A new lower bound on the number of perfect matchings in cubic graphs

Daniel Král'* Jean-Sébastien Sereni† Michael Stiebitz‡

Abstract

We prove that every n-vertex cubic bridgeless graph has at least $n/2$ perfect matchings and give a list of all 17 such graphs that have less than $n/2+2$ perfect matchings.

1 Introduction

Graphs considered in this paper can contain multiple edges but do not contain loops. Finally, a graph is *cubic* if every vertex has degree 3 and a subgraph is *spanning* if it contains all the vertices. A *perfect matching* is a spanning subgraph where every vertex has degree 1. A graph is *bridgeless* if it is connected and stays connected after removing any edge. A classical theorem of Petersen [10] asserts that every cubic bridgeless graph has a perfect matching.

Theorem 1 (Petersen, 1891). *Every cubic bridgeless graph G has a perfect matching.*

*Institute for Theoretical Computer Science, Faculty of Mathematics and Physics, Charles University, Malostranské náměstí 25, 118 00 Prague, Czech Republic. E-mail: kral@kam.mff.cuni.cz. Institute for Theoretical Computer Science (ITI) is supported by the Ministry of Education of the Czech Republic as project 1M0545.

†Department of Applied Mathematics (KAM) and Institute for Theoretical Computer Science (ITI), Faculty of Mathematics and Physics, Charles University, Malostranské náměstí 25, 118 00 Prague, Czech Republic. E-mail: sereni@kam.mff.cuni.cz. This author is supported by the European project IST FET AEOLUS.

‡Technische Universität Ilmenau, Institute of Mathematics, P.O.B. 100 565, D-98684 Ilmenau, Germany. E-mail: Michael.Stiebitz@tu-ilmenau.de.

In fact, for every edge e of G, there is a perfect matching containing e, and for every two edges f and f', there is a perfect matching avoiding both f and f'. Hence, every cubic bridgeless graph has at least three perfect matchings. A natural question is what is the least number of perfect matchings that an n-vertex cubic bridgeless graph contains. Lovász and Plummer conjectured in the mid-1970s that this number grows exponentially with the number of vertices (see the book by Lovász and Plummer [7, Conjecture 8.1.8]).

Conjecture 1 (Lovász and Plummer, 1970s). *Every cubic bridgeless graph with n vertices has at least $2^{\Omega(n)}$ perfect matchings.*

Edmonds, Lovász, and Pulleyblank [3] and, independently, Naddef [9], proved that the dimension of the perfect matching polytope of a cubic bridgeless n-vertex graph is at least $n/4+1$. Since the vertices of the polytope correspond to distinct perfect matchings, we have the following lower bound on the number of perfect matchings of an n-vertex cubic bridgeless graph.

Theorem 2 (Edmonds et al., Naddef, 1982). *Every cubic bridgeless graph with n vertices has at least $n/4+2$ perfect matchings.*

If a cubic graph G has no non-trivial edge-cut of size 3, then the result of Edmonds et al. gives a better bound, since G must be a brick (see Section 2 for the definition). A graph G is *cyclically k-edge-connected* if it has no edge-cut of size at most $k-1$ whose removal yields at least two non-acyclic components. The following result is a simple consequence of a theorem of Edmonds, Lovász, and Pulleyblank [3, Theorem 5.1].

Theorem 3 (Edmonds, Lovász, and Pulleyblank, 1982). *Every cubic cyclically 4-edge-connected graph with n vertices has at least $n/2+1$ perfect matchings.*

Conjecture 1 has been verified for several special classes of graphs, one of them being bipartite graphs. The first non-trivial lower bound on the number of perfect matchings in cubic bridgeless bipartite graphs was obtained in 1969 by Sinkhorn [14] who proved a bound of $\frac{n}{2}$, thereby establishing a conjecture of Marshall. The same year, Minc [8] increased this lower bound by 2. Then, a bound of $\frac{3n}{2}-3$ was proved by Hartfiel and Crosby [5]. The first exponential bound, $6\cdot\left(\frac{4}{3}\right)^{n/2-3}$, was obtained in 1979 by Voorhoeve [15]. This was generalized to all regular bipartite graphs in 1998 by Schrijver [11], who thereby proved a conjecture of himself and Valiant [13]. His argument is involved, and we note that, as a particular case of a different and more general approach (using hyperbolic polynomials), Gurvits [4] managed to slightly improve the bound, as well as simplify the proof.

Recently, an important step towards a proof of Conjecture 1 has been made by Chudnovsky and Seymour [2] who proved the conjecture for planar graphs.

Theorem 4 (Chudnovsky and Seymour, 2008). *Every cubic bridgeless planar graph with n vertices has at least $2^{n/655978752}$ perfect matchings.*

In this paper, we focus on proving a bound matching that stated in Theorem 3 for all cubic bridgeless graphs, i.e., we remove the assumption that G is cyclically 4-edge-connected. In particular, we prove that every n-vertex cubic bridgeless graph G has at least $n/2$ perfect matchings and provide complete lists of such graphs with exactly $n/2$ and $n/2+1$ perfect matchings. Our main result is the following theorem.

Theorem 5. *Let G be a cubic bridgeless graph with n vertices. The graph G contains at least $n/2+2$ perfect matchings unless it is is one of the 17 exceptional graphs $I_1, \ldots, I_{10}$ or $H_0, \ldots, H_6$ which are depicted in Figures 2, 3, 4 and 6. The graph H_0 contains $n/2$ perfect matchings and the other exceptional graphs contain $n/2+1$ perfect matchings.*

2 Brick and brace decomposition

The brick and brace decomposition is one of the essential notions in the theory of perfect matchings. We explain the notion in general though we apply it only to cubic bridgeless graphs. We refer to the monograph of Schrijver [12, Chapter 37] for further exposition. Given a graph G and a subset X of vertices, $G-X$ is the subgraph obtained from G by removing the vertices of X. A graph G is *matching covered* if every edge of G is contained in a perfect matching. If V_1 and V_2 is a partition of a vertex set of G, then the edges with one end-vertex in V_1 and the other in V_2 form an *edge-cut*. An edge-cut is *non-trivial* if both V_1 and V_2 contain at least two vertices. An edge-cut E is *tight* if every perfect matching contains exactly one edge of E.

Let G be a matching covered graph with a non-trivial tight edge-cut E, which partitions the vertices of G into two sets V_1 and V_2. We decompose

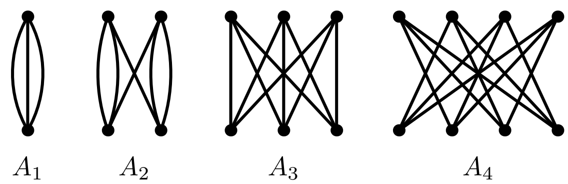


Figure 1: Cubic braces of order at most 4.

G into two simpler graphs G_1 and G_2 by *splitting along* E as follows: the graph G_i is obtained by contracting all the vertices of V_i to a single vertex, for $i \in \{1, 2\}$. Note that the structure of perfect matchings of G_1 and G_2 reflects the structure of perfect matchings of G: no matchings are lost by the splitting since every perfect matching uses exactly one edge of E. In particular, the graphs G_1 and G_2 are matching covered. If one or both of the new graphs contain a non-trivial tight edge-cut, we can again split along it. We continue until we obtain a multiset of graphs with no tight edge-cuts. The following theorem of Lovász [6] states that splitting along tight edge-cuts is independent of the order in that the edge-cuts were chosen.

Theorem 6 (Lovász, 1987). *Let G be a matching covered graph. The multiset of graphs with no tight edge-cuts obtained by splitting along tight edge-cuts of G depends neither on the chosen edge-cuts nor on the order in which the splittings are performed.*

The graphs in the multiset obtained by splitting along tight edge-cuts are of two kinds. Bipartite graphs with no tight edge-cut are referred to as *braces*. They are characterized by the following property [3].

Theorem 7 (Edmonds, Lovász, and Pulleybank, 1982). *A bipartite matching covered graph G has no tight edge-cut if and only if for every subsets V and W of its color classes such that $|V| = |W| \leq 2$, the graph $G-(V \cup W)$ has a perfect matching.*

If G is a cubic bridgeless graph that is a brace, shortly a *cubic brace*, we call the number of vertices in each color class of G the *order* of the brace. There is a unique cubic brace A_n of order n for each $n \in \{1, 2, 3, 4\}$. The braces $A_1, \ldots, A_4$ can be found in Figure 1.

Non-bipartite graphs that appear in the decomposition of a matching-covered graph along its tight edge-cuts are known as *bricks*. They are characterized as follows [3].

Theorem 8 (Edmonds, Lovász, and Pulleybank, 1982). *A non-bipartite matching covered graph G has no tight edge-cut if and only if it is* 3*-connected and for every two-element subset V of its vertices, the graph $G - V$ has a perfect matching.*

As in the case of braces, we refer to bricks that are cubic bridgeless graphs as to *cubic bricks*. Examples of cubic bricks can be found in Figure 2. Note that the graph A_1 is often considered to be a brick, but we prefer viewing it as a brace throughout our exposition. Since the decomposition of a graph G along its tight edge-cuts is formed by bricks and braces, it is called the *brick and brace decomposition* of G. Recall that this decomposition is unique by Theorem 6. The brick and brace decomposition is *non-trivial* if it contains at least two graphs, i.e., the brick and brace decomposition of G is non-trivial if and only if G is neither a brick nor a brace.

In the rest of this section, we deal with cubic bridgeless graphs only. Before our further considerations, let us state the following consequence of the structure of the perfect matching polytope of a cubic bridgeless graph G: *every tight edge-cut of G has size* 3. In particular, the graphs forming the brick and brace decomposition of a cubic bridgeless graph are also cubic and bridgeless. Furthermore, it follows from Theorem 8 that every cubic brick is a simple graph.

We now prove several rather simple facts on the brick and brace decompositions of cubic bridgeless graphs, on cubic bricks and cubic braces. Though the reader can be familiar with some of these facts, we give their short proofs for completeness. Before our first lemma, we need two more definitions. A vertex v of a cubic graph G is *tricovered* if there exists a spanning subgraph H of G such that the degree of v in H is 3 and the degrees of other vertices of G are 1. A cubic graph G is *well-covered* if every vertex of G is tricovered. If G is a simple graph, a vertex v with neighbors v_1, v_2 and v_3 is tricovered if and only if the graph $G - \{v, v_1, v_2, v_3\}$ has a perfect matching.

Lemma 9. *Every cubic brick G is well-covered.*

Proof. Let v be any vertex of G and v_1, v_2 and v_3 its neighbors. By Theorem 8, the graph $G - \{v_2, v_3\}$ has a perfect matching M. Since G is cubic, this perfect matching includes the edge vv_1. Since every cubic brick is simple, the perfect matching M together with the edges vv_2 and vv_3 is the sought spanning subgraph of G. □

Using Lemma 9, we show that every non-trivial brick and brace decomposition contains a brace.

Lemma 10. *Every non-trivial brick and brace decomposition of a cubic bridgeless graph contains a brace.*

Proof. It is enough to prove that there is no graph whose brick and brace decomposition consists of two bricks. Suppose on the contrary that G is such a graph. Let $E := \{v_1w_1, v_2w_2, v_3w_3\}$ be a tight edge-cut of G, and let G_1 and G_2 be the two bricks obtained by splitting along E. We may assume that G_1 contains the vertices v_i and we let u_1 be the vertex corresponding to the contracted part. Similarly, G_2 contains the vertices w_i and we let u_2 be the vertex corresponding to the contracted part.

By Lemma 9, both bricks G_1 and G_2 are well-covered. In particular, the vertex u_i is tricovered in G_i for $i \in \{1, 2\}$. Let H_i be a spanning subgraph of G_i such that u_i has degree 3 in H_i and the other vertices have degree 1. The subgraphs H_1 and H_2 combine to a perfect matching of G including all three edges of E, which contradicts our assumption that E is tight. □

Let us now turn our attention to cubic braces. Again, we have to introduce a definition. An edge of a matching-covered graph G is a *solo-edge* if it is contained in exactly one perfect matching. A matching-covered graph is *double-covered* if it has no solo-edges.

Lemma 11. *Every cubic brace different from A_1 and A_2 is double-covered.*

Proof. Let G be a cubic brace. Since A_1 and A_2 are the only cubic braces of order at most 2, the order of the brace G is at least 3. Let uv be an edge of G and M a matching containing uv. Since the order of G is not 1, there exists an edge $u'v'$ not in M and not adjacent with uv. By Theorem 7, the graph $G - \{u, v, u', v'\}$ has a perfect matching M'. We can extend M' to G by adding the edges uv and $u'v'$. Thus, M and M' are two distinct perfect matchings of G containing the edge uv. Consequently, G has no solo-edge. □

We finish this section with a lemma on cubic graphs whose decomposition contains a brace different from A_1 and A_2.

Lemma 12. *Every cubic bridgeless graph G whose brick and brace decomposition contains a brace different from A_1 and A_2 is double-covered.*

Proof. We proceed by induction on the number k of graphs in the brick and brace decomposition of G. If $k = 1$, then G is double-covered by Lemma 11. Assume that $k \geq 2$ and let us show that G is double-covered. To this end, let e be an edge of G. Consider any tight edge-cut E of G. Let G_1 and G_2 be the graphs obtained from G by splitting along this edge-cut.

By Theorem 6, the brick and brace decomposition of G_1 or G_2 contains a brace different from A_1 and A_2. Assume that G_1 has this property. Thus, G_1 is double-covered by induction.

If e is in G_1, then G_1 contains two distinct perfect matchings containing e, and each of them can be extended to a perfect matching of G since G_2 is matching-covered. Hence, e is not a solo-edge.

If e is in G_2, then a perfect matching of G_2 containing e can be extended to a perfect matching of G in at least two different ways, since G_1 is double-covered. Consequently, e is not a solo-edge either. □

3 Good cubic graphs

In this section, we present most of our tools for proving the lower bounds on the number of perfect matchings in a cubic bridgeless graph. Let us start with some terminology. An n-vertex cubic bridgeless graph G is *α-good* if G has $n/2+\alpha$ perfect matchings, and G is *$(\geq \alpha)$-good* if it has at least $n/2+\alpha$ perfect matchings. Since the dimension of the perfect matching polytope of an n-vertex cubic brick is $\frac{n}{2}$, a theorem of de Carvalho, Lucchesi, and Murty [1] on perfect matching polytopes with the lowest possible dimension implies that every brick is (≥ 2)-good except the bricks $I_1, \dots, I_{10}$ depicted in Figure 2.

Theorem 13 (de Carvalho, Lucchesi, and Murty, 2005). *Every brick different from the* 10 *bricks $I_1, \dots, I_{10}$ depicted in Figure 2 is* (≥ 2)*-good. All the bricks $I_1, \dots, I_{10}$ are* 1*-good.*

Our lower bound argument is based on the analysis of the brick and brace decompositions of cubic bridgeless graphs. We have introduced the operation of *splitting along tight edge-cuts* in Section 2. We now define the inverse operation. Let G_1 and G_2 be cubic bridgeless graphs, u a vertex of G_1 with neighbors u_1, u_2 and u_3 and v a vertex of G_2 with neighbors v_1, v_2 and v_3. Let G be the graph obtained from G_1 and G_2 by removing the vertices u and v and adding the edges u_1v_1, u_2v_2 and u_3v_3. We say that G is obtained by

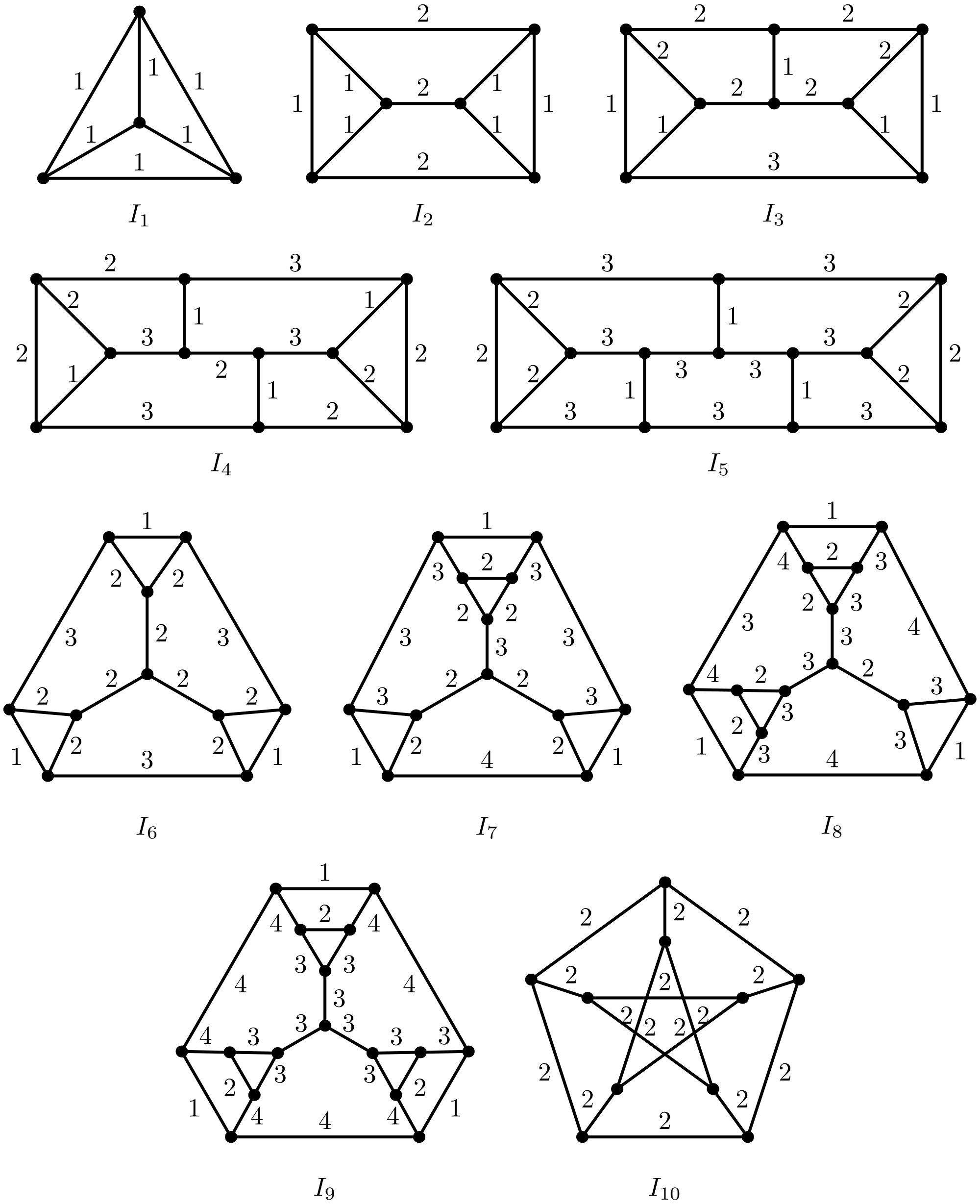


Figure 2: All 1-good bricks. The number near each edge indicates the number of perfect matchings containing this edge.

gluing the graphs G_1 and G_2, or more precisely from G_1 by *gluing* G_2 *through the vertex* u, or from G_2 by *gluing* G_1 *through the vertex* v. The gluing is a *solo-gluing* if for every $i \in \{1,2,3\}$, the edge uu_i is a solo-edge in G_1 or the edge vv_i is a solo-edge in G_2.

We now prove two lemmas giving lower bounds on the number of perfect matchings in graphs obtained by gluing smaller graphs. Before doing so, let us introduce one more definition. If G is a cubic bridgeless graph and v a vertex of G with neighbors v_1, v_2 and v_3, then the *pattern* of v is the triple (m_1, m_2, m_3) where m_i is the number of perfect matchings of G containing the edge vv_i for $i \in \{1,2,3\}$. We are now ready to prove the two lemmas.

Lemma 14. *Let G be a cubic bridgeless graph obtained by gluing an α-good graph G_a and a β-good graph G_b. The graph G is $(\geq \alpha+\beta-1)$-good unless G is obtained by a solo-gluing, in which case G is $(\alpha+\beta-2)$-good.*

Proof. Let n_a be the number of vertices of G_a and n_b the number of vertices of G_b. Next, let v_a be the vertex of G_a such that G is obtained from G_a by gluing G_b through v_a. Similarly, v_b is the vertex of G_b such that G is obtained from G_b by gluing G_a through v_b. Finally, let $(m_{a,1}, m_{a,2}, m_{a,3})$ be the pattern of v_a in G_a and $(m_{b,1}, m_{b,2}, m_{b,3})$ the pattern of v_b in G_b.

Since G_a is α-good and G_b is β-good,

$$n_a/2 + \alpha = m_{a,1} + m_{a,2} + m_{a,3} \tag{1}$$

and

$$n_b/2 + \beta = m_{b,1} + m_{b,2} + m_{b,3}\,. \tag{2}$$

Observe that $xy \geq x + y - 1$ for every positive integers x and y, with equality if and only $x = 1$ or $y = 1$. Hence, the definition of gluing and the fact that $m_{a,i} \geq 1$ and $m_{b,i} \geq 1$ yield that the number of perfect matchings of G is at least

$$m_{a,1}m_{b,1} + m_{a,2}m_{b,2} + m_{a,3}m_{b,3} \geq m_{a,1} + m_{a,2} + m_{a,3} + m_{b,1} + m_{b,2} + m_{b,3} - 3 \tag{3}$$

with equality if and only if for every $i \in \{1,2,3\}$, at least one of the numbers $m_{a,i}$ and $m_{b,i}$ equals 1. Since G has $n_a + n_b - 2$ vertices, (1), (2) and (3) imply that G is $(\geq \alpha+\beta-2)$-good. Moreover, G is $(\geq \alpha+\beta-1)$-good unless at least one of the numbers $m_{a,i}$ and $m_{b,i}$ equals 1 for every $i \in \{1,2,3\}$, i.e. unless G is obtained by a solo-gluing. □

In the final lemma of this section, we show that the bound from Lemma 14 can be improved if one of the glued graphs is double-covered.

Lemma 15. *Let G be a cubic bridgeless graph obtained by gluing an α-good graph G_a and a β-good graph G_b. If G_a is double-covered and G_b has at least five perfect matchings, then G is $(\alpha+\beta)$-good.*

Proof. Let us keep the notation from the proof of Lemma 14. Assume that $m_{b,1} \geq m_{b,2} \geq m_{b,3}$, and let p be the number of perfect matchings of G. It still holds that

$$p = m_{a,1}m_{b,1} + m_{a,2}m_{b,2} + m_{a,3}m_{b,3}\,. \tag{4}$$

First, assume that $m_{b,2} = m_{b,3} = 1$. Hence, $m_{b,1} \geq 3$ since G_b has at least five perfect matchings. Note that $m_{a,i} \geq 2$ for every $i \in \{1,2,3\}$ since G_a is double-covered. In particular, $m_{a,1}m_{b,1} \geq m_{a,1}+m_{b,1}+1$ since $xy \geq x+y+1$ for any $x \geq 2$ and $y \geq 3$. Thus, the bound (4) translates to

$$\begin{aligned} p &\geq m_{a,1} + m_{b,1} + 1 + m_{a,2} + m_{a,3} \\ &= m_{a,1} + m_{a,2} + m_{a,3} + m_{b,1} + m_{b,2} + m_{b,3} - 1 \\ &= \frac{n_a}{2} + \frac{n_b}{2} + \alpha + \beta - 1\,. \end{aligned}$$

Since the number of vertices of G is $n = n_a + n_b - 2$, the equalities (1) and (2) imply that G is $(\alpha+\beta)$-good.

We next assume that both $m_{b,1}$ and $m_{b,2}$ are at least 2. Recalling that $xy \geq x+y-1$ for two positive integers x and y, with equality if and only if $x = 1$ or $y = 1$, we deduce from (4) that

$$p \geq m_{a,1} + m_{b,1} + m_{a,2} + m_{b,2} + m_{a,3} + m_{b,3} - 1 \geq \frac{n_a}{2} + \frac{n_b}{2} - 1.$$

Therefore, G is $(\alpha+\beta)$-good. □

4 Bipartite cubic graphs

In this section, we will revisit a simple bound on the number of perfect matchings in bipartite graphs, which can be found in the book of Lovász and Plummer [7]. We need to slightly tune up the constants so that they

n	1	2	3	4	5	6	7	8	9	10
$f(n)$	3	5	6	8	12	17	23	30	41	54
$g(n)$	2	3	4	6	8	11	15	20	27	36

Table 1: The values $f(n)$ and $g(n)$ for $n \in \{1, \ldots, 10\}$.

are good enough for our later considerations. Let us start by defining two auxiliary functions $f, g : \mathbb{N} \to \mathbb{N}$ recursively, as follows.

$$g(n) = \begin{cases} 2 & \text{if } n = 1, \\ \left\lceil \frac{4}{3} g(n-1) \right\rceil & \text{otherwise,} \end{cases}$$

$$f(n) = \left\lceil \frac{3}{2} g(n) \right\rceil \text{ for every } n \geq 1.$$

The values of the functions $f(n)$ and $g(n)$ for small n can be found in Table 1.

We follow the lines of the proof of Theorem 8.7.1 from the book of Lovász and Plummer [7, Chapter 8] to prove the next lemma. In our further considerations, a bipartite graph is *near-cubic* if all its vertices have degree 3 except one vertex in each color class that has degree 2.

Lemma 16. *For each positive integer n, every cubic bipartite $2n$-vertex graph contains at least $f(n)$ perfect matchings and every near-cubic bipartite $2n$-vertex graph contains at least $g(n)$ perfect matchings.*

Proof. The proof proceeds by induction on n. The only cubic bipartite 2-vertex graph is the brace A_1, which has $3 = f(1)$ perfect matchings. The only near-cubic bipartite 2-vertex graph is obtained from A_1 by removing an edge: it has $2 = g(1)$ perfect matchings. Thus, the bounds claimed in the statement of the lemma hold if $n = 1$.

Assume that $n \geq 2$. Let us first consider a near-cubic bipartite $2n$-vertex graph G and let u and v be its vertices of degree 2. If u and v are adjacent, we show that G contains at least $f(n-1)$ perfect matchings. Indeed, let v' be the neighbor of u distinct from v and u' the neighbor of v distinct from u. Let G' be the graph obtained from G by removing the vertices u and v, and adding an edge between u' and v'. Since G' is a cubic bipartite graph, it contains at least $f(n-1)$ perfect matchings by the induction hypothesis. The perfect matchings of G' that contain the edge $u'v'$ can be converted to perfect matchings of G by replacing the edge $u'v'$ with the edges uv' and $u'v$, and those matchings of G' that avoid the edge $u'v'$ can be extended to perfect matchings of G by adding the edge uv. Since different perfect matchings of G' yield different perfect matchings of G, we deduce that G has at least $f(n-1)$ perfect matchings. The desired bound follows since $f(n-1) \geq \frac{4}{3}g(n-1)$.

We now consider the case where the vertices u and v are not adjacent. Let v_1 and v_2 be the two neighbors of u, and for $i \in \{1,2\}$ let u_i and u'_i be the two neighbors of v_i different from u. Finally, let G_1, G_2, G_3 and G_4 be the four graphs obtained from G by removing the vertex u, identifying the vertices v_1 and v_2 and removing one of the four edges u_1v_1, u'_1v_1, u_2v_2 and u'_2v_2. Each of the four graphs G_i is a near-cubic bipartite graph.

Every perfect matching of G_i corresponds to a perfect matching of G, e.g., any perfect matching of G_1 can be completed to a perfect matching of G by adding the edge uv_1 or uv_2. On the other hand, a perfect matching of G corresponds to perfect matchings in exactly three of the graphs $G_1, \ldots, G_4$ since it includes exactly one of the four edges u_1v_1, u'_1v_1, u_2v_2 and u'_2v_2. Hence, G has at least $4g(n-1)/3$ perfect matchings.

We have shown that G contains at least $4g(n-1)/3$ perfect matchings. Since the number of perfect matchings of G is an integer, G contains at least $g(n)$ perfect matchings, as asserted.

Assume now that H is a bipartite cubic graph. Let v be a vertex of H and v_1, v_2 and v_3 the three neighbors of v. For $i \in \{1,2,3\}$, let H_i be the near-cubic bipartite graph obtained by removing the edge vv_i. As shown before, H_i contains at least $g(n)$ perfect matchings. If M is a perfect matching of H, then M is also a perfect matching of exactly two of the graphs H_1, H_2 and H_3. Hence, H contains at least $3g(n)/2$ perfect matchings. Since the number of perfect matchings is an integer, H contains at least $f(n) = \lceil 3g(n)/2 \rceil$ perfect matchings. □

Lemma 17. *For each $n \geq 5$, every brace G of order n, is $(\geq n+2)$-good.*

Proof. Since $g(5) = 8$, we infer that for all $n \geq 5$,

$$f(n) \geq \frac{3}{2} \cdot \left(\frac{4}{3}\right)^{n-5} \cdot 8 = \frac{4^{n-4}}{3^{n-6}} \geq 2n+2\,.$$

By Lemma 16, G has at least $f(n) \geq 2n+2$ perfect matchings and thus G is $(\geq n+2)$-good. □

We finish this section with a simple constant lower bound on the number of perfect matchings in cubic bridgeless graphs which turns out to be useful in our further considerations.

Lemma 18. *Every cubic bridgeless graph different from A_1, I_1 and I_2 has at least five perfect matchings.*

Proof. Let G be a cubic bridgeless graph. If G is a brace, then G has at least five perfect matchings unless $G = A_1$ by Lemma 16. If G has a non-trivial brick and brace decomposition, then its decomposition contains a brace by Lemma 10, which cannot be A_1. Hence, the brace in the decomposition of G has at least five perfect matchings. Since the number of perfect matchings of a graph is at least the minimum of the number of perfect matchings of the graphs in its brick and brace decomposition (because every perfect matching of a graph in the decomposition can be extended to a perfect matching of the original graph), G has at least five perfect matchings.

It remains to consider the case where G is a brick. By Theorem 13, every n-vertex brick has at least $n/2 + 1$ perfect matchings. Hence, if G has less than five perfect matchings, then G has at most six vertices. The only two bricks with at most six vertices are the bricks I_1 and I_2, which have three and four perfect matchings, respectively. □

5 Single-brace cubic graphs

In this section, we analyze the number of perfect matchings in graphs whose brick and brace decomposition contains exactly one brace. Such cubic bridgeless graphs are referred to as *single-brace* graphs. Before we proceed further, let us state a simple lemma on tricovered vertices in cubic graphs.

Lemma 19. *If G is a cubic graph obtained from G' by gluing a graph G'' through a vertex v, then every vertex $w \neq v$ of G' that is tricovered in G' is also tricovered in G.*

Proof. Let H' be a spanning subgraph of G' such that the vertex w has degree 3 in H' and the other vertices of G' have degree 1. Let e be the edge of H' incident with v and let f be the edge corresponding to e in G''. Let M be a perfect matching of G'' that contains the edge f (recall that every cubic bridgeless graph is matching covered). The subgraph H' and the matching M combine to a spanning subgraph H of G where the degree of w is 3 and the degrees of other vertices are 1. Hence, the vertex w is tricovered in G. □

Let us now apply Lemma 19 to establish the following auxiliary lemma restricting the set of vertices through which a brick can be glued to a brace.

Lemma 20. *Let G be a single-brace graph. If the brick and brace decomposition consists of a brace B of order n and bricks $B_1, \dots, B_k$, and the brace B is not A_2, then $k \leq n$ and G can be obtained from B by gluing B_i through a vertex v_i of B for each $i \in \{1, \dots, k\}$ such that all the vertices v_i are in the same color class of B.*

Proof. The proof proceeds by induction on k, the conclusion holding trivially when $k = 1$. Assume that $k \geq 2$. Let us consider a tight edge-cut E of G and let G_1 and G_2 be the two graphs obtained by splitting along the edge-cut E. By Theorem 6 and Lemma 10, one of the graphs G_1 and G_2 is a brick. By symmetry, we can assume that G_2 is the brick B_k. Let w be the vertex such that G is obtained from G_1 by gluing B_k through w.

By the induction hypothesis, G_1 is obtained from the brace B by gluing B_i for each $i \in \{1, \dots, k-1\}$ through a vertex v_i, and the vertices $v_1, \dots, v_{k-1}$ are in the same color class of B. In order to finish the proof of the lemma, we have to exclude the following two cases.

- The vertex w is a vertex of one of the bricks $B_1, \dots, B_{k-1}$.
- The vertex w is in the other color class than the vertices $v_1, \dots, v_{k-1}$.

To this end, we show if w is one of the above two types, then w is tricovered in G_1. Since G_2 is well-covered by Lemma 9, this will imply that the edge-cut E is not tight. If w is a vertex of one of the bricks, then it is tricovered by Lemma 19 (apply this lemma several times while gluing the bricks to construct G_1). Hence, we have to focus on the case where w is in the other color class of B.

Since the brace A_1 does not appear in any non-trivial brick and brace decomposition and $B \neq A_2$, the brace B is simple (by Theorem 7). Let w' and w'' be two neighbors of w distinct from v_1, and let v' and v'' be two neighbors of v_1 distinct from w. By Theorem 7, the graph $B - \{v', v'', w', w''\}$ has a perfect matching. Adding the edges v_1v', v_1v'', ww' and ww'' to this perfect matching yields a spanning subgraph H_B of B, all of whose vertices have degree 1 except for the vertices v_1 and w, which both have degree 3. Along the brick and brace decomposition, using the fact that the bricks are well-covered by Lemma 9, the subgraph H_B can be extended to a spanning

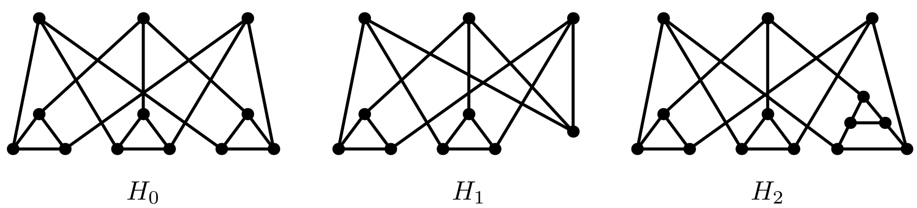


Figure 3: The exceptional graphs H_0, H_1 and H_2.

subgraph H of G_1 in which every vertex has degree 1 but the vertex w, which has degree 3. Hence, w is tricovered in G_1.

Since gluing a brick to a graph through a tricovered vertex does not create a new tight edge-cut, the vertex w must belong to the same color class as $v_1, \dots, v_{k-1}$. In particular, G can be obtained from the brace B by gluing the bricks $B_1, \dots, B_k$ through vertices $v_1, \dots, v_k$ contained in the same color class. Since each color class of B contains n vertices, the number k of bricks is at most n. The proof of the lemma is now finished. □

With Lemma 20, we are ready to consider single-brace graphs whose decomposition contains the brace A_3.

Lemma 21. *If G is a single-brace graph that contains A_3 in its brick and brace decomposition, then G is (≥ 2)-good unless it is one of the graphs H_0, H_1 and H_2 depicted in Figure 3.*

Proof. By Lemma 20, the graph G is obtained from the brace A_3 by gluing at most three bricks through vertices of the same color class of A_3. Let i_1 be the number of bricks I_1 glued to A_3, i_2 the number of bricks I_2 glued to A_3, and i the number of other bricks glued to A_3. Thus, $i_1 + i_2 + i \leq 3$.

The graph A_3 is 3-good and double-covered (the latter being implied by Lemma 11). Since I_1 is 1-good, the graph G_1 obtained by gluing i_1 bricks I_1 to A_3 is $(\geq 3 - i_1)$-good by Lemma 14.

Let G_2 be the graph obtained from G_1 by gluing i_2 bricks I_2 according to the brick and brace decomposition of G. Note that I_2 is 1-good, and no vertex of I_2 is incident with three solo-edges. Moreover, the graph G_1 is double covered by Lemma 12. Consequently, none of these i_2 gluings is a solo-gluing. Hence, the graph G_2 is $(\geq 3 - i_1)$-good by Lemma 14.

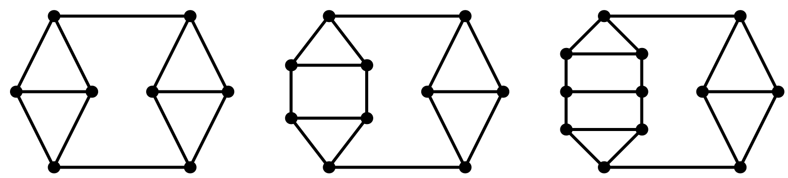


Figure 4: The exceptional graphs H_3, H_4 and H_5.

Finally, each of the remaining i bricks contains at least five perfect matchings by Lemma 18 and is (≥ 1)-good by Theorem 13. Since the graph G_2 is double-covered by Lemma 12, the final graph G is $(\geq 3 - i_1 + i)$-good by Lemma 15. Hence, if G is not (≥ 2)-good, then $i_1 \geq 2+i$. Since $i_1+i_2+i \leq 3$, we deduce that $i = 0$ and $i_2 \in \{0, 1\}$. So, either $i_1 = 3$ and $i_2 = 0$, or $i_1 = 2$ and $i_2 = 0$, or $i_1 = 2$ and $i_2 = 1$. The graph G is then either H_0, H_1, or H_2, respectively. It is straightforward to verify that H_0 is 0-good and the graphs H_1 and H_2 are 1-good. □

Before we proceed with analyzing single-brace graphs whose brick and brace decomposition contains a brace of order at least 4, let us deal with those whose decomposition contains the brace A_2.

Lemma 22. *If G is a single-brace graph that contains A_2 in its brick and brace decomposition, then G is (≥ 2)-good unless it is one of the graphs H_3, H_4 and H_5 depicted in Figure 4.*

Proof. Let $B = A_2$ be the brace and $B_1, \ldots, B_k$ the bricks forming the brick and brace decomposition of G. As in the proof of Lemma 20, it is possible to argue using Lemma 19 that G is obtained by gluing $B_1, \ldots, B_k$ through distinct vertices $v_1, \ldots, v_k$ of the brace B (this part of the proof was only using the fact that every brick is well-covered). However, since B is not simple, it is not possible to argue that the vertices $v_1, \ldots, v_k$ lie in the same color class of A_2 as in the proof of Lemma 20. In fact, they do not have to, as we shall see in what follows.

Although the vertices $v_1, \ldots, v_k$ do not have to be contained in the same color class of B, it still holds that $k \leq 2$. Suppose on the contrary that $k \geq 3$. Then, two of the vertices v_i, say v_1 and v_2, are in the same color class of B. We show that the graph G' obtained from B by gluing the brick B_1 through the vertex v_1 and the brick B_2 through the vertex v_2 is well-covered. Since B_3 is a brick, and thus is well-covered by Lemma 9, this will eventually contradict that the edge-cut of G used to split off B_3 is tight.

Let u and u' be the vertices of the other color class of B than v_1 and v_2. By Lemma 19, all the vertices of G' except possibly u and u' are tricovered. Let us establish that the vertices u and u' are also tricovered in G'.

By symmetry, we can assume that u is joined by two parallel edges to v_1. Let u_1 and u_2 be the neighbors (in G) of u inside the brick B_1, u_3 the vertex of B_1 adjacent to u' and u_0 the remaining neighbor of u. Observe that u_0 is in the brick B_2. Since B_1 is well-covered, there exists a subgraph H' of G spanning B_1 that contains the edges uu_1, uu_2 and $u'u_3$ and every vertex of B_1 has degree 1 in H'. Adding to H' a perfect matching of B_2 containing the edge uu_0 yields a spanning subgraph H of G', in which the vertex u has degree 3 and the remaining vertices have degree 1. Since the case of the vertex u' is symmetric, we have proved that G' is well-covered. As argued before, the number of bricks in the brick and brace decomposition of G is at most 2, i.e. $k \leq 2$.

If $k = 0$, then $G = A_2$ which is 3-good. If $k = 1$, then G is (≥ 2)-good by Lemma 14 since every brick is (≥ 1)-good. If $k = 2$, then G is again (≥ 2)-good by Lemma 14 unless both B_1 and B_2 are 1-good bricks and both gluings are solo-gluings. Since the pattern of every vertex of A_2 is $(1, 2, 2)$, a gluing can be a solo-gluing only if the brick B_i contains a vertex of pattern $(1, 1, x)$ for some $x \in \mathbb{N}$. However, there are only three 1-good bricks containing a vertex of pattern $(1, 1, x)$; see Figure 2. In particular, both the bricks B_1 and B_2 must be one of the bricks I_1, I_2 and I_3.

Let us now argue that at least one of the bricks B_1 and B_2 is I_1. To this end, we prove that one of the two solo-gluings must be through a vertex of a brick with pattern $(1, 1, 1)$. This will yield the desired conclusion since, among I_1, I_2 and I_3, only I_1 contains a vertex with such a pattern. Let G' be the graph obtained from $B = A_2$ by solo-gluing I_2 or I_3. As argued before, the solo-gluing is through a vertex of the brick with pattern $(1, 1, x)$. By the structure of I_2 and I_3, it holds that $x \geq 2$. Let v be a vertex of G' that is not contained in the glued brick and e an edge incident with v. If e is contained in two different perfect matchings of A_2, then e is also contained in at least two different perfect matchings of G'. If e is contained in a single perfect matching of A_2, then this perfect matching can be extended in x different ways to the glued brick. Hence, every edge incident with v is in at least

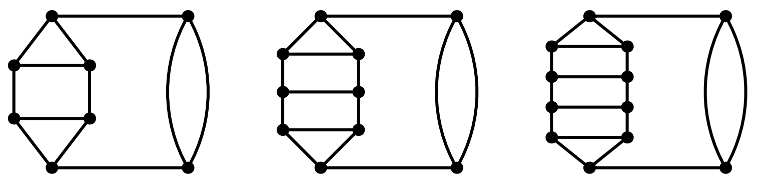


Figure 5: The graphs that can be obtained from the brace A_2 by solo-gluing the brick I_1 and one of the bricks I_1, I_2 and I_3 through vertices joined by parallel edges in A_2.

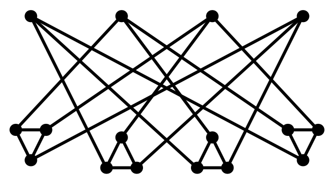
two different perfect matchings of G'. Since the choice of v was arbitrary among the vertices not contained in the brick, we deduce that only a brick containing a vertex with pattern $(1, 1, 1)$ can be solo-glued to G' (recall that gluing the second brick through a vertex contained in the first one would not yield a tight edge-cut). Hence, at least one of the bricks B_1 and B_2 is I_1.

By symmetry, we can assume in the rest that $B_1 = I_1$ and $B_2 \in \{I_1, I_2, I_3\}$. Let u and u' be vertices of one of the color classes of $B = A_2$. Let v be the vertex of the other color class joined by two parallel edges to u, and v' the vertex joined by two parallel edges to u'. By symmetry, the brick $B_1 = I_1$ is glued to $B = A_2$ through the vertex u. If the brick B_2 is glued through the vertex u' or the vertex v', we obtain one of the three 1-good graphs depicted in Figure 4. Note that although the brick B_2 can be glued in several non-symmetric ways, there is a unique way how it can be solo-glued. Finally, if the brick B_2 is glued through the vertex v, then the resulting graph is (≥ 2)-good. See Figure 5 for the three graphs that can be obtained in this way. □

It remains to analyze single-brace graphs whose brace-decomposition contains a brace of order at least four.

Lemma 23. *If G is a single-brace graph that contains neither A_2 nor A_3 in its brick and brace decomposition, then G is (≥ 2)-good unless it is the graph H_6 depicted in Figure 6.*

Proof. Let B be the brace in the decomposition of G, n the order of B and $B_1, \ldots, B_k$ the bricks in the decomposition. By Lemma 20, $k \leq n$. Let i_1 be the number of bricks $B_1, \ldots, B_k$ isomorphic to the brick I_1. If the brace B is A_4, then B is 5-good. After gluing the i_1 bricks I_1, the resulting graph G'



H_6

Figure 6: The exceptional graph H_6.

is $(\geq 5 - i_1)$-good by Lemma 14. Since G' is double-covered by Lemma 12, none of the gluings of the other $k - i_1$ bricks to G' is a solo-gluing. Hence, G is $(\geq 5 - i_1)$-good. We conclude that if G is not (≥ 2)-good, then $i_1 = 4$ and G is the exceptional graph H_6 depicted in Figure 6.

Assume now that B is not the brace A_4. Since B is also neither A_2 nor A_3 by the assumption of the lemma, B is $(n+2)$-good by Lemma 17. As in the previous paragraph, we argue that G is $(\geq n + 2 - i_1)$-good. Since $i_1 \leq k \leq n$ (the latter inequality is implied by Lemma 20), it follows that G is (≥ 2)-good. □

Lemmas 21–23 imply the following theorem. Note that every brace is (≥ 2)-good as shown in Section 4.

Theorem 24. *A single-brace graph G is (≥ 2)-good with the following exceptions:*

- *the graph H_0 which is 0-good, and*
- *the graphs $H_1, \ldots, H_6$ which are 1-good.*

The exceptional graphs are depicted in Figures 3, 4 and 6.

6 More-brace cubic graphs

In this section, we analyze cubic bridgeless graphs whose brick and brace decompositions contain at least two braces. Before we do so, we have to establish two auxiliary lemmas. The first one asserts that almost every single-brace graph that is not (≥ 2)-good is well-covered.

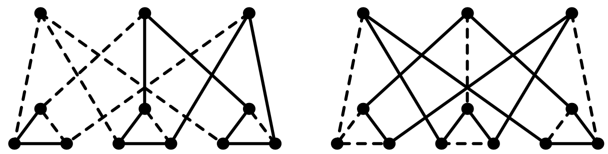


Figure 7: Spanning subgraphs of the graph H_0 witnessing that every vertex of H_0 is tricovered (symmetric cases are omitted). The edges contained in the subgraphs are dashed.

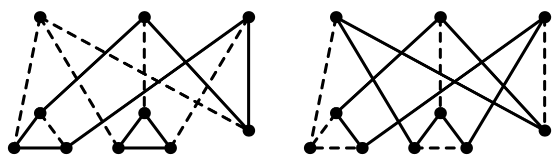


Figure 8: Spanning subgraphs of the graph H_1 witnessing that all but one vertices of H_1 are tricovered (symmetric cases are omitted). The edges contained in the subgraphs are dashed.

Lemma 25. *The cubic graphs $H_0, \ldots, H_6$ are well-covered with the exception of H_1 which contains a single vertex that is not tricovered. The pattern of this vertex of H_1 is $(2,2,2)$.*

Proof. It is enough to exhibit spanning subgraphs of the graphs $H_0, \ldots, H_6$ witnessing the statement of the lemma. Such subgraphs can be found in Figures 7–13; the exceptional vertex of H_1 is the vertex of A_3 of the color class where the brick I_1 was glued through the other two vertices. $\square$

In the next lemma, we restrict the structure of cubic bridgeless graphs that are not double-covered.

Lemma 26. *If G is a cubic bridgeless graph that is neither a brick nor the brace A_1, then every vertex of G is incident with at most one solo-edge.*

Proof. We proceed by induction on the number of vertices of G. If G has no tight edge-cuts, then it must be a brace. If G is the brace A_2, then every vertex of G has pattern $(1,2,2)$ and the statement holds. Otherwise, G is double-covered by Lemma 12 and thus G has no solo-edges at all.

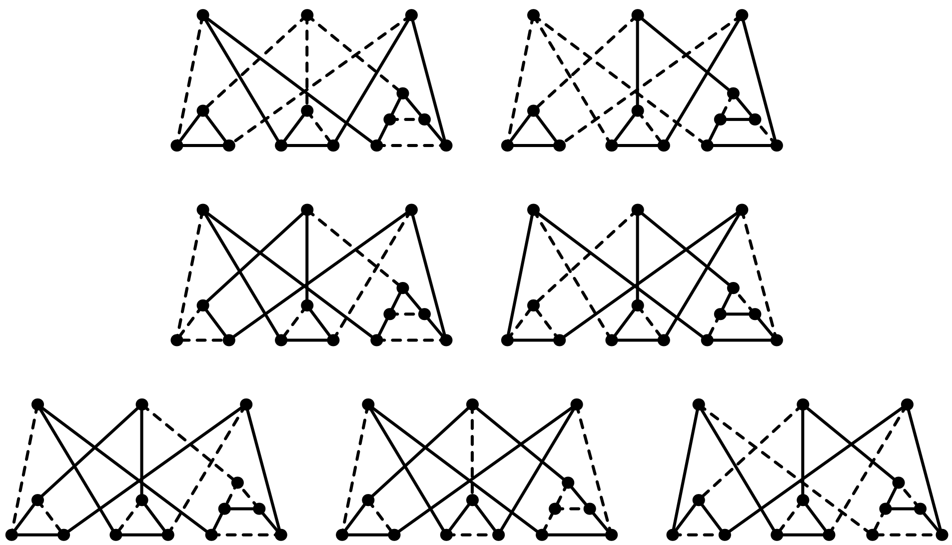


Figure 9: Spanning subgraphs of the graph H_2 witnessing that every vertex of H_2 is tricovered (symmetric cases are omitted). The edges contained in the subgraphs are dashed.

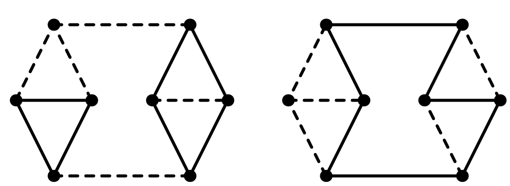


Figure 10: Spanning subgraphs of the graph H_3 witnessing that every vertex of H_3 is tricovered (symmetric cases are omitted). The edges contained in the subgraphs are dashed.

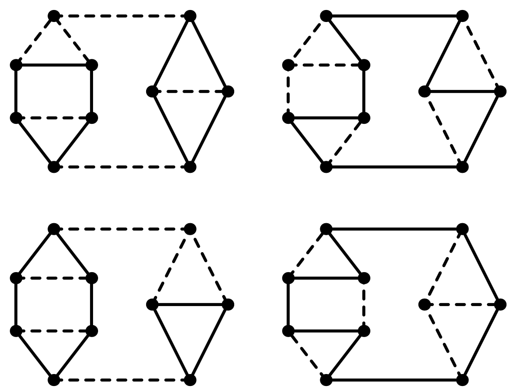


Figure 11: Spanning subgraphs of the graph H_4 witnessing that every vertex of H_4 is tricovered (symmetric cases are omitted). The edges contained in the subgraphs are dashed.

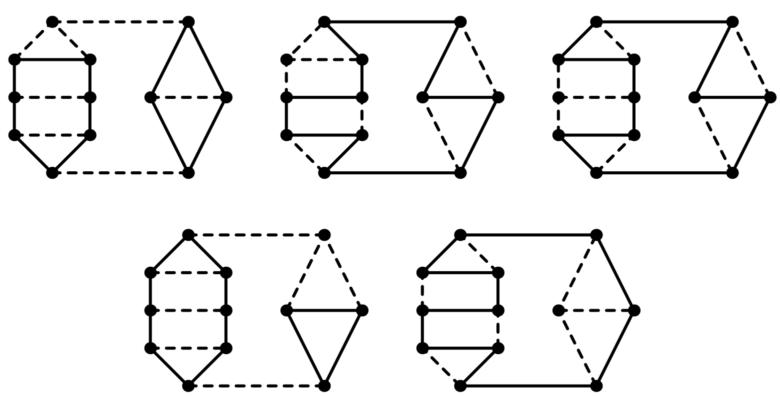


Figure 12: Spanning subgraphs of the graph H_5 witnessing that every vertex of H_5 is tricovered (symmetric cases are omitted). The edges contained in the subgraphs are dashed.

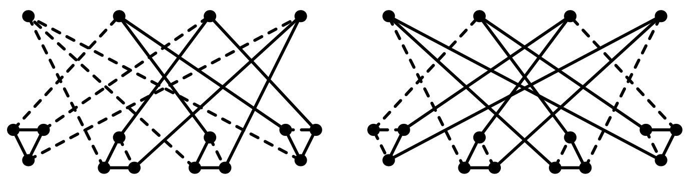


Figure 13: Spanning subgraphs of the graph H_6 witnessing that every vertex of H_6 is tricovered (symmetric cases are omitted). The edges contained in the subgraphs are dashed.

Assume that G has a non-trivial tight edge-cut $E = \{e_1, e_2, e_3\}$, and let G_1 and G_2 be the graphs obtained by splitting along E. Lemma 10 ensures that any non-trivial brick and brace decomposition contains at least one brace. Thus, we can assume that the brick and brace decomposition of G_1 contains a brace. By the induction hypothesis, every vertex of G_1 is incident with at most one solo-edge.

For $i \in \{1, 2\}$, let V_i be the set of vertices of G contained in G_i. Further, let v be the vertex of G_2 such that G is obtained from G_2 by gluing G_1 through v. In particular, $v \notin V_2$. Note that the edges e_1, e_2 and e_3 one-to-one correspond to the edges of G_2 incident with v. Since every vertex of G_1 is incident with at most one solo-edge, we can assume that, for each $i \in \{1, 2\}$, the graph G_2 admits a perfect matching that contains the edge e_i and can be extended to G_1 in at least two different ways.

Let w be any vertex of V_2 and let f_1, f_2 and f_3 be the three edges incident with w. We aim to show that at most one of these edges is a solo-edge. Since a cubic bridgeless graph is matching covered, there exists a perfect matching M_1 of G_2 containing the edge e_1. By symmetry, we can assume that M_1 also contains the edge f_1. Since any matching containing the edge e_1 can be extended to G_1 in at least two different ways, the edge f_1 is not a solo-edge.

On the other hand, as noted after Theorem 1, there exists a perfect matching M_2 of G_2 avoiding both the edges e_3 and f_1. By symmetry, we may assume that M_2 contains the edge f_2. Since M_2 also contains the edge e_1 or e_2, it can be extended to G_1 in at least two different ways. Hence, the edge f_2 is not a solo-edge either. We conclude that every vertex of V_2 is incident with at least two edges that are not solo-edges.

Since the number of perfect matchings containing a given edge can only increase by gluing a graph through a vertex, every vertex of V_1 is incident with at most one solo-edge. This finishes the proof of the lemma. □

In the next lemma, we show that every cubic bridgeless graph G that is neither a brick nor a single-brace graph contains a tight edge-cut with a useful property.

Lemma 27. *Let G be a cubic bridgeless graph that is neither a brick nor a single-brace graph. Then, G contains a tight edge-cut E such that neither of the graphs obtained by splitting along E is a brick.*

Proof. We proceed by induction on the number K of graphs in the brick and brace decomposition of G. The result is true if $K = 2$. Assume now that $K > 2$ and the theorem holds for smaller values of K. Since G is neither a brick nor a brace, G contains a tight edge-cut E. Let G_1 and G_2 be the two graphs obtained from G by splitting along E. By symmetry, we may assume that G_2 is a brick (otherwise E is the sought tight edge-cut). Hence, as G is not a single-brace graph, the brick and brace decomposition of G_1 contains at least two braces. Thus, by induction, G_1 contains a tight edge-cut E' that splits G_1 into two graphs G'_1 and G'_2 such that neither of them is a brick, i.e., the brick and brace decomposition of both G'_1 and G'_2 contains a brace. Let v be the vertex of G_1 such that G is obtained from G_1 by gluing G_2 through v. By symmetry, we can assume that the vertex v is contained in G'_2.

We assert that E' is also a tight edge-cut of G. Indeed, if G contains a perfect matching containing all three edges of E', then this matching uses exactly one edge of E because E is a tight edge-cut. Hence, the edge contained in E can be replaced with an edge of G_1 incident with v yielding a perfect matching of G_1 containing all three edges of E'.

Split now the graph G along the tight edge-cut E'. One of the obtained graphs is the graph G'_1, which is not a brick. The other graph cannot be a brick either, since its brick and brace decomposition must contain a brace contained in the decomposition of G'_2 (recall that Theorem 6 ensures that the brick and brace decomposition of G is unique). □

We are now ready to analyze cubic bridgeless graphs whose brick and brace decomposition contains two or more braces. We start with the case of two braces, which will be the core of our inductive argument later.

Theorem 28. *If the brick and brace decomposition of a cubic bridgeless graph G contains two braces, then G is (≥ 2)-good.*

Proof. Since the brick and brace decomposition of G is non-trivial, G has a tight edge-cut E. Let G_1 and G_2 be two graphs obtained from G by splitting along E. By Lemma 27, we can assume that neither G_1 nor G_2 is a brick. Hence, both G_1 and G_2 are single-brace graphs. By the definition of the brick and brace decomposition, neither G_1 nor G_2 can be the brace A_1. Note that both G_1 and G_2 have at least five perfect matchings by Lemma 18.

Assume first that G_1 is (≥ 2)-good. By Lemma 26, the gluing of G_1 and G_2 resulting to G is not a solo-gluing. Hence, if G_2 is (≥ 1)-good, then G is (≥ 2)-good by Lemma 14. If G_2 is not (≥ 1)-good, then G_2 must be the graph H_0 by Theorem 24. In particular, G_2 is double-covered. Consequently, G is (≥ 2)-good by Lemma 15 since G_1 has at least five perfect matchings. A symmetric arguments applies if G_2 is (≥ 2)-good.

It remains to consider the case where neither G_1 nor G_2 is (≥ 2)-good. Theorem 24 yields that each of G_1 and G_2 is one of the graphs $H_0, \ldots, H_6$. For $i \in \{1, 2\}$, let v_i be the vertex of G_i such that G is obtained from G_i by gluing G_{3-i} through v_i. At least one of the vertices v_1 and v_2 is not tricovered, since the edge-cut E used to split G is tight. By Lemma 25 and symmetry, we can assume that G_1 is the graph H_1 and the pattern of v_1 in G_1 is $(2, 2, 2)$.

If G_2 is 1-good, then G is (≥ 2)-good by Lemma 15 since G_1 is 1-good and double-covered. The other case is that G_2 is not 1-good. Then Theorem 24 implies that G_2 is the graph H_0. Consequently, the pattern of v_2 is also $(2, 2, 2)$, and the graph G has at least $3 \cdot (2 \cdot 2) = 12$ perfect matchings. Since the number of vertices of G is $10 + 12 - 2 = 20$, the graph G is 2-good. □

Finally, we can prove the main theorem of this section.

Theorem 29. *If the brick and brace decomposition of a cubic bridgeless graph G contains at least two braces, then G is (≥ 2)-good.*

Proof. The proof proceeds by induction on the number of braces in the brick and brace decomposition of G. If the brick and brace decomposition of G contains exactly two braces, then G is (≥ 2)-good by Theorem 28. Assume now that the decomposition of G contains at least three braces. Let G_1 and G_2 be two graphs that can be obtained from G by splitting along a tight edge-cut. By Lemma 27, we can assume that neither G_1 nor G_2 is a brick. By the definition of the brick and brace decomposition, neither G_1 nor G_2 is the brace A_1.

Since the brick and brace decomposition of G contains at least three braces, at least one of G_1 and G_2 is not a single-brace graph. By symmetry, we can assume that G_1 is not a single-brace graph, and thus G_1 is (≥ 2)-good by the induction hypothesis. The graph G_2 is (≥ 0)-good. This follows from Theorem 24 if G_2 is a single-brace graph, and from the induction hypothesis otherwise. By Lemma 26, the gluing of G_1 and G_2 resulting in G is not a solo-gluing. So, if G_2 is (≥ 1)-good, then G is (≥ 2)-good by Lemma 14. If G_2 is 0-good, then G_2 must be the graph H_0 by Theorem 24 and the induction hypothesis. In particular, G_2 is double-covered. Moreover, G_1 has at least five perfect matchings by Lemma 18. Hence, Lemma 15 implies that G is (≥ 2)-good. □

Theorems 13, 24 and 29 imply Theorem 5, the main result of this paper.

Acknowledgment

This research was done while the first two authors were visiting the third one at Technische Universität Ilmenau. They thank their host for providing a perfect working environment.

References

[1] M. H. de Carvalho, C. L. Lucchesi, and U. S. R. Murty. Graphs with independent perfect matchings. *J. Graph Theory*, 48(1):19–50, 2005.

[2] M. Chudnovsky and P. Seymour. Perfect matchings in planar cubic graphs. Submitted for publication, 2008.

[3] J. Edmonds, L. Lovász, and W. R. Pulleyblank. Brick decompositions and the matching rank of graphs. *Combinatorica*, 2(3):247–274, 1982.

[4] L. Gurvits. Hyperbolic polynomials approach to Van der Waerden/Schrijver-Valiant like conjectures: sharper bounds, simpler proofs and algorithmic applications. In *STOC'06: Proceedings of the 38th Annual ACM Symposium on Theory of Computing*, pages 417–426, New York, 2006. ACM.

[5] D. J. Hartfiel and J. W. Crosby. On the permanent of a certain class of (0, 1)-matrices. *Canad. Math. Bull.*, 14:507–511, 1971.

[6] L. Lovász. Matching structure and the matching lattice. *J. Combin. Theory Ser. B*, 43(2):187–222, 1987.

[7] L. Lovász and M. D. Plummer. *Matching theory*, volume 121 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1986. Annals of Discrete Mathematics, 29.

[8] H. Minc. On lower bounds for permanents of (0, 1) matrices. *Proc. Amer. Math. Soc.*, 22:117–123, 1969.

[9] D. Naddef. Rank of maximum matchings in a graph. *Math. Programming*, 22(1):52–70, 1982.

[10] J. Petersen. Die Theorie der regulären graphs. *Acta Math.*, 15(1):193–220, 1891.

[11] A. Schrijver. Counting 1-factors in regular bipartite graphs. *J. Combin. Theory Ser. B*, 72(1):122–135, 1998.

[12] A. Schrijver. *Combinatorial optimization. Polyhedra and efficiency. Vol. A*, volume 24 of *Algorithms and Combinatorics*. Springer-Verlag, Berlin, 2003. Paths, flows, matchings, Chapters 1–38.

[13] A. Schrijver and W. G. Valiant. On lower bounds for permanents. *Nederl. Akad. Wetensch. Indag. Math.*, 42(4):425–427, 1980.

[14] R. Sinkhorn. Concerning a conjecture of Marshall Hall. *Proc. Amer. Math. Soc.*, 21:197–201, 1969.

[15] M. Voorhoeve. A lower bound for the permanents of certain (0, 1)-matrices. *Nederl. Akad. Wetensch. Indag. Math.*, 41(1):83–86, 1979.