list coloring counterpart of Theorem 1. The main part of the proof consists in examining properties of a possible counterexample. For an integer $D \geq 1$, a graph is said to be *D-bad* if it is K_4-minor free, its maximum degree is at most D, and its list chromatic number is greater than $\lfloor 3D/2 \rfloor + 1$. Further, a graph is said to be *D-minimal* if it is D-bad and there is no D-bad graph of smaller order.

Clearly, our main result (see Theorem 16) for $D \geq 4$ is equivalent to stating that there are no D-bad graphs and no D-minimal graphs in particular for $D \geq 4$. In what follows, we exhibit a (long) series of lemmas that yield the proof of the statement.

3.1 Even Maximum Degree

Let us start with observations on the structure of D-minimal graphs which lead straightforwardly to a proof of the desired bound for graphs with maximum degree that is even (but they are also useful for graphs with odd maximum degree). Clearly, every D-minimal graph is connected. In the next lemma, we show that D-minimal graphs do not contain vertices of degree one and neighboring vertices of degree two.

Lemma 4. *No D-minimal graph G, for $D \geq 4$, contains a vertex of degree one or two adjacent vertices of degree two.*

Since proofs of most of the lemmas use the same technique, we explain the notation and principle in more detail here. This will allow us to make the other proofs more compact and to concentrate on their main aspects.

Proof. Fix a list assignment L giving each vertex of G a list of at least $\lfloor 3D/2 \rfloor + 1$ colors.

We first consider the case that G contains a vertex v of degree one. Remove the vertex v, and find a proper list-coloring c of the square of the resulting graph using the lists given by L which exists by the D-minimality of G. We now aim to extend c to v. To show that this is always possible, we count the number of colors in $L(v)$ that cannot be used to color the vertex v since the color is already used to color a vertex at distance at most two from v. We say that the colors that cannot be used to color v because of the above reason are *forbidden* for v; the remaining colors in $L(v)$ are said to be *available* for v. In particular, if we show that the number of available colors for v is at least one (i.e., the number of forbidden colors is at most $\lfloor 3D/2 \rfloor$), we can conclude that the coloring c can be extended to v. Let us proceed with counting. The only neighbor of v forbids at most one color, and its neighbors excluding v forbid at most additional $D-1$ colors. In total, there are at most $D \leq \lfloor 3D/2 \rfloor$ forbidden colors. Hence, c can be extended to v.

Next, we show that there are no two vertices u and v of degree two joined by an edge, i.e., there is no path $xuvy$ where u and v have degree two. As before, we find a list-coloring c of the square of $G \setminus \{u, v\}$. Now, we find a color for u: the neighbors of x forbid at most $D-1$ colors, and two more colors can be forbidden by x a y. Hence, there are at most $D+1$ forbidden colors for the vertex u and therefore, u can be properly colored. The case of the vertex v is similar: the number of forbidden colors will be at most $D+2$, since the vertex u has already been colored. We conclude that the coloring c can be extended to a proper list-coloring of the square of G, which contradicts the D-minimality of G. □

For the rest of the section, we consider a block-decomposition of a D-minimal graph G, and focus on one of its endblocks (we consider the entire graph G if it is itself 2-connected). The chosen block will be referred to as to the *final block* of G, and will be denoted by $G^\star$.

By Lemma 2, $G^\star$ is a series-parallel graph. If $G \neq G^\star$, $G^\star$ is connected to the rest of G through a vertex $v^\star$. If $G = G^\star$ (i.e., G itself is already 2-connected), $v^\star$ will be an arbitrarily chosen vertex of $G^\star$. By Lemma 3, there is an SP-decomposition $T^\star$ of $G^\star$ such that $v^\star$ is one of the poles corresponding to the root of of the decomposition $T^\star$. Note that the root of $T^\star$ always corresponds to a P-node since $G^\star$ is 2-connected.

We adopt the notation of A-subgraphs introduced in Section 2, and we say that an A-subgraph is *contained* in $G^\star$, if there is a subtree T_A of the form described by A with root r in $T^\star$ such that there is no descendant w of r in $T^\star$ whose depth is greater than the maximum depth of a descendant of r in T_A (in other words, we allow the subtree of the node r to be more complex than just a subtree corresponding to an A-subgraph, but we do not want it to be higher).

An immediate consequence of Lemma 4 is that all P-subgraphs contained in the final block are crystals. The following lemma shows that sizes and types of crystals in D-minimal graphs are quite restricted.

Lemma 5. *In a D-minimal graph, $D \geq 2$, the size of each crystal is at most $\lceil D/2 \rceil$, with the equality holding only for diamonds.*

Proof. Fix a D-minimal graph G, $D \geq 2$, that contains a crystal C with poles u and v whose size is S, and a list-assignment L of G giving each vertex a list of at least $\lfloor 3D/2 \rfloor + 1$ colors. Further, let w be an arbitrary inner vertex of C. Let G' be a graph obtained from G by deleting w. Since

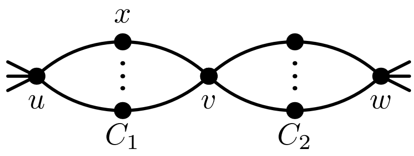

Figure 2: The proof of Lemma 6.

G is D-minimal, there exists a proper list-coloring c of the square of G' using the list-assignment L. We now show that if the statement of the lemma is violated, then c can be extended to the entire graph G.

First assume that C is a diamond and $S \geq \lceil D/2 \rceil + 1$. As in Lemma 4, we count the number of forbidden colors for w. The neighbors of u and v outside C forbid at most $2(D-S)$ colors, the inner vertices of $C \setminus w$ forbid at most another $S-1$ colors, and 2 more colors may be forbidden by the vertices u and v themselves. Summing up, there are at most $2D - S + 1 \leq \lfloor 3D/2 \rfloor$ forbidden colors. Hence, the list $L(w)$ contains at least one available color.

Next, assume that C is not a diamond and $S \geq \lceil D/2 \rceil$. This time, the number of forbidden colors for w is at most $2D - S$: the neighbors of u and v outside C forbid at most $2(D-S)$ and the inner vertices of $C \setminus w$ forbid at most another $S-2$ colors (recall that there are only $S-1$ inner vertices if C is not a diamond). In particular, there are at most $\lfloor 3D/2 \rfloor$ forbidden colors, and therefore there exists a color that can be used to color w. □

The next lemma complements Lemma 5 by showing that the size of certain crystals can be bounded from below as well.

Lemma 6. *Let C_1 and C_2 be two crystals in a D-minimal graph G sharing a common pole v and v has no other neighbors in G except for the vertices in C_1 and C_2. If $D \geq 4$, then the sizes of C_1 and C_2 belong to the set $\{\lfloor D/2 \rfloor, \lceil D/2 \rceil\}$. Moreover, one of the following properties must hold:*

- *both C_1 and C_2 are diamonds, or*
- *D is odd, one crystal is a diamond of size $\lceil D/2 \rceil$, and the other crystal is of size $\lfloor D/2 \rfloor$ and is not a diamond.*

Proof. First, we fix a D-minimal graph G and crystals C_1 and C_2 as specified above. By Lemma 5, the size of each of the crystals is bounded by $\lceil D/2 \rceil$ from above. Hence, it suffices to show that the size is also bounded by $\lfloor D/2 \rfloor$ from below and to discuss the possibilities whether a certain crystal is a diamond or not. In particular, we show that C_2 has the required property, depending on properties of C_1; the rest will follow by symmetry.

Before we start distinguishing the cases, let S_1 and S_2 be the sizes of crystals C_1 and C_2, respectively, u and v the poles of C_1, v and w the poles of C_2, and x an arbitrary inner vertex of C_1 (see Figure 2). We now fix an arbitrary list-assignment L giving each vertex at least $\lfloor 3D/2\rfloor + 1$ colors and find a proper list-coloring c of the square $G \setminus x$ using those lists. Next, the coloring c is extended to the square of the entire graph G as follows.

Case 1. *C_1 is not a diamond and S_2 is at most $\lfloor D/2\rfloor$.* Let us calculate the number of forbidden colors of x. At most S_2 colors are forbidden by the inner vertices of C_2 and possibly by w, $S_1 - 2$ by the inner vertices of C_1 except for x, $D - S_1$ by neighbors of u outside of C_1, and finally 2 more colors may be forbidden by the vertices u and v. Altogether, there are at most $D + S_2 \leq \lfloor 3D/2\rfloor$ forbidden colors. Hence, there is at least one color available for x.

Case 2. *C_1 is a diamond and S_2 is at most $\lfloor D/2\rfloor$ - 1.* The number of forbidden colors for x is again bounded by $\lfloor 3D/2\rfloor$: at most S_2 colors are forbidden by the inner vertices of C_2 and possibly by w, $S_1 - 1$ by the inner vertices of C_1 except for x, $D - S_1$ by neighbors of u outside of C_1, and finally 2 more colors may be forbidden by the vertices u and v. Therefore, there is always at least one color available for x, i.e., the coloring can be extended to the entire graph G.

Finally, let us show the two possibilities mentioned at the end of the statement of the lemma are the only possible cases. First assume that both C_1 and C_2 are not diamonds. Case 1 immediately yields that the sizes of both C_1 and C_2 are at least $\lfloor D/2\rfloor + 1$, hence the degree of v is at least $D + 1$ which is not possible. Next assume that D is even, C_1 is a diamond, and C_2 is not. By Case 1, $S_1 \geq \lfloor D/2\rfloor + 1$, and by Case 2, $S_2 \geq \lfloor D/2\rfloor$. In particular, this yields that the degree of the vertex v exceeds D, which is, again, not possible. □

While the preceding two lemmas focused on the interior of the crystals, the next lemma describes something about their neighborhood.

Lemma 7. *If $D \geq 4$, then each pole of a crystal C in a D-minimal graph has at least 2 neighbors outside of C.*

Proof. Fix a D-minimal graph G, $D \geq 4$, and a crystal C with a pole v that has at most one neighbor outside of C. Further fix a list-assignment L giving each vertex a list of at least $\lfloor 3D/2\rfloor + 1$ colors. Next, remove an arbitrary inner vertex w from the crystal C and find a list-coloring c of the square of the new graph with respect to the list-assignment L. To show that c can be extended to w, we again calculate the number of forbidden colors for w: at most $D - 1$ colors are forbidden by the neighbors of the pole of C different

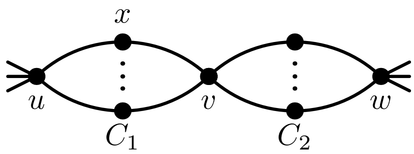

Figure 3: The proof of Lemma 8.

from v, at most two colors are forbidden by the two poles, and at most one more color are forbidden by the neighbor (if any) of the vertex v outside of C. Again, this gives at most $D + 2 \leq \lfloor 3D/2 \rfloor$, hence a suitable color for w exists. □

We now turn our attention back to the final block $G^\star$ and its SP-decomposition $T^\star$. A corollary of Lemma 7 is that $G^\star$ cannot be just a crystal (i.e., just a P-subgraph) since at least one of the poles of the crystal would have no neighbors outside of the crystal.

Let us further examine the properties of the deepest P-nodes in $T^\star$. Since they do not form the entire $T^\star$, they must have an S-node parent. By Lemma 7, this S-node can only have P-subgraphs as its children. In the next lemma, we further examine properties of such P-subgraphs.

Lemma 8. *If $D \geq 4$, then no final block of a D-minimal graph contains an $S(P, P)$-subgraph whose both P-subgraphs are diamonds.*

Proof. Fix a D-minimal graph G, $D \geq 4$, its final block $G^\star$ containing an $S(P, P)$-subgraph whose both P-subgraphs are diamonds, and a list-assignment L giving each vertex at least $\lfloor 3D/2 \rfloor + 1$ colors. Let the two diamonds be C_1 and C_2, their sizes S_1 and S_2, and their poles u and v, and v and w, respectively (see Figure 3). Without loss of generality, we may assume that $S_2 \leq \lfloor D/2 \rfloor$.

Choose x arbitrarily among the inner vertices of C_1 and find a proper list-coloring c of the square of the graph $G \setminus x$ with respect to L. Next, uncolor the vertex v, obtaining a partial coloring c' of G. We claim that c' can be extended to the entire graph G. The number of forbidden colors for x is bounded by $D + S_2 - 1 \leq \lfloor 3D/2 \rfloor$: there are $S_1 - 1$ colors forbidden by other inner vertices of C_1, at most $D - S_1$ forbidden by the neighbors of u outside of S_1, at most S_2 forbidden by the inner vertices of C_2, and at most one more color forbidden by the vertex u.

Next, we find a suitable color for the vertex v. The number of forbidden colors is bounded by $D + 2 \leq \lfloor 3D/2 \rfloor$: at most D colors are forbidden by the inner vertices of the two crystals, and additional two colors may be forbidden by the vertices u and w. □

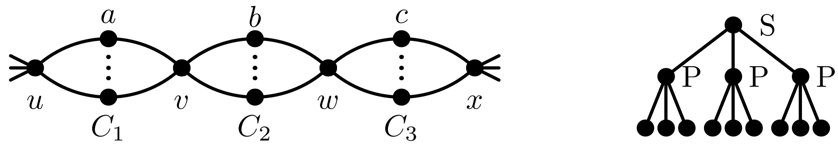

Figure 4: An $S(P, P, P)$-subgraph and the subtree corresponding to it.

Lemmas 6 and 8 imply that there there is no D-minimal graph for even D. The case when D is odd, requires additional work, and is dealt with in the following subsection.

3.2 Odd Maximum Degree

In this subsection, we state and prove the lemmas necessary for proving that there are no D-minimal graphs for odd $D \geq 5$.

The first lemma excludes a possibility that the deepest S-node in $T^\star$ corresponds to an $S(P, P, P)$-subgraph.

Lemma 9. *If $D \geq 5$, then no final block of a D-minimal graph contains an $S(P, P, P)$-subgraph.*

Proof. Let G be a D-minimal graph for $D \geq 5$ whose final block contains an $S(P, P, P)$-subgraph. In particular, there exist three crystals C_1, C_2, and C_3 with poles u, v, w, and x, as depicted in Figure 4. Further, choose vertices a, b, and c among the inner vertices of C_1, C_2, and C_3, respectively. Next, fix an arbitrary list-assignment L assigning each vertex a list of at least $\lfloor 3D/2 \rfloor + 1$ colors. We show that G can be properly list-colored with respect to L.

First, by Lemma 8, we know that crystals and non-crystals must alternate among C_is, i.e., both C_1 and C_3 have the same properties (size and being diamond) which differ from those of C_2. We now delete all the three crystals except for the vertices u and x from G and find a list-coloring c' of the square of the new graph using the lists given by L. We aim to extend c' to the entire graph G.

Two cases have to be distinguished, based on which crystals are diamonds.

Case 1. *C_1 and C_3 are diamonds, C_2 is not.* By Lemma 6, the sizes of C_1 and C_2 are $\lceil D/2 \rceil$, and the size of C_2 is $\lfloor D/2 \rfloor$. Consequently, each of the vertices u and x has at most $\lfloor D/2 \rfloor$ neighbors outside the three crystals.

First, we aim to find colors for a and c in such a way that the number of colors available for b will remain at least $\lfloor 3D/2 \rfloor$. There are at least $\lfloor 3D/2 \rfloor + 1 - 1 - \lfloor D/2 \rfloor = D$ colors available for a (and, by symmetry, for c), as the are at most $\lfloor D/2 \rfloor$ colors forbidden by the neighbors of the vertex u and one more color may be forbidden by the vertex u itself. As there are no colors forbidden for the vertex b so far, there are $\lfloor 3D/2 \rfloor + 1$ colors available for it. If there is a color α available for both a and c, we color both the vertices by such a color α. Otherwise, as $2D > \lfloor 3D/2 \rfloor + 1$, there is a color β available for one of those vertices, say a, which is not available for b. In that case, color a by β and color c using any color available for it. Let this partial coloring be $c^\star$.

Next, we color the vertices v and w—the number of colors forbidden for each of them, say w, is at most 4: those forbidden by the vertices v, a, c, and x. We continue by coloring of the remaining inner vertices of C_1. The number of forbidden colors for those vertices is at most $D + 2 \leq \lfloor 3D/2 \rfloor$: at most $D - 1$ colors are forbidden by the neighbors of u and additional three colors may be forbidden by the vertices u, v, and w. Next, we color the inner vertices of C_2 except for b. The number of forbidden colors is bounded by $D \leq \lfloor 3D/2 \rfloor$: there are at most $\lfloor D/2 \rfloor - 3$ colors forbidden by the inner vertices of C_2, at most $\lceil D/2 \rceil$ by the inner vertices of C_1, and three more colors may be forbidden by the vertices c, v, and w. Then, we color the remaining inner vertices of C_3. The number of forbidden colors is bounded by $\lfloor 3D/2 \rfloor$: at most $D - 1$ colors are forbidden by the neighbors of x, at most $\lfloor D/2 \rfloor - 2$ by the inner vertices of C_2 and at most three other colors are forbidden by the vertices v, w, and x.

As the last step, we have to find the color for b. To show that we can color the vertex b, we calculate the number of vertices we colored since the coloring $c^\star$ was made; each of those vertices can forbid at most one color that was available for vertex b at the time $c^\star$ was made. In particular, we show that number of such vertices is less than the number of available colors, hence it follows that there must be at least one color that can be used for coloring the vertex b. Let us calculate the vertices that were colored since $c^\star$ was made: there are $\lceil D/2 \rceil - 1$ such inner vertices in both C_1 and C_3, $\lfloor D/2 \rfloor - 2$ inner vertices in D_2, and the remaining two colored vertices are v and w; we conclude that exactly $\lfloor 3D/2 \rfloor - 1$ vertices have been colored. Since the number of colors available for b in $c^\star$ was at least $\lfloor 3D/2 \rfloor$, we infer that b can be properly colored.

Case 2. *C_2 is a diamond, C_1 and C_3 are not.* By Lemma 6, the sizes of C_1 and C_2 are $\lfloor D/2 \rfloor$, the size of C_2 is $\lceil D/2 \rceil$, and each of the vertices u and x has at most $\lceil D/2 \rceil$ neighbors outside the three crystals.

As in the previous case, we calculate the numbers of colors available to the vertices a, b, and c. In particular, there are at least $D-1$ colors available for a (the same holds for c), as the only colors forbidden for a could be those assigned to at most $\lceil D/2 \rceil$ neighbors of the vertex u and the color of the vertex u itself. Similarly, the vertex b has at least $\lfloor 3D/2 \rfloor - 1$ available colors since at most two colors can be forbidden—the colors of the vertices u and x. Hence, we can use the same approach to color the vertices a and c in such a way that the vertex b has at least $\lfloor 3D/2 \rfloor - 2$ available colors after the coloring. This partial coloring will be denoted by $c^\star$.

Continue by choosing the colors for the vertices v and w. The number of forbidden colors for each of them, say w, will be at most $\lceil D/2 \rceil + 3$: at most $\lceil D/2 \rceil$ colors are forbidden by the neighbors of the vertex x outside C_3, and at most three other colors may be forbidden by the vertices c, v, and x.

The remaining inner vertices of C_1 are colored next; the number of forbidden colors is bounded by D: at most $\lfloor D/2 \rfloor - 2$ colors are forbidden by the inner vertices of C_1, at most $\lceil D/2 \rceil$ colors are forbidden by the neighbors of the vertex u outside C_1, and at most two more colors are forbidden by the vertices u and v. Next, color the remaining inner vertices of C_3, the situation is symmetrical. Then, color the inner vertices of C_2 except for b. The number of forbidden colors is at most $\lfloor 3D/2 \rfloor$: at most $\lceil D/2 \rceil - 2$ colors are forbidden by the inner vertices of C_2, at most $2(\lfloor D/2 \rfloor - 1)$ colors are forbidden by the inner vertices of C_1 and C_4, and at most four colors are forbidden by the vertices u, v, w, and x. Now, the only vertex missing a color is b. Let us calculate the number of vertices that are now colored now did not have a color in $c^\star$. There are $\lfloor 3D/2 \rfloor - 3$ such vertices: $\lfloor D/2 \rfloor - 2$ vertices in both C_1 and C_3, $\lceil D/2 \rceil - 1$ vertices in C_2, and the vertices v and w. Since the vertex b had at least $\lfloor 3D/2 \rfloor - 2$ available colors in $c^\star$, there is still at least one available color which can be used to color it.

In both cases, we proved that c' can be extended to the entire graph G, hence the statement of lemma follows. □

Lemmas 7 and 9 imply that the deepest S-nodes in $T^\star$ must correspond to roots of $S(P, P)$-subgraphs having exactly two P-subgraphs as their children. Since the root of $T^\star$ is a P-node, those S-nodes must have a P-node parent. This node corresponds either to a $P(S(P, P), S(P, P))$-subgraph (Figure 5) or to a $P(S(P, P), \ell*)$-subgraph (Figure 6). The next lemma excludes the former case.

Lemma 10. *If $D \geq 5$, then no final block of a D-minimal graph contains an $P(S(P, P)S(P, P))$-subgraph.*

Proof. Consider four vertices x, y, u, and v and four crystals C_{UL}, C_{UR}, C_{LL}, and C_{LR} as depicted in Figure 5. Finally, the entire structure may be connected to the rest of the graph through the vertices u and v.

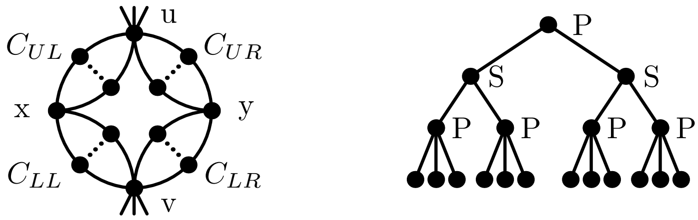

Figure 5: A $P(S(P,P),S(P,P))$-subgraph and the subtree corresponding to it.

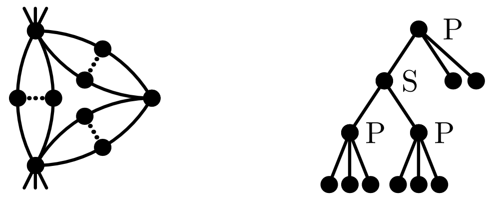

Figure 6: A $P(S(P,P),\ell*)$-subgraph and the subtree corresponding to it.

Applying Lemma 8 to C_{UL} and C_{LL}, we get that one of the crystals is of size $\lfloor D/2 \rfloor$ and the size of other one is $\lceil D/2 \rceil$; the same holds for C_{UR} and C_{LR}. Since both C_{UL} and C_{UR} cannot be of size $\lceil D/2 \rceil$ at the same time, we infer that one of C_{LL} and C_{LR} must be of size $\lceil D/2 \rceil$; the same again for C_{UL} and C_{UR}. Hence, the four crystals comprise the entire graph G as degrees of the vertices u and v cannot exceed D. However, we may find another decomposition of G as a parallel join of an $S(P,P,P)$-subgraph (corresponding to the crystals C_{UL}, C_{UR}, and C_{LR}) with several paths (corresponding to the crystal C_{LL}). Since the existence of an $S(P,P,P)$-subgraph was already excluded in Lemma 9, the statement of the lemma follow. □

We now consider $P(S(P,P),\ell*)$-subgraphs.

Lemma 11. *If $D \geq 5$, then no final block of a D-minimal graph is a $P(S(P,P),\ell*)$-subgraph.*

Proof. Let G be a D-minimal graph for $D \geq 5$ whose final block $G^\star$ consists of three vertices u, v, and w joined by three crystals C_1, C_2, and C_3 as in Figure 7. $G^\star$ may be connected to the rest of G through the vertex u. As in Lemma 10, one can quickly infer that C_2 is a diamond of size $\lceil D/2 \rceil$, while C_1 and C_3 are not, and their sizes are $\lfloor D/2 \rfloor$. In particular, u may have at most one neighbor outside of $G^\star$.

In the rest of the proof, we show that G^2 can be properly colored using lists containing at least $\lfloor 3D/2 \rfloor + 1$ colors. First, fix a list-assignment L

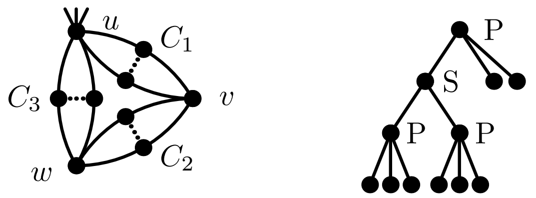

Figure 7: A $P(S(P,P),\ell*)$-subgraph with one of the poles being the cut-vertex in G.

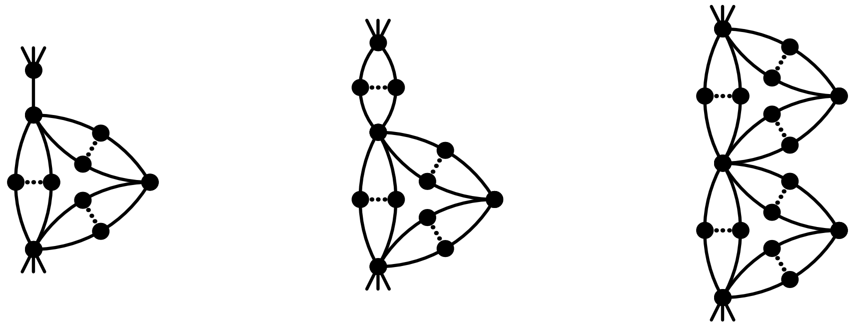

Figure 8: An $S(P(S(P,P),\ell*),\ell*)$-subgraph, an $S(P(S(P,P),\ell*),P)$-subgraph, and an $S(P(S(P,P),\ell*),P(S(P,P),\ell*))$-subgraph.

assigning each vertex a list of $\lfloor 3D/2 \rfloor + 1$ colors. Next, remove all the vertices of $G^\star$ except for u and find a proper list-coloring c of the square of the new graph with respect to L. We extend the coloring c to the rest of the graph G. First, color the vertices v and w, which have at most 3 forbidden colors. Next, we color the inner vertices of C_1 and C_3. These vertices have at most $D - 1 + 2 \leq \lfloor 3D/2 \rfloor$ forbidden colors: at most $D - 1$ because of the colors of the neighbors of the vertex u and at most 2 because of the two poles of the crystal they belong to. Finally, we finish with coloring of the inner vertices of C_2. The number of forbidden color in this case is at most $D - 1 + \lfloor D/2 \rfloor - 1 + 2 = \lfloor 3D/2 \rfloor$: at most $D - 1$ colors are forbidden by the neighbors of the vertex v, at most $\lfloor D/2 \rfloor - 1$ colors are forbidden by the inner vertices of the crystal C_3, and additional 2 colors can be forbidden by the vertices v and w. Hence, c can be extended to the entire G, contradicting the minimality of G. $\square$

Since the $P(S(P,P),\ell*)$-subgraph is not the entire final block $G^\star$, the P-node corresponding to it must have an S-node parent in $T^\star$. In particular, any $P(S(P,P),\ell*)$-subgraph of the maximum depth must be contained either in an $S(P(S(P,P),\ell*),\ell*)$-subgraph, an $S(P(S(P,P),\ell*),P)$-subgraph, or an $S(P(S(P,P),\ell*),P(S(P,P),\ell*))$-subgraph. The structures are depicted

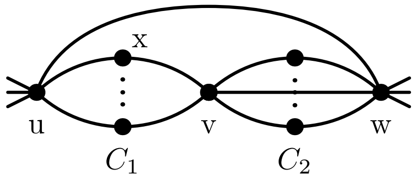

Figure 9: The proof of Lemma 12.

in Figure 8.

It is not hard to see that the final block of a D-minimal graph cannot contain an $S(P(S(P,P),\ell*),P(S(P,P),\ell*))$-subgraph, as the degree of the pole connecting the two $P(S(P,P),\ell*)$-subgraphs would have degree at least $D+1$. In the following two lemmas, we consider the remaining two possibilities.

Lemma 12. *If $D \geq 5$, then no final block of a D-minimal graph contains an $S(P,P)$-subgraph whose poles are joined by an edge.*

Proof. Fix a D-minimal graph G, $D \geq 5$, its final block $G^\star$ containing an $S(P,P)$-subgraph whose poles are joined by an edge, and a list-assignment L giving each vertex at least $\lfloor 3D/2 \rfloor + 1$ colors. Let the two P-subgraphs (crystals) of the $S(P,P)$-subgraph mentioned above be C_1 and C_2, and their poles u and v, and w, as depicted in Figure 9. By Lemma 6, we can assume that C_1 is a diamond of size $\lceil D/2 \rceil$ and C_2 is a crystal of size $\lfloor D/2 \rfloor$ that is not a diamond.

We pick an arbitrary inner vertex x in diamond C_1. By minimality, there exists a proper list-coloring c of the square of the graph $G \setminus x$ with respect to L; we fix one such coloring and extend it to x. The number of forbidden colors for x can be bounded as follows: there are at most $D-1$ colors forbidden by neighbors of u, additional $\lfloor D/2 \rfloor - 1$ colors may be forbidden by the inner vertices of C_2, and two more colors may be forbidden by the vertices u and v. This gives at most $\lfloor 3D/2 \rfloor$ colors forbidden altogether, i.e., c can be extended to vertex x. □

Having established Lemma 12, we are ready to prove the final lemma of this subsection.

Lemma 13. *If $D \geq 5$, then the final block of a D-minimal graph contains neither an $S(P(S(P,P),\ell*),\ell*)$-subgraph nor an $S(P(S(P,P),\ell*),P)$-subgraph.*

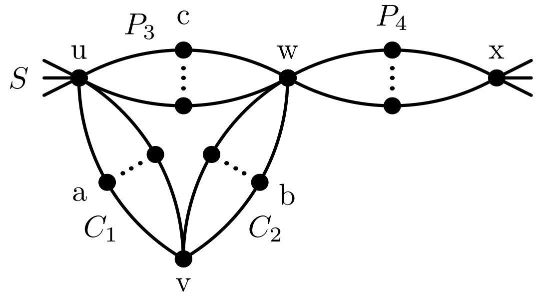

Figure 10: The proof of Lemma 13.

Proof. Fix a D-minimal graph G, $D \geq 5$, its final block $G^\star$ containing one of the subgraphs from the statement of the lemma, and a list-assignment L giving each vertex at least $\lfloor 3D/2 \rfloor + 1$ colors. In particular, $G^\star$ contains four vertices u, v, w, and x; a crystal C_1 with poles u and v, a crystal C_2 with poles v and w, S_3 vertex-disjoint paths of length at most 2 connecting the vertices u and w, and S_4 vertex-disjoint paths of length at most 2 connecting the vertices w and x (see Figure 10). The described subgraph is connected to the rest of G through the vertices u and x. By Lemma 12, none of the S_3 paths connecting u and w is an edge. On the other hand, the S_4 paths between w and x may or may not be crystals. In order to simplify the notation used in the proof, P_3 and P_4 denote the two sets of the paths between u and w and the paths between w and x, respectively. Finally, let S be the number of neighbors of u except for the vertices in C_1 and the paths in P_3.

We now choose an arbitrary inner vertex a of the crystal C_1, an arbitrary inner vertex b of the crystal C_2, and an arbitrary internal vertex c of one of the paths in P_3. We find a list-coloring c' of the square of $G \setminus b$ with respect to L. In each of the following three cases, we show how to use c' to obtain a proper list-coloring of the square of the entire graph G.

Case 1. *C_2 is a diamond, C_1 is not.* In this case, $S + S_3 \leq \lceil D/2 \rceil$ and $S_4 \leq \lfloor D/2 \rfloor - 1$. First, we uncolor the inner vertices of both C_1 and C_2, together with the vertices v, w, and c. At this point, vertex a has at least $\lfloor 3D/2 \rfloor + 1 - S - (S_3 - 1) - 1 = D$ available colors, as the only colors forbidden for it are the colors used on the S neighbors of the vertex u outside the structure, on the $S_3 - 1$ colored internal vertices of the paths in P_3, and on the vertex u. Similarly, the vertex b has at least $\lfloor 3D/2 \rfloor + 1 - (\lfloor D/2 \rfloor - 1) - 1 = D + 1$ available colors as the only forbidden colors are used on the $\lfloor D/2 \rfloor - 1$ neighbors of the vertex w in P_3 and P_4 and one more color can be forbidden by the vertex u. Finally, the vertex w has at most $\lfloor 3D/2 \rfloor - 1$ forbidden colors (and thus at least 2 available colors): at most D colors are forbidden by the vertex x and its neighbors (notice that if there is no edge wx then at most $\lfloor D/2 \rfloor - 1$ neighbors of x are at distance at most 2 from the vertex w), at most $S_3 - 1$ colors are forbidden by the internal vertices of the paths in P_3, and one more color can be forbidden by the vertex u. In particular, we can choose the colors for w and a in such a way that there will remain at least D colors available for b: either the previously stated bound on available colors for b is not sharp, or there is a color α available for both a and w, or, as $D+2 > D+1$, there is a color α available for a or w which is not available for b.

We now choose suitable colors for the vertices c and v. The number of colors forbidden for the vertex c is bounded by $\lceil D/2 \rceil - 1 + \lfloor D/2 \rfloor - 1 + 3 = D + 1$: there are at most $S + S_3 - 1$ colors forbidden by the neighbors of the vertex u, at most S_4 colors forbidden by the remaining neighbors of the vertex w, and three more colors may be forbidden by the vertices u, w and a. Similarly, the number of colors forbidden for the vertex v is bounded by $\lceil D/2 \rceil + 3 \leq \lfloor 3D/2 \rfloor$: there are at most $S + S_3$ colors forbidden by the neighbors of the vertex u and at most three other colors are forbidden by the vertices u, w and a.

The remaining inner vertices of C_1 are colored next—there are at most $D - 2 + 2$ forbidden colors: at most $S + S_3 + \lfloor D/2 \rfloor - 2$ colors are forbidden by the neighbors of the vertex u and two more colors may be forbidden by the vertices u and v. We continue with the inner vertices of C_2 except for b. This time, the number of forbidden colors is at most $D - 2 + \lfloor D/2 \rfloor - 1 + 3 = \lfloor 3D/2 \rfloor$: at most $\lceil D/2 \rceil - 2$ colors are forbidden by the inner vertices of C_2, at most additional $S_3 + S_4 \leq \lfloor D/2 \rfloor$ colors are forbidden by the remaining neighbors of the vertex w, at most $\lfloor D/2 \rfloor - 1$ colors are forbidden by the inner vertices of C_1, and three more colors may be forbidden by the vertices u, v, and w. Let us now calculate the number of vertices we have colored after coloring the vertices a and w: $\lfloor D/2 \rfloor - 2$ inner vertices of C_1, $\lceil D/2 \rceil - 1$ inner vertices of C_2, and the vertices c and v. In particular, at most $D - 1$ additional colors might have been forbidden for b. Since b had at least D available colors after coloring the vertices a and w, there is still an available color for b.

Case 2a. *C_1 is a diamond, C_2 is not, and there is at least one path of length two in P_4.* In this case, $S + S_3 \leq \lfloor D/2 \rfloor$. In particular, $S_3 \leq \lfloor D/2 \rfloor - 1$ as $S \geq 1$. Further, we choose an arbitrary vertex d among the inner vertices of the paths of length 2 in P_4.

We proceed similarly to the previous case. We uncolor v, w, c, d, and all the inner vertices of C_1 and C_2. Next, we would like to choose colors for the vertices a and d such that there will remain at least $D + 1$ colors available for b. We proceed as in Case 1: there are at least $D + 2$ colors available of the vertex b (the only forbidden colors are those used to color the $\lceil D/2 \rceil - 2$ neighbors of w in P_3 and P_4, excluding the vertices c and d), at least $D+1$ colors are available for a (there are at most $S + S_3 - 1$ colors forbidden by the neighbors of the vertex u and one more color can be forbidden by the vertex u itself), and at least 3 colors are available for d (there are at most $D - 1$ colors forbidden by the neighbors of the vertex x, at most $S_3 - 1$ colors are forbidden by the internal vertices of the paths in P_3, and one more color can be forbidden by the vertex x). We follow the discussion in Case 1 and conclude that either we can color a and d with the same color, or, as $(D+1)+3 > D+2$, there exists a color β that is not available for b, but is available for one of the vertices a and d.

Vertex w is colored next—there are at most $D-1+\lfloor D/2 \rfloor-2+3 = \lfloor 3D/2 \rfloor$ colors forbidden for it: at most $D - 1$ colors are forbidden by the neighbors of the vertex x (if vx is an edge) or the internal vertices of the paths in P_4, at most $\lfloor D/2 \rfloor - 2$ colors are forbidden the remaining neighbors of the vertex w, and three more colors may be forbidden by the vertices a, u, and x.

Then, we continue exactly as in Case 1: we start with c and v, continue with the rest of the inner vertices of C_1, and the inner vertices of C_2 except for b. In particular, we color $3+\lceil D/2 \rceil-1+\lfloor D/2 \rfloor-2 = D$ vertices (including the vertex w discussed above). The only vertex without a color is the vertex b, and because it had at least $D+1$ available colors after coloring the vertices a and d, there must be at least one color that is still available for b.

Case 2b. *C_1 is a diamond, C_2 is not, and there is no path of length two in P_4 (i.e., P_4 consists of just a single path of length one).* As in the previous case, we observe that $S_3 \leq \lfloor D/2 \rfloor - 1$.

In this case, the coloring c' can directly be extended to the vertex b. Let us count the forbidden colors for b: there at most $D - 1$ colors forbidden by the neighbors of v, at most two more colors are forbidden by the vertices v and x, and at most $\lfloor D/2 \rfloor$ colors are forbidden by the inner vertices of the paths of length two connecting u and w. In particular, there is at least one color available for b. □

3.3 The Final Step

Lemmas 4–13 exclude the existence of a D-minimal graph for $D \geq 4$ (see the discussion before Lemma 12). In order to complete the proof of Theorem 16, it is necessary to consider K_4-minor free graphs with maximum degree two and three. Such graphs are considered in the next two propositions.

Proposition 14. *The list chromatic number of the square of a graph of maximum degree 2 is at most 5.*

Proof. The statement follows easily from the fact that if G has maximum degree 2, then G^2 has maximum degree at most 4. $\square$

Proposition 15. *The list chromatic number of the square of a K_4-minor free graph of maximum degree 3 is at most 6.*

Proof. Fix a vertex-minimal K_4-minor free graph G with maximum degree three for that there exists a list-assignment L giving each vertex a list of 6 colors such that G^2 cannot be properly colored.

By considering the last level of an SP-decomposition tree of a final block of G, we obtain that G contains a vertex of degree one, two adjacent vertices of degree two or a crystal. If G contains a vertex v of degree one or a vertex v of degree two adjacent to another vertex of degree two, contract an edge incident with v and find a proper list-coloring c of the square of the resulting graph. The vertices of G preserve their colors and the vertex v can be assigned a color from its list since there are at most $3 + 2 = 5 \leq 6$ colors forbidden for the vertex v.

If G contains a crystal C of size $S \geq 2$, remove an arbitrarily chosen inner vertex w of the crystal C and find a list-coloring c of the square of $G \setminus w$. Let us calculate the number of colors forbidden for w: there are at most $S - 1$ colors forbidden by the inner vertices of C, at most $2 \cdot (3 - S)$ colors are forbidden by the neighbors of the poles of C that are outside the crystal, and two more colors may be forbidden by the poles. In particular, there are $7 - S \leq 5$ forbidden colors, therefore c can be extended to the entire graph G. $\square$

Lemmas 4–13 and Propositions 14 and 15 yield our main result:

Theorem 16. *The list chromatic umber of the square of a K_4-minor free graph G of maximum degree Δ is at most $\lfloor 3\Delta/2 \rfloor + 1$ if $\Delta \geq 4$ and at most $\Delta + 3$ if $\Delta \in \{2, 3\}$.*

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