A Unified Approach to Distance-Two Colouring of Graphs on Surfaces

Omid Amini *, Louis Esperet †,
and Jan van den Heuvel ‡

* CNRS – DMA, *École Normale Supérieure, Paris, France*

† *Institute of Theoretical Computer Science, Charles University, Prague, Czech Republic*

‡ *Department of Mathematics, London School of Economics, London, U.K.*

Abstract

In this paper we introduce the notion of (A,B)-colouring of a graph: For given vertex sets A,B, this is a colouring of the vertices in B so that both adjacent vertices and vertices with a common neighbour in A receive different colours. This concept generalises the notion of colouring the square of graphs and of cyclic colouring of graphs embedded in a surface. We prove a general result which implies asymptotic versions of Wegner's and Borodin's Conjecture on the planar version of these two colourings. Using a recent approach of Havet *et al.*, we reduce the problem to edge-colouring of multigraphs and then use Kahn's result that the list chromatic index is close to the fractional chromatic index.

Our results are based on a strong structural lemma for graphs embedded in a surface which also implies that the size of a clique in the square of a graph

The research for this paper was started during a visit of LE and JvdH to the Mascotte research group at INRIA Sophia-Antipolis, where OA was a PhD student. The authors like to thank the members of Mascotte for their hospitality.

Part of this research has been conducted while OA was visiting McGill University in Montreal. He is extremely grateful to the members of the CS and Math Departments, especially Bruce Reed. He is also grateful to the hospitality of Max-Planck Institut in Saarbrücken.

JvdH's visit was partly supported by a grant from the Alliance Programme of the British Council.

Email: oamini@dma.ens.fr, esperet@kam.mff.cuni.cz, jan@maths.lse.ac.uk

of maximum degree Δ embeddable in some fixed surface is at most $\frac{3}{2}\Delta$ plus a constant.

1 Introduction

Most of the terminology and notation we use in this paper is standard and can be found in any text book on graph theory (such as [2] or [8]). All our graphs and multigraphs will be finite. A *multigraph* can have multiple edges; a *graph* is supposed to be simple. We will not allow loops. The vertex and edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively (or just V and E, if the graph G is clear from the context).

Given a graph G, the *chromatic number* of G, denoted $\chi(G)$, is the minimum number of colours required so that we can properly colour its vertices using those colours. If we colour the edges of G, we get the *chromatic index*, denoted $\chi'(G)$. The *list chromatic number* or *choice number* $ch(G)$ is the minimum value k, so that if we give each vertex v of G a list $L(v)$ of at least k colours, then we can find a proper colouring in which each vertex gets assigned a colour from its own private list. The list chromatic index is defined analogously for edges.

The *square* G^2 of a graph G is the graph with vertex set $V(G)$, with an edge between any two different vertices that have distance at most two in G. A proper vertex colouring of the square of a graph can also be seen as a vertex colouring of the original graph satisfying:

- vertices that are adjacent must receive different colours, and
- vertices that have a common neighbour must receive different colours.

Another way to formulate these conditions is as 'vertices at distance one or two must receive different colours'. This is why the name *distance-two colouring* is also used in the literature.

In this paper we consider a colouring concept that generalises the concept of colouring the square of a graph, but that also can be used to study different concepts such as *cyclic colouring of plane graphs* (definition will be given later).

Let A and B be two subsets of the vertex set V. (Note that we do not require A and B to be disjoint.) An (A, B)*-colouring* of G is an assignment of colours to the vertices in B so that:

- vertices of B that are adjacent must receive different colours, and
- vertices of B that have a common neighbour from A must receive different colours.

When each vertex $v \in B$ has its own list $L(v)$ of colours from which its colour must be chosen, we talk about a *list* (A, B)*-colouring.*

We denote by $\chi(G; A, B)$ the minimum number of colours required for an (A, B)-colouring to exist. Its list variant is denoted by $ch(G; A, B)$, and is defined as the minimum integer k so that for each assignment of a list $L(v)$ of at least k colours to vertices $v \in B$, there exists a proper (A, B)-colouring of G in which the vertices in B are assigned colours from their own lists.

Notice that we trivially have $\chi(G) = \chi(G; \varnothing, V)$ and $\chi(G^2) = \chi(G; V, V)$; and the same relations holds for the list variant.

For a vertex $v \in V$, let $N(v)$ be the set of vertices adjacent to v, and define $N_B(v) = N(v) \cap B$, and $d_B(v) = |N_B(v)|$ (so $d_G(v) = d_V(v)$). If we set $\Delta(G; A, B) = \max\{\, d_B(v) \mid v \in A\,\}$, then it is clear that we always need at least $\Delta(G; A, B)$ colours in a proper (A, B)-colouring.

In the case $A = B = V$, there exist plenty of graphs G that require $O(\Delta(G)^2)$ colours (where $\Delta(G) = \Delta(G; V, V)$ is the normal maximum degree of a graph). But for planar graphs, it is known that a constant times $\Delta(G)$ colours is enough (even for list colouring). We'll take a closer look at this in Subsection 1.1 below.

Following Wegner's Conjecture on colouring the square of planar graphs (see also next subsection), we propose the following conjecture.

Conjecture 1.1
There exist constants c_1, c_2, c_3 such that for all planar graphs G and $A, B \subseteq V$ we have

$$\begin{aligned} \chi(G; A, B) &\le \left\lfloor \tfrac{3}{2}\,\Delta(G; A, B) \right\rfloor + c_1; \\ ch(G; A, B) &\le \left\lfloor \tfrac{3}{2}\,\Delta(G; A, B) \right\rfloor + c_2; \\ ch(G; A, B) &\le \left\lfloor \tfrac{3}{2}\,\Delta(G; A, B) \right\rfloor + 1, \qquad \text{if } \Delta(G; A, B) \ge c_3. \end{aligned}$$

If $A = \varnothing$ (hence $\Delta(G; A, B) = 0$) and $B = V$, then the Four Colour Theorem means that the smallest possible value for c_1 is four; while the fact that planar graphs are always 5-list colourable but not always 4-list colourable, shows that the smallest possible value for c_2 is five.

Our main result is that Conjecture 1.1 is asymptotically correct. In fact, we can prove a more general asymptotic version, which holds for general surfaces.

Theorem 1.2
Let S be a fixed surface, G a graph embeddable in S, and $A, B \subseteq V$. Then $ch(G; A, B) \le (1 + o(1))\, \frac{3}{2}\, \Delta(G; A, B)$.

In other words, for all $\varepsilon > 0$, there exists $D_{S,\varepsilon}$, so that for all $D \ge D_{S,\varepsilon}$ we have: If G is a graph embeddable in S, with $A, B \subseteq V$ so that $\Delta(G; A, B) \le D$, and L is a list assignment so that each vertex v in B gets a list $L(v)$ of at least $\left(\frac{3}{2} + \varepsilon\right) D$ colours, then there exists an (A, B)-colouring of G in which the vertices in B are assigned colours from their own lists.

A trivial lower bound for the (list) chromatic number of a graph G is the *clique number* $\omega(G)$, the maximal size of a clique in G. For (A, B)-colourings, where $A, B \subseteq V$, we can define the following related concept. An (A, B)*-clique* is a subset $C \subseteq B$ so that every two different vertices in C are adjacent or have a common neighbour in A. Denote by $\omega(G;A,B)$ the maximal size of an (A,B)-clique in G. Then we trivially have $ch(G;A,B) \geq \omega(G;A,B)$, and so Theorem 1.2 means that for a graph G embeddable in some fixed surface S we have $\omega(G;A,B) \leq (1+o(1))\,\frac{3}{2}\,\Delta(G;A,B)$.

But in fact, the structural result we use to prove Theorem 1.2 fairly easily gives a better estimate.

Theorem 1.3
Let S be a fixed surface, G a graph embeddable in S, and $A,B \subseteq V$. Then $\omega(G;A,B) \leq \frac{3}{2}\,\Delta(G;A,B) + O(1)$.

To prove Theorems 1.2 and 1.3 we can as well assume that A contains all the vertices adjacent to at most $\Delta(G;A,B)$ vertices of B. To simplify things, define $B^\beta = \{\, v \in V \mid d_B(v) \leq \beta \,\}$. So to prove Theorems 1.2 and 1.3 it is enough to prove the following theorems.

Theorem 1.4
Let S be a fixed surface. For all real $\varepsilon > 0$, there exists a $\beta_{S,\varepsilon}$ so that the following holds for all $\beta \geq \beta_{S,\varepsilon}$. Let G be a graph that can be embedded in S, with $B \subseteq V$ a set of vertices, and suppose every vertex $v \in B$ has a list $L(v)$ of at least $\left(\frac{3}{2}+\varepsilon\right)\beta$ colours. Then a list (B^β,B)-colouring of G with those colours exist.

Theorem 1.5
Let S be a fixed surface. There exist constants β_S, γ_S so that the following holds for all $\beta \geq \beta_S$. Let G be a graph that can be embedded in S, with $B \subseteq V$ a set of vertices. Then every (B^β,B)-clique in G has size at most $\frac{3}{2}\,\beta + \gamma_S$.

In the next two subsections we discuss two special consequences of these results. These special versions of the Theorems 1.4 and 1.5 also show that the term $\frac{3}{2}\,\beta$ in these results is best possible.

The main steps in the proof of Theorem 1.4 can be found in Section 2. The proof relies on two technical lemmas; the proofs of those can be found in Section 3. After that we use one of those lemmas to provide the relatively short proof of Theorem 1.5 in Section 4. In Section 5 we discuss some of the aspects of our work and discuss open problems related to (list) (A,B)-colouring of graphs. The final section provides some background regarding the proof of Kahn [17] on the asymptotical equality of the fractional chromatic index and the list chromatic index of multigraphs. A more general result, contained implicitly in Kahn's work, is of crucial importance to our proof in this paper.

1.1 Colouring the Square of Graphs

Recall that the *square* of a graph G, denoted G^2, is the graph with the same vertex set as G and with an edge between any two different vertices that have distance at most two in G. If G has maximum degree Δ, then a vertex colouring of its square will need at least $\Delta + 1$ colours, but the greedy algorithm shows that it is always possible to find a colouring of G^2 with Δ^2+1 colours. Diameter two cages such as the 5-cycle, the Petersen graph and the Hoffman-Singleton graph (see [2, page 239]) show that there exist graphs that in fact require Δ^2+1 colours.

Regarding the chromatic number of the square of a planar graph, Wegner [31] posed the following conjecture (see also the book of Jensen and Toft [14, Section 2.18]), suggesting that for planar graphs far less than $\Delta^2 + 1$ colours suffice.

Conjecture 1.6 (Wegner [31])

For a planar graph G of maximum degree Δ : $\chi(G^2) \leq \begin{cases} 7, & \text{if } \Delta = 3, \\ \Delta + 5, & \text{if } 4 \leq \Delta \leq 7, \\ \left\lfloor \frac{3}{2}\,\Delta \right\rfloor + 1, & \text{if } \Delta \geq 8. \end{cases}$

Wegner also gave examples showing that these bounds would be tight. For even $\Delta \geq 8$, these examples are sketched in Figure 1(a).

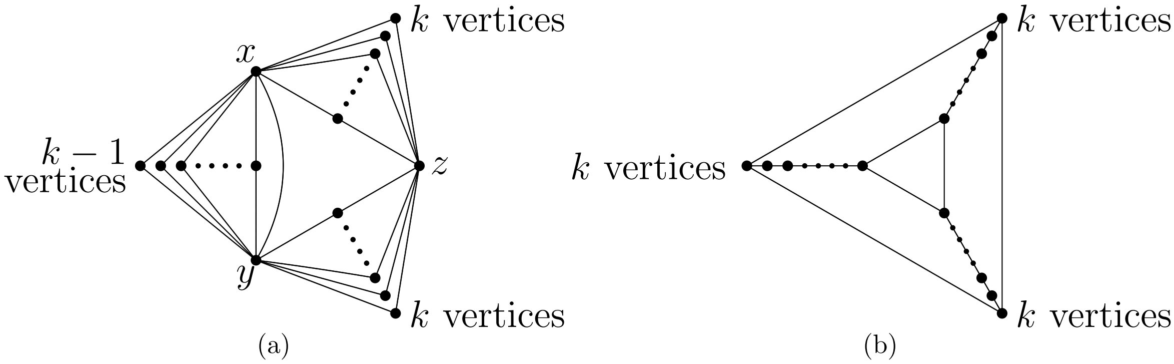

Figure 1: (a) A planar graph G with maximum degree $\Delta = 2\,k$ and $\omega(G^2) = \chi(G^2) = 3\,k + 1 = \left\lfloor \frac{3}{2}\,\Delta \right\rfloor + 1$.
(b) A planar graph H with maximum face degree $\Delta^* = 2\,k$ and $\chi^*(H) = 3\,k = \left\lfloor \frac{3}{2}\,\Delta^* \right\rfloor$ (see Subsection 1.2 below).

The graph in the picture has maximum degree $2\,k$ and yet all the vertices except z are pairwise adjacent in its square. Hence to colour these $3\,k + 1$ vertices, we need at least $3\,k + 1 = \frac{3}{2}\,\Delta + 1$ colours. Note that the same arguments also show that the graph G in the picture has $\omega(G^2) = \frac{3}{2}\,\Delta + 1$.

Kostochka and Woodall [19] conjectured that for every square of a graph the chromatic number equals the list chromatic number. This conjecture and Wegner's one together imply the conjecture that for planar graphs G with $\Delta \geq 8$ we have $ch(G^2) \leq \lfloor \frac{3}{2}\Delta \rfloor + 1$.

The first upper bound on $\chi(G^2)$ for planar graphs in terms of Δ, $\chi(G^2) \leq 8\Delta - 22$, was implicit in the work of Jonas [15]. This bound was later improved by Wong [32] to $\chi(G^2) \leq 3\Delta + 5$ and then by Van den Heuvel and McGuinness [13] to $\chi(G^2) \leq 2\Delta + 25$. Better bounds were then obtained for large values of Δ. It was shown that $\chi(G^2) \leq \lceil \frac{9}{5}\Delta \rceil + 1$ for $\Delta \geq 750$ by Agnarsson and Halldórsson [1], and the same bound for $\Delta \geq 47$ by Borodin *et al.* [4]. Finally, the best known upper bound so far has been obtained by Molloy and Salavatipour [24]: $\chi(G^2) \leq \lceil \frac{5}{3}\Delta \rceil + 78$. As mentioned in [24], the constant 78 can be reduced for sufficiently large Δ. For example, it was improved to 24 when $\Delta \geq 241$.

Surprisingly, no independent results for the clique number of the square of planar graphs, other than those that follow from their chromatic numbers, have been published so far.

Since $ch(G^2) = ch(G; V, V)$, as an immediate corollary of Theorem 1.2 we obtain.

Corollary 1.7
Let S be a fixed surface. Then the square of every graph G embeddable in S and of maximum degree Δ has list chromatic number at most $(1 + o(1))\,\frac{3}{2}\Delta$.

In fact, the same asymptotic upper bound as in Corollary 1.7 can be proved for even larger classes of graphs. Additionally, a stronger conclusion on the colouring is possible. For the following result we assume that colours are integers, which allows us to talk about the 'distance' $|\alpha_1 - \alpha_2|$ between two colours α_1, α_2.

Theorem 1.8 (Havet, van den Heuvel, McDiarmid & Reed [10])
Let k be a fixed positive integer. The square of every $K_{3,k}$-minor free graph G of maximum degree Δ has list chromatic number (and hence clique number) at most $(1 + o(1))\,\frac{3}{2}\Delta$. Moreover, given lists of this size, there is a proper colouring in which the colours on every pair of adjacent vertices of G differ by at least $\Delta^{1/4}$.

Note that planar graphs do not have a $K_{3,3}$-minor. In fact, for every surface $\mathcal{S}$, there is a constant k so that all graphs embeddable in $\mathcal{S}$ do not have $K_{3,k}$ as a minor (see also Lemma 4.1). That shows that Theorem 1.8 is stronger than our Corollary 1.7. On the other hand, Theorem 1.8 gives a weaker bound for the clique number than the one we obtain in Corollary 1.9 below.

Both Corollary 1.7 and Theorem 1.8 can be applied to K_4-minor free graphs, since these graphs are planar and don't have $K_{3,3}$ as a minor. But for this class we actually know the exact results. Lih, Wang and Zhu [21] showed that the square of K_4-minor free graphs with maximum degree Δ has chromatic number at most $\left\lfloor \frac{3}{2}\,\Delta \right\rfloor + 1$ if $\Delta \geq 4$ and $\Delta + 3$ if $\Delta = 2, 3$. The same bounds, but then for the list chromatic number of K_4-minor free graphs, were proved by Hetherington and Woodall [12].

Regarding the clique number of the square of graphs, we get the following corollary of Theorem 1.3.

Corollary 1.9
Let S be a fixed surface. Then the square of every graph G embeddable in S and of maximum degree Δ has clique number at most $\frac{3}{2}\,\Delta + O(1)$.

From the proof of Theorem 1.3, it can be deduced that for planar graphs we know that every planar graph G of maximum degree $\Delta \geq 1056$ has clique number at most $\frac{3}{2}\,\Delta + 109$.

Very recently, this was improved by the following result.

Theorem 1.10 (Cohen & van den Heuvel [7])
For a planar graph G of maximum degree $\Delta \geq 41$ we have $\omega(G^2) \leq \left\lfloor \frac{3}{2}\,\Delta \right\rfloor + 1$.

Apart from the bound $\Delta \geq 41$, this theorem is best possible, as is shown by the same graphs that show that Wegner's Conjecture 1.6 is best possible for $\Delta \geq 8$ (see also Figure 1(a)). The proof of Theorem 1.10 in [7] in fact uses the concept of (A, B)-cliques, together with the main result from Hell and Seyffarth [11]: *For $\Delta \geq 8$, the maximum number of vertices of a planar graph with maximum degree Δ and diameter two is $\left\lfloor \frac{3}{2}\,\Delta \right\rfloor + 1$.*

1.2 Cyclic Colourings of Embedded Graphs

Given a surface S and a graph G embeddable in S, we denote by G^S that graph with a prescribed embedding in S. If the surface S is the sphere, we talk about a *plane graph G^P*. The size (number of vertices in its boundary) of a largest face of G^S is denoted by $\Delta^*(G^S)$.

A *cyclic colouring* of an embedded graph G^S is a vertex colouring of G such that any two vertices incident to the same face have distinct colours. The minimum number of colours required in a cyclic colouring of an embedded graph is called the *cyclic chromatic number* $\chi^*(G^S)$. This concept was introduced for plane graphs by Ore and Plummer [25], who also proved that for a plane graph G^P we have $\chi^*(G^P) \leq 2\,\Delta^*$. Borodin [3] (see also Jensen and Toft [14, page 37]) conjectured the following.

Conjecture 1.11 (Borodin [3])
For a plane graph G^P of maximum face size Δ^* we have $\chi^*(G^P) \le \left\lfloor \frac{3}{2}\,\Delta^* \right\rfloor$.

The bound in this conjecture is best possible. Consider the plane graph depicted in Figure 1(b) : it has $3\,k$ vertices and has three faces of size $\Delta^* = 2\,k$. Since all pairs of vertices have a face they are both incident with, we need $3\,k = \left\lfloor \frac{3}{2}\,\Delta^* \right\rfloor$ colours in a cyclic colouring.

Borodin [3] also proved Conjecture 1.11 for $\Delta^* = 4$. For general values of Δ^*, the original bound $\chi^*(G^P) \le 2\,\Delta^*$ of Ore and Plummer [25] was improved by Borodin *et al.* [6] to $\chi^*(G^P) \le \left\lfloor \frac{9}{5}\,\Delta^* \right\rfloor$. The best known upper bound in the general case is due to Sanders and Zhao [27] : $\chi^*(G^P) \le \left\lceil \frac{5}{3}\,\Delta^* \right\rceil$.

Although Wegner's and Borodin's Conjectures seem to be closely related, nobody has ever been able to bring to light a direct connection between them. Most of the results approaching these conjectures use the same ideas, but up until this point (as far as the authors know) no one had proved a general theorem implying both a result on the colouring of the square and a result on the cyclic colouring of plane graphs (let alone on embedded graphs).

In order to show that our Theorem 1.2 provides an asymptotically best possible upper bound for the cyclic chromatic number for a graph G with some fixed embedding G^S, we need some extra notation. For each face f of G^S, add a vertex x_f and call X_F the set of vertices that were added to G. For any face f of G^S, and any vertex v incident with f, add an edge between v and x_f. We denote by G_F the graph obtained from G^S by this construction, so $V(G_F) = V(G) \cup X_F$. Observe that a (list) $(X_F, V(G))$-colouring of G_F is exactly a cyclic (list) colouring of G^S and that $\Delta(G_F; X_F, V(G)) = \Delta^*(G^S)$. We get the following corollary of Theorem 1.2.

Corollary 1.12
Let S be a fixed surface. Every embedded graph G^S of maximum face size Δ^* has cyclic list chromatic number at most $(1 + o(1))\,\frac{3}{2}\,\Delta^*$.

For an embedded graph G^S, the *cyclic clique number* $\omega^*(G^S)$ is the maximal size of a set $C \subseteq V$ so that every two vertices in C have some face they are both incident with. Note that the plane graph depicted in Figure 1(b) satisfies $\omega^*(G^P) = 3\,k = \left\lfloor \frac{3}{2}\,\Delta^* \right\rfloor$. This shows that the following corollary of Theorem 1.3 is best possible, up to the constant term.

Corollary 1.13
Let S be a fixed surface. Every embedded graph G^S of maximum face size Δ^* has cyclic clique number at most $\frac{3}{2}\,\Delta^* + O(1)$.

For plane graphs, the proof of Theorem 1.3 guarantees that a plane graph G^P of maximum face size $\Delta^* \ge 1056$ has cyclic clique number at most $\frac{3}{2}\,\Delta^* + 109$.

2 Proof of Theorem 1.4

Throughout this section we assume that $G = (V, E)$ is a graph embedded in a surface S, with $B \subseteq V$, and β is a positive integer. Recall the notation $U^\beta = \{\, v \in V \mid d_U(v) \le \beta \,\}$ for a subset $U \subseteq V$. Note that this means that V^β is the set of all vertices of degree at most β

Our goal is to show that for all $\varepsilon > 0$, if we take β large enough, then for every assignment $L(v)$ of at least $\left(\frac{3}{2} + \varepsilon\right)\beta$ colours to the vertices $v \in B$, there is a list (B^β, B)-colouring of G where each vertex in B receives a colour from its own list. In other words, we want an assignment $c(v)$ for each $v \in B$ so that:

- for all $v \in B$ we have $c(v) \in L(v)$;
- for all $u, v \in B$ with $uv \in E$ we have $c(u) \neq c(v)$; and
- for all $u, v \in B$ with a common neighbour in B^β (i.e., with a common neighbour w with $d_B(w) \le \beta$) we have $c(u) \neq c(v)$.

Before we start the actual proofs, we recall some of the important terminology, notation and facts concerning embeddings of graph in surfaces.

2.1 Graphs in Surfaces

In this subsection, we give some background about graphs embedded in a surface. For more details, the reader is referred to [22]. Here, by a *surface* we mean a compact 2-dimensional surface without boundary. An *embedding* of a graph G in a surface S is a drawing of G on S so that all vertices are distinct and every edge forms a simple arc connecting in S the vertices it joins, so that the interior of every edge is disjoint from other vertices and edges. A *face* of this embedding (or just a *face of* G, for short) is an arc-wise connected component of the space obtained by removing the vertices and edges of G from the surface S. We say that an embedding is *cellular* if every face is homeomorphic to an open disc in $\mathbb{R}^2$.

A surface can be orientable or non-orientable. The *orientable surface* $\mathbb{S}_h$ *of genus* h is obtained by adding $h \ge 0$ 'handles' to the sphere; while the *non-orientable surface* $\mathbb{N}_k$ *of genus* k is formed by adding $k \ge 1$ 'crosscaps' to the sphere. The *genus* $\mathbf{g}(G)$ and *non-orientable genus* $\widetilde{\mathbf{g}}(G)$ of a graph G is the minimum h and the minimum k, resp., so that G has an embedding in $\mathbb{S}_h$, resp. in $\mathbb{N}_k$.

The following result will allow us to suppose that a graph G with known genus $\mathbf{g}(G)$ or non-orientable genus $\widetilde{\mathbf{g}}(G)$ can be assumed to be embedded in a cellular way.

Lemma 2.1 ([22, Propositions 3.4.1 and 3.4.2])
(i) *Every embedding of a connected graph G in $\mathbb{S}_{\mathbf{g}(G)}$ is cellular.*
(ii) *There exists an embedding of a connected graph G in $\mathbb{N}_{\widetilde{\mathbf{g}}(G)}$ that is cellular.*

The *Euler characteristic* $\chi(S)$ of a surface S is $2 - 2\,h$ if $S = \mathbb{S}_h$, and $2 - k$ if $S = \mathbb{N}_k$.

The crucial result connecting all these concepts is *Euler's Formula* : If G is a graph with an embedding in S, with vertex set V, edge set E and face set F, then

$$|V| - |E| + |F| \;\geq\; \chi(S).$$

Moreover, if the embedding is cellular, then we have equality in Euler's Formula.

Finally, if v is a vertex of a graph G embedded in a surface, then that embedding imposes two circular orders of the edges incident with v. Since we assume graphs to be simple, this corresponds to two circular orders of the neighbours of v. If the surface is orientable, then we can consistently choose one of the two clockwise orders for *all* vertices; if the surface is non-orientable, then such a choice is not possible. In our proofs that follow it is not important that we can choose a consistent circular order : we only require that for each vertex v there is at least one circular order of the neighbours around v.

2.2 The First Steps

A *β-neighbour* of v is a vertex $u \neq v$, so that u and v are adjacent, or u and v have a common neighbour in B^β. Denote the set of β-neighbours of v by $N^\beta(v)$, and its number by $d^\beta(v)$. Note that we have

$$d^\beta(v) \;\leq\; d(v) + \sum_{u \in N(v) \cap B^\beta} (d(u) - 1).$$

For $P, Q \subseteq V$, the set of edges between P and Q is denoted by $E(P, Q)$, and the number of edges between P and Q by $e(P, Q)$ (edges with both ends in $P \cap Q$ are counted twice).

An important tool in our proof of Theorem 1.4 is the following structural result.

Lemma 2.2
Let S be a fixed surface. Set $\zeta_S^* = 132\,(3 - \chi(S))$ and $\beta_S^* = 8\,\zeta_S^*$. Then for all $\beta \geq \beta_S^*$ and any connected graph G with a given cellular embedding in S, one of the following holds.

(S1) There is a vertex with degree zero or one.

(S2) There is a face f and two vertices u, v on the boundary of f with $d(u) + d(v) \le \beta$ and $d^{\beta}(u) \le \frac{3}{2}\,\beta$.

(S3) There are two disjoint non-empty sets $X, Y \subseteq V^{\beta}$ with the following properties:

(i) Every vertex $y \in Y$ has degree at most four. Moreover, y is adjacent to exactly two vertices of X and the other neighbours of y have degree at most four as well.

For $y \in Y$, let X^y be the set of its two neighbours in X. And for $W \subseteq X$, let Y^W be the set of vertices $y \in Y$ with $X^y \subseteq W$ (that is, the set of vertices of Y having their two neighbours from X in W).

(ii) For all pairs of vertices $y, z \in Y$, if y and z are adjacent or have a common neighbour $w \notin X$, then $X^y = X^z$.

(iii) For all non-empty subsets $W \subseteq X$, we have the following inequality:

$$e(W, V \setminus Y) \;\le\; e(W, Y \setminus Y^W) + \zeta^*_S\,|W|.$$

The proof of Lemma 2.2 can be found in Subsection 3.1. Observe that the values we use for β^*_S and ζ^*_S are probably far from best possible. The important point, to our mind, is that they only depend on (the Euler characteristic of) the surface S.

We continue with a description how to apply the lemma to prove Theorem 1.4, assuming that $\beta \ge \beta^*_S$. We use induction on the number of vertices of G. We obviously can assume that G is connected, otherwise we use induction on each of the components.

We first show that we can apply Lemma 2.2 to G. If G has a cellular embedding in S, there is nothing to prove. Suppose that it is not the case, and assume that S is orientable with genus h. By Lemma 2.1 we must have $\mathbf{g}(G) < h$, and hence $\chi(\mathbb{S}_{\mathbf{g}(G)}) = 2 - 2\,\mathbf{g}(G) > 2 - 2\,h = \chi(S)$. But that also means that the constants in Lemma 2.2 satisfy $\beta^*_{\mathbb{S}_{\mathbf{g}(G)}} < \beta^*_S$ and $\zeta^*_{\mathbb{S}_{\mathbf{g}(G)}} < \zeta^*_S$. We will see later (see the end of the proof in Subsection 2.3), that then also $\beta_{S,\varepsilon} \ge \beta_{\mathbb{S}_{\mathbf{g}(G)},\varepsilon}$. So we can use Lemma 2.2 with the surface $\mathbb{S}_{\mathbf{g}(G)}$ instead of S, and use a cellular embedding of G in $\mathbb{S}_{\mathbf{g}(G)}$.

If S is non-orientable, then exactly the same argument can be applied, this time using the surface $\mathbb{N}_{\tilde{\mathbf{g}}(G)}$.

So we can assume that G has a given cellular embedding in S, and so by Lemma 2.2, G contains one of (S1), (S2) or (S3).

(S1) If G contains a vertex v of degree at most one, we consider the graph G_1 obtained from G by removing v. This graph is clearly embeddable in S.

If $v \notin B$, then a list (B^β, B)-colouring of G_1 is also a list (B^β, B)-colouring of G. Otherwise set $B_1 = B \setminus \{v\}$. Now find a list (B_1^β, B_1)-colouring of G_1, and give an appropriate colour to v at the end. This is always possible since v is in conflict with at most β other vertices, and we have $\left(\frac{3}{2} + \varepsilon\right)\beta \geq \beta + 1$ colours available for v.

(S2) Let f be a face with two vertices u, v on its boundary such that $d(u) + d(v) \leq \beta$ and $d^\beta(u) \leq \frac{3}{2}\beta$. In this case we construct a new graph G_2 embeddable in S by identifying u and v into a new vertex w. Set $V_2 = (V \setminus \{u, v\}) \cup \{w\}$, and notice that G_2 has strictly fewer vertices than G, and w has degree at most $d_G(u) + d_G(v) \leq \beta$ in G_2. In other words, $w \in V_2^\beta$. If $v \notin B$, then set $B_2 = B$. Otherwise, set $B_2 = (B \setminus \{u, v\}) \cup \{w\}$ and give w a list of colours $L(w)$ with $L(w) = L(v)$.

By induction there exists a list (B_2^β, B_2)-colouring of G_2. We define a colouring of G as follows: every vertex different from u and v keeps its colour from the colouring of G_2. If $v \in B$, then we colour v with the colour given to w in G_2. And if $u \in B$, then we use the assumption, $d_G^\beta(u) \leq \frac{3}{2}\beta$, and hence there exists a colour for u different from the colour of all the vertices in conflict with u. We colour u with one of these colours. It is easy to verify that this defines a list (B^β, B)-colouring of G.

(S3) This is the only non-trivial case. In the remaining of this subsection we describe how to reduce this case to a list edge-colouring problem. In the next subsection, we then describe how Kahn's approach to prove that the list chromatic index is asymptotically equal to the fractional chromatic index, can be used to conclude the proof of Theorem 1.4.

Let X and Y be the two disjoint sets as in (S3). This means that every vertex in X has degree at most β. Also recall that by (S3) (i), every vertex $y \in Y$ has degree at most four. Moreover, y is adjacent to exactly two vertices of X and the other neighbours of y have degree at most four as well. As in (S3), let X^y be the set of the two neighbours of y in X.

Suppose there is a vertex $y \in Y$ with $y \notin B$. If $N(y) = X^y$, then contract y to one of its two neighbours in X^y. If y has a neighbour u outside X^y, then contract the edge uy. Call the resulting graph G_3. It is easy to check that a list (B^β, B)-colouring of G_3, which exists by induction, also is a proper list (B^β, B)-colouring of G.

So from now on we assume that all vertices in Y are contained in B.

Let Y_0 be the set of vertices from Y with no neighbour outside $X \cup Y$. Consider the graph $G[V \setminus Y_0]$ induced on the set of vertices outside Y_0. For every vertex $y \in Y \setminus Y_0$ with a unique neighbour u outside $X \cup Y$, or with exactly two neighbours u and v outside $X \cup Y$, contract the edge yu into a new vertex u^*. The graph obtained is denoted by G_0. And let B_0 be the union of $B \setminus Y_0$ and all new vertices u^* that originated from an edge yu with $u \in B$.

By the construction of G_0, it is easy to verify the following statement.

Claim 2.3 *For all $u \in V(G_0) \cap V(G)$ we have $(N_G^\beta(u) \setminus Y) \subseteq N_{G_0}^\beta(u)$.*

For each vertex u^* of B_0 corresponding to the contraction of an edge uy ($y \in Y \setminus Y_0$) in G, set $L_0(u^*) = L(u)$ and for all other vertices v of B_0 set $L_0(v) = L(v)$. By the induction hypothesis, the graph G_0 admits a list (B_0^β, B_0)-colouring c_0 with respect to the list assignment L_0.

We now transform this colouring into a list (B^β, B)-colouring of G with respect to the original list assignment L. For each vertex $u \in B \setminus Y$, if an edge incident to u has been contracted in the construction of G_0 to form a new vertex u^*, set $c(u) = c_0(u^*)$. Otherwise set $c(u) = c_0(u)$. Using Claim 2.3, this is a good partial (B^β, B)-colouring of all the vertices of $B \setminus Y$. The difficult part of the proof is to show that c can be extended to Y.

By assumption, at the beginning every vertex in Y has a list of at least $\left(\frac{3}{2} + \varepsilon\right)\beta$ available colours. For each vertex y in Y, let us remove from $L(y)$ the colours which are forbidden for y according to the partial (B^β, B)-colouring c of G. At the worst case, these forbidden colours are exactly the colours of the vertices of $V \setminus Y$ at distance at most two from y.

Let us define the multigraph H as follows: H has vertex set X. And for each vertex $y \in Y$ we add an edge e_y between the two neighbours of y in X (in other words, between the two vertices from X^y). We associate a list $L(e_y)$ to e_y in H by taking the list of y obtained after removing the set of forbidden colours for y from the original list $L(y)$. Finally, for every edge e in $G[X]$, we add the same edge e to H and associate a list $L(e)$ of at least $\left(\frac{3}{2} + \varepsilon\right)\beta$ colours to such an edge. (The colours within these lists are irrelevant for what follows, we just have to make sure that the lists of these specific edges of H are large enough.)

We now prove the following lemma.

Lemma 2.4
A list edge-colouring for H, with the list assignment L defined as above, provides an extension of c to a list (B^β, B)-colouring of G by giving to each vertex $y \in Y$ the colour of the edge e_y in H.

Proof This follows from property (S3) (ii) in Lemma 2.2: for every two vertices $y, z \in Y$, if y and z are adjacent or have a common neighbour $w \notin X$, then $X^y = X^z$. This proves that the two vertices adjacent in Y or with a common neighbour not in X define parallel edges in H and so will have different colours. If two vertices y_1 and y_2 of Y have a common neighbour in X, e_{y_1} and e_{y_2} will be adjacent in H and so will get different colours. Since we have already removed from the list of vertices in Y the set of forbidden colours (defined by the colours of the vertices in $V\setminus Y$), there will be no conflict between the colours of a vertex from Y and a vertex from $V\setminus Y$. We conclude that the edge-colouring of H will provide an extension of c to a list (B^β, B)-colouring of G. $\square$

The following lemma provides a lower bound on the size of $L(e)$ for the edges e in H.

Lemma 2.5
Let $e = uv$ be an edge in H. Then we have

$$|L(e)| \ \geq\ \left(\tfrac{3}{2}+\varepsilon\right)\beta - (d_G(u)-d_H(u)) - (d_G(v)-d_H(v)) - 10.$$

Proof If e originated because there was already an edge in $G[X]$, then by construction we have $|L(e)| \geq \left(\tfrac{3}{2}+\varepsilon\right)\beta$. On the other hand, suppose that $e = e_y$, i.e., e originated because of a vertex $y \in Y$ in G with $X^y = \{u, v\}$. Let Z be the set of vertices adjacent in G to y in $V\setminus X$. Then by (S3), $|Z| \leq 2$ and $|N_G(Z)\setminus Y| \leq 6$. The colours that are forbidden for y are the colours of $\{u, v\}$, plus the colours of vertices in $(N_G(u)\cup N_G(v))\setminus Y$, plus the colours of vertices in $(Z\cup N_G(Z))\setminus Y$. The number of vertices in these three sets add up to $(d_G(u)-d_H(u)) + (d_G(v)-d_H(v)) + 10$. The lemma follows. $\square$

In the remainder of this subsection, we apply Lemma 2.2 to obtain information on the density of subgraphs in H, which we will need in the next subsection. As in Lemma 2.2, for all non-empty subsets $W\subseteq X$, we define Y^W as the set of vertices $y\in Y$ with $X^y\subseteq W$ (that is, the set of vertices of Y having their two neighbours from X in W). By (S3) (iii) we have for all non-empty $W\subseteq X$:

$$e_G(W, V\setminus Y) \ \leq\ e_G(W, Y\setminus Y^W) + \zeta_S^*\,|W|.$$

This inequality has the following interpretation in H.

Lemma 2.6
For all non-empty subsets $W\subseteq X(= V(H))$ we have

$$\sum_{w\in W}(d_G(w)-d_H(w)) \ \leq\ e_H(W, X\setminus W) + \zeta_S^*\,|W|.$$

Proof First note that $\sum_{w \in W} (d_G(w) - d_H(w)) = e_G(W, V \setminus (X \cup Y)) \leq e_G(W, V \setminus Y)$. We also have $e_G(W, Y \setminus Y^W) = e_H(W, X \setminus W) - e_G(W, X \setminus W) \leq e_H(W, X \setminus W)$. Combining these two observations with the formula in (S3) (iii) immediately gives the required inequality. □

At this point, our aim will be to apply Kahn's approach to the multigraph H with the list assignment L, to prove the existence of a proper list edge-colouring for H. This is described in the next subsection.

We summarise the properties we assume are satisfied by the multigraph H and the list assignment L of the edges of H. For these conditions we just consider $d_G(v)$ as an integer with certain properties, assigned to each vertex of H.

(H1) For all vertices v in H we have $d_H(v) \leq d_G(v) \leq \beta$.

(H2) For all edges $e = uv$ in H: $|L(e)| \geq \left(\frac{3}{2} + \varepsilon\right) \beta - (d_G(u) - d_H(u)) - (d_G(v) - d_H(v)) - 10$.

(H3) For all non-empty subsets $W \subseteq V(H)$: $\sum_{w \in W} (d_G(w) - d_H(w)) \leq e_H(W, X \setminus W) + \zeta_S^* \, |W|$, for some constant ζ_S^*.

2.3 The Matching Polytope and Edge-Colourings

We briefly describe the matching polytope of a multigraph. More about this subject can be found in [28, Chapter 25].

Let H be a multigraph with m edges. Let $\mathcal{M}(H)$ be the set of all matchings of H, including the empty matching. For each $M \in \mathcal{M}(H)$, let us define the m-dimensional *characteristic vector* $\mathbf{1}_M$ as follows: $\mathbf{1}_M = (x_e)_{e \in E(H)}$, where $x_e = 1$ for an edge $e \in M$, and $x_e = 0$ otherwise. The *matching polytope* of H, denoted by $\mathcal{MP}(H)$, is the polytope defined by taking the convex hull of all the vectors $\mathbf{1}_M$ for $M \in \mathcal{M}(H)$. Also, for any real number λ, we set $\lambda \, \mathcal{MP}(H) = \{ \lambda \, x \mid x \in \mathcal{MP}(H) \}$.

Edmonds [9] gave the following characterisation of the matching polytope:

Theorem 2.7 (Edmonds [9])
A vector $\vec{x} = (x_e)$ is in $\mathcal{MP}(H)$ if and only if $x_e \geq 0$ for all x_e and the following two types of inequalities are satisfied:

- *For all vertices $v \in V(H)$: $\sum\limits_{e:\ v \text{ incident to } e} x_e \leq 1$;*
- *for all subsets $W \subseteq V(H)$ with $|W| \geq 3$ and $|W|$ odd: $\sum\limits_{e \in E(W)} x_e \leq \frac{1}{2}\,(|W| - 1)$.*

The significance of the matching polytope and its relation with list edge-colouring is indicated by the following important result.

Theorem 2.8 (Kahn [17])
For all real numbers δ, μ, $0 < \delta < 1$ and $\mu > 0$, there exists a $\Delta_{\delta,\mu}$ so that for all $\Delta \geq \Delta_{\delta,\mu}$ the following holds. If H is a multigraph and L is a list assignment of colours to the edges of H so that

- *H has maximum degree at most Δ;*
- *for all edges $e \in E(H)$: $|L(e)| \geq \mu\,\Delta$;*
- *the vector $\vec{x} = (x_e)$ with $x_e = \dfrac{1}{|L(e)|}$ for all $e \in E(H)$ is an element of $(1-\delta)\,\mathcal{MP}(H)$.*

Then there exists a proper edge-colouring of H where each edge gets a colour from its own list.

The theorem above is actually not explicitly stated this way in [17], but can be obtained from the appropriate parts of that paper. We give some further details about that in the final section of this paper.

The next lemma shows how to use Theorem 2.8 to complete the induction.

Lemma 2.9
Let ζ be a real number. Then there exists a K_ζ, so that for all $K \geq K_\zeta$ the following holds. Let H be a multigraph, and suppose that for each vertex v an integer $D(v)$, and for each edge e a positive real number b_e is given. Suppose that the following three conditions are satisfied:

(H1') For all vertices v in H: $d(v) \leq D(v) \leq \beta$.
(H2') For all edges $e = uv$ in H: $b_e \geq \bigl(\frac{3}{2}\,\beta + K\bigr) - (D(u)-d(u)) - (D(v)-d(v))$.
(H3') For all non-empty subsets $W \subseteq V(H)$: $\sum\limits_{w \in W} (D(w)-d(w)) \leq e_H(W, V(H) \setminus W) + \zeta\,|W|$.

Then for all edges $e \in E(H)$ we have $b_e \geq \frac{1}{2}\,\beta$. And the vector $\vec{x} = (x_e)$ defined by $x_e = \dfrac{1}{b_e}$ for $e \in E(H)$ is an element of $\mathcal{MP}(H)$.

The proof of Lemma 2.9 will be given in Subsection 3.2. This lemma guarantees that for all $\varepsilon > 0$, there exists a β_ε, so that for all $\beta \geq \beta_\varepsilon$ Theorem 2.8 can be applied to a multigraph H with an edge list assignment L satisfying properties (H1) – (H3) stated at the end of the previous subsection.

To see this, take $\delta_\varepsilon = \dfrac{\varepsilon}{3 + 2\,\varepsilon}$, so $0 < \delta_\varepsilon < 1$. In order to be able to apply Theorem 2.8, we want to prove the existence of $\beta_{\varepsilon,\zeta_S^*}$ such that for any $\beta \geq \beta_{\varepsilon,\zeta_S^*}$, the vector $\vec{x} = (x_e)$, $x_e = \dfrac{1}{|L(e)|}$, is in $(1 - \delta_\varepsilon)\,\mathcal{MP}(H)$. Let ζ_S^* be the real number described in condition (H3) and let $K_{\zeta_S^*}$ be the number given by Lemma 2.9. By condition (H2) we have

$$\begin{aligned}(1 - \delta_\varepsilon)\,|L(e)| \;&\geq\; (1 - \delta_\varepsilon)\,\Big(\big(\tfrac{3}{2} + \varepsilon\big)\,\beta - (D(u) - d(u)) - (D(v) - d(v)) - 10\Big)\\ &\geq\; (1 - \delta_\varepsilon)\,\big(\tfrac{3}{2} + \varepsilon\big)\,\beta - (D(u) - d(u)) - (D(v) - d(v)) - 10\\ &=\; \big(\tfrac{3}{2}\,\beta + \tfrac{1}{2}\,\varepsilon\,\beta\big) - (D(u) - d(u)) - (D(v) - d(v)) - 10.\end{aligned}$$

Let $\beta_{\varepsilon,\zeta_S^*} = \dfrac{2\,(K_{\zeta_S^*} + 10)}{\varepsilon}$. For $\beta \geq \beta_{\varepsilon,\zeta_S^*}$ we have

$$(1 - \delta_\varepsilon)\,|L(e)| \;\geq\; \big(\tfrac{3}{2}\,\beta + K_{\zeta_S^*}\big) - (D(u) - d(u)) - (D(v) - d(v)).$$

So by Lemma 2.9, taking $b_e = (1 - \delta_\varepsilon)\,|L(e)|$, the vector $\vec{x}' = (x'_e)$, $x'_e = \dfrac{x_e}{1 - \delta_\varepsilon}$, is in $\mathcal{MP}(H)$. We infer that $\vec{x} \in (1 - \delta_\varepsilon)\,\mathcal{MP}(H)$ and the lemma follows.

Now set $\beta_{S,\varepsilon} = \max\{\,\beta_S^*,\, \beta_{\varepsilon,\zeta_S^*},\, \Delta_{\delta_\varepsilon,1/2}\,\}$ (where β_S^*, ζ_S^* are determined by Lemma 2.2, $\beta_{\varepsilon,\zeta_S^*}$ and δ_ε are related to $K_{\zeta_S^*}$ from Lemma 2.9 as explained above, and $\Delta_{\delta_\varepsilon,1/2}$ is according to Theorem 2.8), and assume $\beta \geq \beta_{S,\varepsilon}$. Then using Lemma 2.9, we can now apply Theorem 2.8 which implies that the multigraph H defined in Subsection 2.2 has a list edge-colouring corresponding to the list assignment L. Lemma 2.4 then implies that the colouring c can be extended to a list (B^β, B)-colouring of the original graph G. This concludes the induction and also completes the proof of Theorem 1.4. ■

3 Proofs of the Main Lemmas

We use the terminology and notation from the previous sections.

3.1 Proof of Lemma 2.2

Let S be a surface, and set $\zeta_S^* = 132\,(3 - \chi(S))$ and $\beta_S^* = 8\,\zeta_S^*$. We take $\beta \geq \beta_S^*$ and consider a connected graph G with a given cellular embedding in S. We need some further notation and terminology.

The set of faces of G is denoted by F. Recall that since the embedding in S is cellular, every face is homeomorphic to an open disk in $\mathbb{R}^2$. For a face f, a *boundary walk of* f is a walk consisting of vertices and edges as they are encountered when walking along the whole boundary of f, starting at some vertex. The *degree of a face* f, denoted $d(f)$, is the number of edges on the boundary walk of f. Note that this means that if f is incident with a bridge (cut edge) of G, that bridge will be counted twice in $d(f)$. The *size of a face* f is the number of vertices on its boundary. We always have that the size of f is at most $d(f)$, with strict inequality if and only if the face has a cut vertex on its boundary.

We start by proving that we can assume that any two vertices on the boundary of the same face are adjacent in G. For suppose this is not the case for some pair u, v on the boundary of a face f. Form the graph G' by adding the edge uv to G, with the natural embedding of that edge inside f. Then G' is still a connected simple graph, and the embedding of G' is cellular.

Suppose G' contains one of the structures (S1) – (S3) in the lemma. We claim that then also G contains one of these structures. This is obvious if G' contains (S1) or (S2). So suppose G' has sets X, Y according to (S3).

It is easy to check that exactly the same pair X, Y works for G as well in the following cases : if $\{u, v\} \cap (X \cup Y) = \varnothing$, or if $u, v \in X$, or if $u, v \in Y$, or if $u \in Y$ and $v \in V \setminus (X \cup Y)$. If $u \in X$ and $v \in V \setminus (X \cup Y)$, then going from G' to G for $W \subseteq X$ with $u \in W$, we loose one on the left hand side of the inequality in (iii). Hence the pair X, Y also works for G. If $u \in X$ and $v \in Y$, then in G either v has degree at most one, and then G contains structure (S1), or v is adjacent to one vertex $x \in X$ and at most two more vertices of degree at most four. But then v has a neighbour w with $d(v) + d(w) \leq 7 \leq \beta$. Moreover, since $x \in X \subseteq V^{\beta}$, we have $d^{\beta}(v) \leq 8 + \beta \leq \frac{3}{2}\beta$. Hence in this case G contains structure (S2). Finally, the possibilities $v \in Y$ and $u \in V \setminus (X \cup Y)$, or $v \in X$ and $u \in V \setminus (X \cup Y)$, or $v \in X$ and $u \in Y$, can be done by symmetry with the cases above.

So, by adding edges we can transform G to a simple connected graph G^* with a cellular embedding in S so that any two vertices on the boundary of the same face are adjacent in G^*, and so that if G^* satisfies the lemma, then so does G. Hence we might as well assume the following :

(a) *The graph G with a cellular embedding in S has the property that any two vertices on the boundary of the same face are adjacent.*

Now suppose that G does not contain any of the structures (S1) or (S2). In order to prove Lemma 2.2, we only need to prove that G contains structure (S3). We can observe that :

(b) *All vertices have degree at least three.* (Since G does not contain (S1), degrees must be at least two. As G is not a triangle, we cannot have a vertex of degree two, since otherwise, for each face to have at degree at least three, we have a multiple edge as well.)

(c) *For all pairs of adjacent vertices u, v we have $d(u) + d(v) > \beta$ or $d^\beta(u) > \frac{3}{2}\beta$* (otherwise we have structure (S2)).

Let $\mathcal{B} \subseteq V$, the *big* vertices, be the vertices of degree at least $\zeta_S^* + 1$; the other vertices are called *small*. Define $\mathcal{B}_\beta = \mathcal{B} \cap V^\beta$ (the big vertices with degree at most β) and $\mathcal{B}_{>\beta} = \mathcal{B} \setminus \mathcal{B}_\beta$.

Note that for a vertex u, a neighbour v that is not in $\mathcal{B}_\beta$ adds at most $\zeta_S^* - 1$ neighbours at distance two from u to $d^\beta(u)$ (none if $v \in \mathcal{B}_{>\beta}$ and at most $\zeta_S^* - 1$ if v is small, where the -1 appears since one of the neighbours of v is u itself).

(d) *If a vertex u of degree three has a small neighbour, then its other two neighbours are in $\mathcal{B}_\beta$.*

This follows since if u has a small neighbour v, then $d(u) + d(v) \leq \beta$. But then, by observation (c), we must have $d^\beta(u) > \frac{3}{2}\beta$, which is only possible if both its other neighbours are in $\mathcal{B}_\beta$.

In the same way we can prove :

(e) *If a vertex of degree four has a small neighbour, then it also has at least two neighbours from $\mathcal{B}_\beta$.*

(f) *A vertex u of degree five has at least two big neighbours.* (otherwise we have $d^\beta(u) \leq 5 + 4 \cdot (\zeta_S^* - 1) + (\beta - 1) = 4\,\zeta_S^* + \beta \leq \frac{3}{2}\beta$, since $\beta \geq 8\,\zeta_S^*$).

We continue our analysis using the classical technique of *discharging*. Give each vertex v an initial charge $\mu(v) = 6\,d(v) - 36$. Since every face has degree at least three, $2\,|E| \geq 3\,|F|$. Hence, by Euler's Formula, $\sum_{v \in V} \mu(v) = 12\,|E| - 36\,|V| \leq -36\,|V| + 36\,|E| - 36\,|F| \leq -36\,\chi(S)$.

We further redistribute charges according to the following rules :

(R1) Each vertex of degree three that is adjacent to three big vertices receives a charge 6 from each of its neighbours.

(R2) Each vertex of degree three that is adjacent to two big vertices receives a charge 9 from each of its big neighbours.

(R3) Each vertex of degree four that is adjacent to four big vertices receives a charge 3 from each of its big neighbours.

(R4) Each vertex of degree four that is adjacent to three big vertices receives a charge 4 from each of its big neighbours.

(R5) Each vertex of degree four that is adjacent to two big vertices receives a charge 6 from each of its big neighbours.

(R6) Each vertex of degree five receives a charge 3 from each of its big neighbours.

Denote the resulting charge of an element $v \in V$ after applying rules (R1)–(R6) by $\mu'(v)$. Since the global charge has been preserved, we have $\sum_{v \in V} \mu'(v) \leq -36\,\chi(S)$. We will show that for most $v \in V$, $\mu'(v)$ is non-negative.

Combining observations (d)–(f) with rules (R1)–(R6) and our knowledge that $\mu(v) = 6\,d(v) - 36$, we find that $\mu'(v) = 0$ if $d(v) = 3, 4$, while $\mu'(v) \geq 0$ if $d(v) = 5$. If v is a small vertex with $d(v) \geq 6$, we have $\mu'(v) = \mu(v) = 6\,d(v) - 36 \geq 0$.

So we are left to consider vertices $v \in \mathcal{B}$. By the final paragraph of Subsection 2.1, the embedding of G in S imposes a circular order on the neighbours of each vertex v. By observation (a) we may assume that two consecutive vertices in this order are adjacent. If u is a neighbour of v, then by u^- (resp. u^+) we indicate the neighbour of v that comes before (resp. after) u in that order. Similarly, we denote by u^{--} (resp. u^{++}) the neighbour of v that comes before u^- (resp. after u^+) in the same order.

Let us take a vertex $v \in \mathcal{B}_{>\beta}$. We distinguish five different types of neighbours of v:

$$\begin{aligned}
N_3(v) &= \{\, u \in N(v) \mid d(u) = 3 \text{ and all neighbours of } u \text{ are big} \,\};\\
N_{4a}(v) &= \{\, u \in N(v) \mid d(u) = 4 \text{ and all neighbours of } u \text{ are big} \,\};\\
N_{4b}(v) &= \{\, u \in N(v) \mid d(u) = 4 \text{ and } u \text{ has exactly one small neighbour} \,\};\\
N_5(v) &= \{\, u \in N(v) \mid d(u) = 5 \,\};\\
N_6(v) &= \{\, u \in N(v) \mid d(u) \geq 6 \,\}.
\end{aligned}$$

Notice that each neighbour of v is in one of these sets. (For a neighbour of degree three, this follows from observation (d). And for a neighbour u of degree four, it follows from observation (e) that, since $v \in \mathcal{B}_{>\beta}$, if u has a small neighbour, then the remaining two neighbours are in $\mathcal{B}_\beta$.)

Moreover, by observation (d) we must have that if $u \in N_3(v)$, then $u^-, u^+ \in N_6(v)$. Similarly, if $u \in N_{4a}(v)$, then we also have $u^-, u^+ \in N_6(v)$. While if $u \in N_{4b}(v)$, then at least one of u^-, u^+ is in $N_6(v)$. Set $n_3 = |N_3(v)|$, $n_{4a} = |N_{4a}(v)|$, $n_{4b} = |N_{4b}(v)|$, $n_5 = |N_5(v)|$, and $n_6 = |N_6(v)|$. From the previous observation, we deduce

$$n_6 \;\geq\; n_3 + n_{4a} + \tfrac{1}{2}\, n_{4b}.$$

We also have, using $\mu(v) = 6\,d(v) - 36$ and applying rules (R1), (R3), (R4) and (R6), that

$$\mu'(v) \;=\; 6\,d(v) - 36 - 6\,n_3 - 3\,n_{4a} - 4\,n_{4b} - 3\,n_5.$$

Combining this with $d(v) = n_3 + n_{4a} + n_{4b} + n_5 + n_6$ and $3\,n_6 \geq 3\,n_3 + 3\,n_{4a} + \frac{3}{2}\,n_{4b}$, we find

$$\begin{aligned} \mu'(v) &= 6\,n_6 + 3\,n_{4a} + 2\,n_{4b} + 3\,n_5 - 36 \\ &\geq 3\,n_6 + 3\,n_3 + 6\,n_{4a} + \tfrac{7}{2}\,n_{4b} + 3\,n_5 - 36 \\ &\geq 3\,(n_6 + n_3 + n_{4a} + n_{4b} + n_5) - 36 \;\geq\; 3\,d(v) - 36 \;\geq\; 0. \end{aligned}$$

So for all $v \notin \mathcal{B}_\beta$ we have $\mu'(v) \geq 0$, and hence we must have

$$\sum_{v \in \mathcal{B}_\beta} \mu'(v) \;\leq\; -36\,\chi(S). \tag{1}$$

To derive the relevant consequence of that formula, we must make a detailed analysis of the neighbours of vertices in $\mathcal{B}_\beta$. We distinguish six different types of neighbours of a vertex $v \in \mathcal{B}_\beta$:

$$\begin{aligned} M_1(v) &= \{\, u \in N(v) \mid \{u^-, u^{--}, u^+, u^{++}\} \cap \mathcal{B}_\beta \neq \varnothing \,\}; \\ M_3(v) &= \{\, u \in N(v) \setminus M_1(v) \mid d(u) = 3 \,\}; \\ M_{4a}(v) &= \{\, u \in N(v) \setminus M_1(v) \mid d(u) = 4 \\ &\qquad \text{and } u^- \text{ or } u^+ \text{ have degree at least five} \,\}; \\ M_{4b}(v) &= \{\, u \in N(v) \setminus M_1(v) \mid d(u) = d(u^-) = d(u^+) = 4 \,\}; \\ M_5(v) &= \{\, u \in N(v) \setminus M_1(v) \mid d(u) = 5 \,\}; \\ M_6(v) &= \{\, u \in N(v) \setminus M_1(v) \mid d(u) \geq 6 \,\}. \end{aligned}$$

First observe that if $u \in N(v) \setminus M_1(v)$ is a small vertex, then u^- and u^+ both have degree at least four: Assume that u^- has degree three, then by observation (d) the neighbour w of u^- distinct from v and u is in $\mathcal{B}_\beta$. By observation (a), $w = u^{--}$, which contradicts the fact that $u \notin M_1(v)$. If u^+ has degree three, we find that $u^{++} \in \mathcal{B}_\beta$, which again contradicts $u \notin M_1(v)$. Also note that if $u \in M_3(v)$, then u^- and u^+ are both in $\mathcal{B}_{>\beta}$, where we use observation (d) and the fact that $u \notin M_1(v)$.

As a consequence, every neighbour of v is in exactly one set. Our aim in the following, in order to prove Lemma 2.2, is to show that most neighbours of vertices $v \in \mathcal{B}_\beta$ are in $M_{4b}(v)$.

We now evaluate the charge that a vertex $v \in \mathcal{B}_\beta$ has given to its neighbours. If $u \in M_1(v)$, then v gave at most $9 + 9 + 9 = 27$ to $\{u^-, u, u^+\}$; if $u \in M_3(v)$, then v gave at most $0 + 9 + 0 = 9$ to $\{u^-, u, u^+\}$; if $u \in M_{4a}(v)$, then v gave at most $3 + 6 + 6 = 15$ to $\{u^-, u, u^+\}$; if $u \in M_{4b}(v)$, then v gave at most $6 + 6 + 6 = 18$ to $\{u^-, u, u^+\}$; if $u \in M_5(v)$, then v gave at most $3 + 6 + 6 = 15$ to $\{u^-, u, u^+\}$; and, finally, if $u \in M_6(v)$, then v gave at most $6 + 0 + 6 = 12$

to $\{u^-, u, u^+\}$. Setting $m_1 = |M_1(v)|$, $m_3 = |M_3(v)|$, $m_{4a} = |M_{4a}(v)|$, $m_{4b} = |M_{4b}(v)|$, $m_5 = |M_5(v)|$, and $m_6 = |M_6(v)|$, we can conclude that v gave at most

$$\begin{aligned}&\tfrac{1}{3}\,(27\,m_1 + 9\,m_3 + 15\,m_{4a} + 18\,m_{4b} + 15\,m_5 + 12\,m_6)\\ &\qquad\qquad \le\; 9\,m_1 + 6\,m_{4b} + 5\,(m_3 + m_{4a} + m_5 + m_6) \;\le\; 5\,d(v) + 4\,m_1 + m_{4b}\end{aligned}$$

to its neighbourhood. This means that the remaining charge $\mu'(v)$ of a vertex $v \in \mathcal{B}_\beta$ must satisfy

$$\mu'(v) \;\ge\; (6\,d(v) - 36) - (5\,d(v) + 4\,m_1 + m_{4b}) \;=\; d(v) - m_{4b} - 4\,m_1 - 36.$$

By definition, $|M_1(v)|$ is at most four times the number of neighbours of v in $\mathcal{B}_\beta$. Consider the subgraph $G[\mathcal{B}_\beta]$ of G induced by $\mathcal{B}_\beta$. This graph can be embedded in S, and no face of such an embedding is incident with two or fewer edges. So Euler's Formula means that $G[\mathcal{B}_\beta]$ has at most $3\,|\mathcal{B}_\beta| - 3\,\chi(S)$ edges, and hence

$$\sum_{v\in\mathcal{B}_\beta} |M_1(v)| \;\le\; \sum_{v\in\mathcal{B}_\beta} 4\,d_{G[\mathcal{B}_\beta]}(v) \;=\; 8\,|E(G[\mathcal{B}_\beta])| \;\le\; 24\,|\mathcal{B}_\beta| - 24\,\chi(S).$$

Combining the last two inequalities with (1) gives

$$-36\,\chi(S) \ge \sum_{v\in\mathcal{B}_\beta} \mu'(v) \ge \sum_{v\in\mathcal{B}_\beta} (d(v) - |M_{4b}(v)|) - 4\,(24\,|\mathcal{B}_\beta| - 24\,\chi(S)) - 36\,|\mathcal{B}_\beta|.$$

Using that $\mathcal{B}_\beta \ne \varnothing$ (otherwise G contains structure (S1) or (S2)) and $\chi(S) \le 2$, this can be written as

$$\begin{aligned}\sum_{v\in\mathcal{B}_\beta} (d(v) - |M_{4b}(v)|) \;&\le\; 132\,|\mathcal{B}_\beta| - 132\,\chi(S)\\ &<\; 132\,|\mathcal{B}_\beta| + 132\,(2 - \chi(S)) \;\le\; 132\,(3 - \chi(S))\,|\mathcal{B}_\beta|.\end{aligned}$$

Define $X_0 = \mathcal{B}_\beta$ and $Y_0 = \bigcup_{v\in\mathcal{B}_\beta} M_{4b}(v)$. Note that the previous inequality can be written

$$e(X_0, V \setminus Y_0) \;<\; \zeta^*_S\,|X_0|. \tag{2}$$

Also observe that the pair (X_0, Y_0) satisfies the conditions (i) and (ii) for X and Y in part (S3) of Lemma 2.2:

(i) For all vertices $u \in M_{4b}(v)$, u, u^- and u^+ have degree four in G, and the fourth neighbour of u is in $\mathcal{B}_\beta = X_0$ by observation (e).

(ii) By observation (a) and the definition of $M_{4b}(v)$, all pairs of adjacent vertices $y, z \in Y_0$, satisfy $X_0^y = X_0^z$. Moreover, if $y, z \in Y_0$ share a neighbour

$w \notin X_0$, then t has degree four and, by observation (a), its neighbours distinct from y and z are in X_0^y and in X_0^z. This again gives $X_0^y = X_0^w = X_0^z$.

So we are done if the pair (X_0, Y_0) also satisfies condition (iii) (with $X = X_0$ and $Y = Y_0$). Suppose this is not the case. So there must exist a set $Z_1 \subseteq X_0$ with

$$e(Z_1, V \setminus Y_0) \ > \ e(Z_1, Y_0 \setminus Y_0^{Z_1}) + \zeta_S^* \, |Z_1|.$$

Define $X_1 = X_0 \setminus Z_1$ and $Y_1 = Y_0^{X_1}$. Again, by construction, (X_1, Y_1) satisfies conditions (i) and (ii) of (S3). If it does not satisfy condition (iii) we iterate the process (see Figure 2) and eventually obtain a pair (X_k, Y_k) satisfying conditions (i), (ii) and (iii) of (S3). We only need to check that $X_k \neq \varnothing$ and $Y_k \neq \varnothing$.

Figure 2: $X_i = X_{i-1} \setminus Z_i$ and $Y_i = Y_{i-1}^{X_i}$.

Let $1 \leq i \leq k$. Since $X_i = X_{i-1} \setminus Z_i$, we have

$$\begin{aligned} e(X_i, V \setminus Y_i) \ &= \ e(X_{i-1}, V \setminus Y_i) - e(Z_i, V \setminus Y_i) \\ &= \ e(X_{i-1}, V \setminus Y_{i-1}) + e(X_{i-1}, Y_{i-1} \setminus Y_i) - e(Z_i, V \setminus Y_{i-1}) - e(Z_i, Y_{i-1} \setminus Y_i) \\ &= \ e(X_{i-1}, V \setminus Y_{i-1}) - e(Z_i, V \setminus Y_{i-1}) + e(X_i, Y_{i-1} \setminus Y_i). \end{aligned}$$

Since $Y_i = Y_{i-1}^{X_i}$, every neighbour $u \in Y_{i-1} \setminus Y_i$ of a vertex from X_i has exactly one neighbour in Z_i (see Figure 2). Hence, $e(X_i, Y_{i-1} \setminus Y_i) = e(Z_i, Y_{i-1} \setminus Y_{i-1}^{Z_i})$. So we have

$$e(X_{i-1}, V \setminus Y_{i-1}) \ = \ e(X_i, V \setminus Y_i) + e(Z_i, V \setminus Y_{i-1}) - e(Z_i, Y_{i-1} \setminus Y_{i-1}^{Z_i}).$$

By the definition of Z_i, we have $e(Z_i, V \setminus Y_{i-1}) > e(Z_i, Y_{i-1} \setminus Y_{i-1}^{Z_i}) + \zeta_S^* \, |Z_i|$. Combining the last two expressions gives

$$e(X_{i-1}, V \setminus Y_{i-1}) \ > \ e(X_i, V \setminus Y_i) + \zeta_S^* \, |Z_i|.$$

Setting $Z^* = \bigcup_{1 \leq i \leq k} Z_i$, we have $e(X_k, V \setminus Y_k) < e(X_0, V \setminus Y_0) - \zeta_S^* \, |Z^*|$. As a consequence, using equation (2),

$$|Z^*| \ < \ \frac{e(X_0, V \setminus Y_0) - e(X_k, V \setminus Y_k)}{\zeta_S^*} \ \leq \ \frac{e(X_0, V \setminus Y_0)}{\zeta_S^*} \ < \ \frac{\zeta_S^* \, |X_0|}{\zeta_S^*} \ = \ |X_0|.$$

Since $X_k = X_0 \setminus Z^*$, this implies $|X_k| > 0$, which leads to $X_k \neq \varnothing$.

Finally, let $v \in X_k \neq \varnothing$ and assume $Y_k = \varnothing$. Taking $W = \{v\}$ in the inequality (iii) of (S3) (which by construction is satisfied by (X_k, Y_k)), we obtain $d(v) \leq \zeta_S^*$. Since v is a big vertex, $d(v) \geq \zeta_S^* + 1$. This contradiction means that we must have $Y_k \neq \varnothing$, which concludes the proof of Lemma 2.2.

■

3.2 Proof of Lemma 2.9

We recall the hypotheses of the lemma: We have a real number ζ; H is a multigraph; each vertex v of H has an associated integer $D(v)$; and for each edge e a positive number b_e is given. Finally, K is a real number satisfying $K \geq K_\zeta$ for a constant K_ζ whose existence will be determined in the proof.

The following three conditions are satisfied:

(H1') For all vertices v in H: $d(v) \le D(v) \le \beta$.

(H2') For all edges $e = uv$ in H: $b_e \ge \bigl(\frac{3}{2}\beta + K\bigr) - (D(u) - d(u)) - (D(v) - d(v))$.

(H3') For all non-empty subsets $W \subseteq V(H)$: $\sum_{w \in W} (D(w) - d(w)) \le e_H(W, V(H) \setminus W) + \zeta\,|W|$.

In the proof that follows, we will show that $K_\zeta = \max\{\,0, \frac{9}{2}\zeta\,\}$ will be a suitable choice. I.e., it will guarantee that under the conditions above, the vector $\vec{x} = (x_e)$, $x_e = 1/b_e$, will be in $\mathcal{MP}(H)$.

For an edge $e = uv$ in H, define

$$a_e \;=\; \bigl(\tfrac{3}{2}\beta + K\bigr) - (D(u) - d(u)) - (D(v) - d(v)) \qquad \text{and} \qquad y_e \;=\; \frac{1}{a_e}. \quad (3)$$

We will in fact prove that the vector $\vec{y} = (y_e)$ is in the matching polytope $\mathcal{MP}(H)$. Since $b_e \ge a_e$, we have $x_e = 1/b_e \le 1/a_e = y_e$. So, by Edmonds' characterisation of the matching polytope, if $\vec{y} \in \mathcal{MP}(H)$, this guarantees that $\vec{x} \in \mathcal{MP}(H)$, as required.

Applying condition (H3') to the set $W = \{v\}$ gives $D(v) - d(v) \le d(v) + \zeta$, which implies:

(a) *For all vertices $v \in V(H)$ we have $d(v) \ge \frac{1}{2}(D(v) - \zeta)$.*

Let $e = uv$ be an edge of H. If we use the estimate above for both u and v in the definition of a_e in (3), and recalling that $D(u), D(v) \le \beta$, we obtain

$$a_e \;\ge\; \tfrac{3}{2}\beta - \tfrac{1}{2}D(u) - \tfrac{1}{2}D(v) + K - \zeta \;\ge\; \tfrac{1}{2}\beta + K - \zeta.$$

On the other hand, if we use observation (a) to u only we get

$$a_e \;\ge\; d(v) + \tfrac{3}{2}\beta - \tfrac{1}{2}D(u) - D(v) + K - \tfrac{1}{2}\zeta \;\ge\; d(v) + K - \tfrac{1}{2}\zeta.$$

Since $K \ge 2\zeta$, the following two conclusions hold.

(b) *For all edges $e = uv$ in $E(H)$ we have $a_e \ge d(v) + \frac{1}{2}K$.*

(c) *For all edges $e \in E(H)$ we have $a_e \ge \frac{1}{2}\beta + \frac{1}{2}K$.*

Note that observation (c) also gives $b_e \ge a_e \ge \frac{1}{2}\beta$ for all $e \in E(H)$, as required.

Next notice that for any $\kappa \ge 0$, the function $x \mapsto \dfrac{x}{x+\kappa}$ is non-decreasing in x. Together with the fact that $d(v) \le \beta$ for all $v \in V(H)$ and observation (b), we find, since $K \ge 0$,

$$\sum_{e \ni v} \frac{1}{a_e} \;\le\; d(v) \cdot \frac{1}{d(v) + \frac{1}{2}K} \;\le\; 1,$$

which shows that

Claim 3.1 *For all vertices $v \in V(H)$ we have* $\sum_{e \ni v} y_e \leq 1$.

Using Theorem 2.7, all that remains is to prove that for all $W \subseteq V(H)$ with $|W| \geq 3$ and $|W|$ odd we have $\sum_{e \in E(W)} y_e \leq \frac{1}{2}(|W|-1)$. We actually will prove this for all $|W| \geq 3$. Note that we certainly can assume $E(W) \neq \varnothing$.

Using observation (b), we infer that:

$$\sum_{e \in E(W)} \frac{1}{a_e} \;\leq\; \tfrac{1}{2} \sum_{u \in W} \frac{d_{H[W]}(u)}{d(u) + \frac{1}{2}K} \;=\; \tfrac{1}{2} \sum_{u \in W} \Big(\frac{d(u)}{d(u) + \frac{1}{2}K} - \frac{d(u) - d_{H[W]}(u)}{d(u) + \frac{1}{2}K} \Big).$$

Since $\dfrac{d(u)}{d(u) + \frac{1}{2}K} \leq \dfrac{\beta}{\beta + \frac{1}{2}K}$ and $\dfrac{d(u) - d_{H[W]}(u)}{d(u) + \frac{1}{2}K} \geq \dfrac{d(u) - d_{H[W]}(u)}{\beta + \frac{1}{2}K}$, this implies

$$\sum_{e \in E(W)} \frac{1}{a_e} \;\leq\; \tfrac{1}{2} |W| \frac{\beta}{\beta + \frac{1}{2}K} - \tfrac{1}{2} \frac{e(W, W^c)}{\beta + \frac{1}{2}K}.$$

Here we used that $\sum_{u \in W} \big(d(u) - d_{H[W]}(u) \big) = e(W, W^c)$, where $W^c = V(H) \setminus W$.

If $e(W, W^c) \geq \beta$, we obtain, since $K \geq 0$,

$$\sum_{e \in E(W)} y_e \;\leq\; \tfrac{1}{2}(|W| - 1) \cdot \frac{\beta}{\beta + \frac{1}{2}K} \;\leq\; \tfrac{1}{2}(|W| - 1).$$

So we can assume in the following that $e(W, W^c) \leq \beta$, in which case Condition (H3') of Lemma 2.9 implies

$$\sum_{u \in W} (D(u) - d(u)) \;\leq\; e(W, W^c) + \zeta\,|W| \;\leq\; \beta + \zeta\,|W|.$$

For a vertex u set $c(u) = D(u) - d(u)$, and for a set of vertices U we define $c(U) = \sum_{u \in U} c(u)$. So we can write the above as $c(W) \leq \beta + \zeta\,|W|$.

In the following we use the fact that all a_e are large enough to find a bound for the sum $\sum_{e \in E(W)} a_e^{-1}$. To this aim, recall from definition (3) that $a_e = \big(\frac{3}{2}\beta + K\big) - c(u) - c(v)$ for all edges $e = uv$ in H. This gives

$$\sum_{e \in E(W)} a_e \;=\; \big(\tfrac{3}{2}\beta + K\big)\,|E(W)| - \sum_{u \in W} c(u)\, d_{H[W]}(u).$$

Since $d_{H[W]}(u) \leq d(u) = D(u) - c(u) \leq \beta - c(u)$, we have

$$\sum_{e \in E(W)} a_e \;\geq\; \big(\tfrac{3}{2}\beta + K\big)\,|E(W)| - \beta\, c(W) + \sum_{u \in W} c(u)^2.$$

Set $p = \min\limits_{uv \in E(W)} \left\{ \left(\frac{3}{2}\beta + K \right) - c(u) - c(v) \right\}$ and $q = \frac{3}{2}\beta + K$. This means that $q - p = \max\limits_{uv \in E(W)} \left\{ c(u) + c(v) \right\}$. Let $e = uv$ be an edge in $E(W)$ so that $c(u) + c(v) = q - p$. Then $c(u)^2 + c(v)^2 \geq \frac{1}{2}(q-p)^2$, and hence we can be sure that

$$\sum_{e \in E(W)} a_e \;\geq\; q\,|E(W)| - \beta\, c(W) + \tfrac{1}{2}(q-p)^2.$$

We now use this inequality and the following claim to bound $\sum\limits_{e \in E(W)} a_e^{-1}$.

Claim 3.2 *Let $r_1, \ldots, r_m$ be m real numbers so that $0 < p \leq r_1, \ldots, r_m \leq q$ and $\sum\limits_{1 \leq i \leq m} r_i \geq q\,m - (q-p)\,S$, for some $S \geq 0$. Then we have $\sum\limits_{1 \leq i \leq m} r_i^{-1} \leq \dfrac{S}{p} + \dfrac{m-S}{q}$.*

Proof The result is trivial if $p = q$, so suppose $p < q$. For any $1 \leq i \leq m$, set $c_i = \dfrac{q - r_i}{q - p}$. Now we have $0 \leq c_i \leq 1$ for all $1 \leq i \leq m$, and $\sum\limits_{1 \leq i \leq m} c_i \leq S$. Since the function $x \mapsto \dfrac{1}{x}$ is convex, we have that for $1 \leq i \leq m$,

$$\frac{1}{r_i} \;=\; \frac{1}{q - c_i\,(q-p)} \;=\; \frac{1}{c_i\,p + (1-c_i)\,q} \;\leq\; c_i\,\frac{1}{p} + (1-c_i)\,\frac{1}{q} \;=\; c_i\left(\frac{1}{p} - \frac{1}{q}\right) + \frac{1}{q}.$$

As a consequence,

$$\sum_{1 \leq i \leq m} \frac{1}{r_i} \;\leq\; \left(\frac{1}{p} - \frac{1}{q}\right) \sum_{1 \leq i \leq m} c_i + \frac{m}{q} \;\leq\; \left(\frac{1}{p} - \frac{1}{q}\right) S + \frac{m}{q} \;\leq\; \frac{S}{p} + \frac{m-S}{q}.$$

□

We set $R = \beta\, c(W) - \frac{1}{2}(q-p)^2$ and $S = \dfrac{R}{q-p}$. Using Claim 3.2, at this point we have

$$\sum_{e \in E(W)} \frac{1}{a_e} \;\leq\; \frac{S}{p} + \frac{|E(W)| - S}{q} \;=\; \frac{S\,(q-p)}{p\,q} + \frac{|E(W)|}{q} \;=\; \frac{R}{p\,q} + \frac{2\,|E(W)|}{3\,\beta + 2\,K}.$$

Notice that by condition (H3') of Lemma 2.9, $2\,|E(W)| \leq \sum\limits_{u \in W} D(u) - 2\,c(W) + \zeta\,|W| \leq \beta\,|W| - 2\,c(W) + \zeta\,|W|$. Hence we find

$$\sum_{e \in E(W)} \frac{1}{a_e} \;\leq\; \frac{\beta\,|W|}{3\,\beta + 2\,K} + \frac{R}{p\,q} - \frac{2\,c(W)}{3\,\beta + 2\,K} + \frac{\zeta\,|W|}{3\,\beta + 2\,K}. \tag{4}$$

Claim 3.3 *For $K \geq \frac{9}{2}\zeta$ we have*

$$\frac{R}{p\,q} - \frac{2\,c(W)}{3\,\beta + 2\,K} + \frac{\zeta\,|W|}{3\,\beta + 2\,K} \leq \frac{2\,K}{3\,(3\,\beta + 2\,K)}\,|W|$$

.

Proof Since $q = \frac{3}{2}\beta + K$, we only have to prove that $\dfrac{2\,R}{p} - 2\,c(W) + \zeta\,|W| \leq \frac{2}{3}\,K\,|W|$.

Let us write $q - p = \alpha\,\beta$, and so $p = \frac{1}{2}\,(3 - 2\,\alpha)\,\beta + K$ and $R = \beta\,c(W) - \frac{1}{2}\,\alpha^2\,\beta^2$. Using that $c(W) \leq \beta + \zeta\,|W|$, we have

$$\begin{aligned}
\frac{2\,R}{p} - 2\,c(W) + \zeta\,|W| &= \frac{2\,\beta\,c(W)}{p} - \frac{\alpha^2\,\beta^2}{p} - 2\,c(W) + \zeta\,|W| \\
&= c(W)\,\frac{2\,\beta - 2\,p}{p} - \frac{\alpha^2\,\beta^2}{p} + \zeta\,|W| \\
&\leq \frac{\beta}{p}\,(2\,\beta - 2\,p - \alpha^2\,\beta) + \zeta\,|W|\,\frac{2\,\beta - p}{p}.
\end{aligned}$$

As $2\,p = (3 - 2\,\alpha)\,\beta + 2\,K$, we have $2\,\beta - 2\,p - \alpha^2\,\beta = (-1 + 2\,\alpha - \alpha^2)\,\beta - 2\,K = -(\alpha - 1)^2\,\beta - 2\,K < 0$. Note that by observation (a), condition (H1'), and $K \geq \zeta$, we have $q - p \leq \beta + \zeta \leq \beta + K$, and hence $p \geq \frac{1}{2}\,\beta$. We can conclude $\dfrac{2\,R}{p} - 2\,c(W) + \zeta\,|W| \leq 3\,\zeta\,|W|$. As soon as $K \geq \frac{9}{2}\,\zeta$, we have $3\,\zeta\,|W| \leq \frac{2}{3}\,K\,|W|$, which completes the proof of the claim. □

Combining (4) and Claim 3.3, we obtain that for any $K \geq \frac{9}{2}\,\zeta$:

$$\begin{aligned}
\sum_{e \in E(W)} y_e \;=\; \sum_{e \in E(W)} \frac{1}{a_e} &\leq \frac{\beta\,|W|}{3\,\beta + 2\,K} + \frac{2\,K\,|W|}{3\,(3\,\beta + 2\,K)} \\
&= \frac{\beta + \frac{2}{3}\,K}{3\,\beta + 2\,K}\,|W| \;=\; \tfrac{1}{3}\,|W|.
\end{aligned}$$

Since $|W| \geq 3$ we have $\frac{1}{3}\,|W| \leq \frac{1}{2}\,(|W| - 1)$, which completes the proof of the lemma. ■

4 Proof of Theorem 1.5

Let β^*_S and ζ^*_S be as given in Lemma 2.2, and take $\beta_S = \beta^*_S$ and $\gamma_S = \left\lceil \frac{1}{4}\,(3\,\zeta^*_S + 37) \right\rceil$. Next take $\beta \geq \beta^*_S$. Suppose the theorem is false, and let the graph G embeddable in S be a counterexample with the minimum number of vertices, for some $B \subseteq V$. Similarly as in the proof of Theorem 1.4, we can assume that G is connected and has a cellular embedding in S.

We will need the following simple lemma.

Lemma 4.1
For any positive integer ℓ and surface S, if the bipartite graph $K_{3,\ell}$ can be embedded in S, then $\ell \le 6 - 2\,\chi(S)$.

Proof Again, first assume that there is a cellular embedding of $K_{3,\ell}$ in S with f faces. Then by Euler's Formula we obtain

$$\chi(S) \;=\; v(K_{3,\ell}) - e(K_{3,\ell}) + f \;=\; (3+\ell) - 3\,\ell + f.$$

Since the graph $K_{3,\ell}$ is bipartite, every face of this cellular embedding should contain an even number of vertices, and so each face has degree at least 4. It follows that $f \le \frac{1}{2}\,e(K_{3,\ell}) = \frac{1}{2}\cdot 3\,\ell$. We obtain that

$$\chi(S) \;\le\; (3+\ell) - 3\,\ell + \tfrac{1}{2}\cdot 3\,\ell \;=\; 3 - \tfrac{1}{2}\,\ell.$$

We infer that $\ell \le 6 - 2\,\chi(S)$, as required.

The case that there is no cellular embedding in S can be done exactly as in the beginning of the proof of Theorem 1.4 and leads to $\ell < 6 - \chi(S)$. $\square$

Suppose that G contains vertices u, v that are incident with a common face, and so that $d(u) + d(v) \le \beta$. Construct a graph G_1 by identifying u and v into a new vertex w, and observe that G_1 also has a cellular embedding in S. Set $V_1 = (V \setminus \{u, v\}) \cup \{w\}$, and notice that G_1 has strictly fewer vertices than G, and w has degree at most $d_G(u) + d_G(v) \le \beta$ in G_1. In other words, $w \in V_1^{\beta}$. If $v \notin B$, then set $B_1 = B$; otherwise, set $B_1 = (B \setminus \{u, v\}) \cup \{w\}$.

Every (B^{β}, B)-clique in G not containing u corresponds to a (B_1^{β}, B_1)-clique in G_1 of the same size. Since G was chosen as the smallest counterexample to Theorem 1.5, this means that every (B^{β}, B)-clique in G of size larger than $\frac{3}{2}\,\beta + \gamma_S$ must contain u. On the other hand, any (B^{β}, B)-clique in G containing u has size at most $1 + d^{\beta}(u)$.

We can conclude that for all pairs of vertices u, v in G incident with a common face and with $d(u) + d(v) \le \beta$, we have that u and v are in every (B^{β}, B)-clique of size larger than $\frac{3}{2}\,\beta + \gamma_S$, and these vertices satisfy $d^{\beta}(u), d^{\beta}(v) \ge \frac{3}{2}\,\beta + \gamma_S$.

Since $\beta \ge \beta_S^*$, we can apply Lemma 2.2. We use the notation from the lemma. Because of the observation above, conclusions (S1) and (S2) of that lemma are not possible. Hence we know that G contains $X, Y \subseteq V^{\beta}$ satisfying (S3) from the lemma. We recall the crucial properties:

(i) *Every vertex $y \in Y$ has degree at most four. Moreover, y is adjacent to exactly two vertices of X and the other neighbours of y have degree at most four as well.*

For $y \in Y$, let X^y be the set of its two neighbours in X. And for $W \subseteq X$, let Y^W be the set of vertices $y \in Y$ with $X^y \subseteq W$ (that is, the set of vertices of Y having their two neighbours from X in W).

(ii) *For all pairs of vertices $y, z \in Y$, if y and z are adjacent or have a common neighbour $w \notin X$, then $X^y = X^z$.*

(iii) *For all non-empty subsets $W \subseteq X$, we have the following inequality:*

$$e(W, V \setminus Y) \;\leq\; e(W, Y \setminus Y^W) + \zeta^*_S\,|W|.$$

By (i), it follows that all vertices in Y are in every (B^β, B)-clique of size larger than $\frac{3}{2}\,\beta + \gamma_S$. Hence in particular:

(a) *For every $y \in Y$ we have $d^\beta(y) \geq \frac{3}{2}\,\beta + \gamma_S$.*

Also by the properties of the vertices in Y according to (i) and (ii) we have for all $y \in Y$ and $X^y = \{x_1, x_2\}$:

$$\begin{aligned} d^\beta(y) \;&\leq\; 4 + 2\cdot(4-1) + (d(x_1)-1) + (d(x_2)-1) - |Y^{\{x_1,x_2\}} \setminus \{y\}| \\ &=\; 9 + d(x_1) + d(x_2) - |Y^{\{x_1,x_2\}}| \end{aligned}$$

(the term $|Y^{\{x_1,x_2\}} \setminus \{y\}|$ is subtracted, since these vertices are counted twice in $(d(x_1)-1) + (d(x_2)-1)$). Since $d(x_1), d(x_2) \leq \beta$, from observation (a) we can conclude that

(b) *for every pair $x_1, x_2 \in X$ we have $|Y^{\{x_1,x_2\}}| \leq \frac{1}{2}\,\beta - \gamma_S + 9$.*

We also must have that all pairs of vertices from Y are adjacent or have a common neighbour from B^β. By (ii), this proves that for every two vertices $y_1, y_2 \in Y$ we have $X^{y_1} \cap X^{y_2} \neq \varnothing$. As a consequence, if X' denotes the set of vertices of X with at least one neighbour in Y, and H denotes the graph with vertex set X' in which two vertices are adjacent if they have a common neighbour in Y, then H is either a triangle or a star.

Case 1. H is a triangle or H is a star with at most two leaves.
First suppose H is a triangle. Let $y \in Y$ with $(X')^y = \{x_1, x_2\}$, where $X' = \{x_1, x_2, x_3\}$. Then $Y = Y^{\{x_1,x_2\}} \cup Y^{\{x_1,x_3\}} \cup Y^{\{x_2,x_3\}}$, hence by observation (b), $|Y| \leq \frac{3}{2}\,\beta - 3\,\gamma_S + 27$. Since all vertices in Y are in every (B^β, B)-clique of size larger than $\frac{3}{2}\,\beta + \gamma_S$, we can estimate, using (i):

$$\begin{aligned} d^\beta(y) \;&\leq\; 2\cdot(4-1) + |X'| + |Y| + e(\{x_1,x_2\}, V \setminus (X' \cup Y)) \\ &\leq\; \tfrac{3}{2}\,\beta - 3\,\gamma_S + 36 + e(\{x_1,x_2\}, V \setminus (X' \cup Y)). \end{aligned}$$

Since $Y^{X'} = Y$ by definition of X', we have $e(X', Y \setminus Y^{X'}) = 0$. So using the inequality in (iii) with $W = X'$ leads to

$$e(\{x_1, x_2\}, V \setminus (X' \cup Y)) \;\leq\; e(X', V \setminus Y) \;\leq\; 3\,\gamma_S.$$

These two estimates give $d^{\beta}(y) \leq \frac{3}{2}\beta + 3\,\zeta^*_S + 36 - 3\,\gamma_S$, which contradicts observation (a), since $4\,\gamma_S \geq 3\,\zeta^*_S + 37$.

If H is a star with at most two leaves, then similar arguments will give a contradiction.

Case 2. H is a star with at least three leaves.
We denote by x the vertex of X which corresponds to the centre of the star H. Let C be a (B^{β}, B)-clique of size larger than $\frac{3}{2}\beta + \gamma_S$. Take a vertex $y \in Y$, and consider the set D of vertices of C that are distinct from y, from the neighbours of y, the neighbours of y's neighbours of degree at most four, and the neighbours of x. Since C is a (B^{β}, B)-clique containing Y, all the vertices of D are β-neighbours of every vertex of Y. Denote by $x_1, \ldots, x_k$ the vertices of X corresponding to the leaves of the star H, and consider the graph obtained from G by contracting the vertices of $Y^{\{x, x_i\}}$ to x_i, for every $i = 1, \ldots, k$. By property (ii) above, this graph contains $K_{k,|D|+1}$ as a minor. Since H has at least three leaves, so $k \geq 3$, and G is embeddable in S, by Lemma 4.1 we have $|D| \leq 5 - 2\,\chi(S)$. So, we can estimate

$$|C| \;\leq\; 1 + 4 + 2 \cdot (4-1) + (\beta - 1) + 5 - 2\,\chi(S) \;\leq\; \beta + 15 - 2\,\chi(S).$$

This contradicts the definition of C, since $\beta \geq \beta^*_S$ guarantees $\frac{1}{2}\beta > 15 - 2\,\chi(S)$. ■

Lemma 2.2 was proved with $\beta^*_S = 1056\,(3 - \chi(S))$ and $\zeta^*_S = 132\,(3 - \chi(S))$. Since the plane P has $\chi(P) = 2$, following the proof above means we can obtain $\beta_P = 1056$ and $\gamma_P = 109$ in Theorem 1.5 for the planar case. But it is clear that these values are far from best possible. Using more elaborate discharging arguments and more careful reasoning in the final parts of the proof of Lemma 2.2 can give significantly smaller values. Since our first goal is to show that we can obtain constant values for these results, we do not pursue this further.

5 Concluding Remarks and Discussion

5.1 About the Proof

The proof of our main theorem for major parts follows the same lines as the proof of Theorem 1.8 in [10]. In particular, the proof of that theorem also starts with a structural lemma comparable to Lemma 2.2, uses the structure of the graph to reduce the problem to edge-colouring a specific multigraph, and then applies (and extends) Kahn's approach to that multigraph. Of course, a difference is that Theorem 1.8 only deals with list colouring the square of a graph, but it is probably possible to generalise the whole proof to the case of list (B^β, B)-colouring. Nevertheless, there are some important differences in the proofs we feel deserve highlighting.

Lemma 2.2 is stronger than the comparable [10, Lemma 3.3]. We obtain a set Y of vertices with degree at most four and with a very specific structure of their neighbourhoods. This structure allows us to construct a multigraph H so that a standard list edge-colouring of H provides the information to colour the vertices in Y (see Lemma 2.4). In the lemma in [10], the vertices in the comparable set Y are only guaranteed to have degree at most $\Delta^{1/4}$, and knowledge about their neighbourhood is far sketchier. This means that the translation to list edge-colouring of a multigraph is not so clean; apart from the normal condition in the list edge-colouring of H (that adjacent edges need different colours), for each edge there may be up to $O(\Delta^{1/2})$ non-adjacent edges that also need to get a different colour. In particular this means that in [10], Kahn's result in Theorem 2.8 cannot be used directly. Instead, a new, stronger, version has to be proved that can deal with a certain number of non-adjacent edges that need to be coloured differently. Lemma 2.2 allows us to use Kahn's Theorem directly.

A second aspect in which our Lemma 2.2 is stronger is that in the final condition (S3) (iii), we have an 'error term' that is a constant times $|W|$. In [10] the comparable term is $\Delta^{9/10}\,|W|$, where Δ is the maximum degree of the graph. This in itself already means that the approach in [10] at best can give a bound of the type $\frac{3}{2}\,\Delta + o(\Delta)$. The fact that we cannot do better with the stronger structural result is because of the limitations of Kahn's Theorem, Theorem 2.8. If it would be possible to replace the condition in that theorem by a condition of the form 'the vector $\vec{x} = (x_e)$ with $x_e = \dfrac{1}{|L(e)| - K}$ for all $e \in E(H)$ is an element of $\mathcal{MP}(H)$', where K is some positive constant, the work in this paper would give an improvement for the bound in Theorem 1.4 to $\frac{3}{2}\,\beta + O(1)$. Note that our version of Lemma 2.9 is also already strong enough to support that case.

Lemma 2.2 also allows us to prove a bound $\frac{3}{2}\,\beta + O(1)$ for the (B^β, B)-clique number in Theorem 1.5. The important corollary that the square of a graph embeddable on a fixed surface has clique number at most $\frac{3}{2}\,\Delta + O(1)$ would have been impossible without the improved bound in the lemma.

Also Lemma 2.9 is stronger than its compatriot [10, Lemma 5.9]. The lemma in [10] only deals with the case $D(v) = \beta$ for all vertices v in H. Because of this, it can only be applied to the case that all vertices in H have maximum degree Δ. Some non-trivial trickery then has to be used to deal with the case that there are vertices in H of degree less than Δ. Apart from that difference, the proof of Lemma 2.9 is completely different from the proof in [10]. We feel that our new proof is more natural and intuitive, giving a clear relation between the lower bounds on the sizes of the lists and the upper bound of the sum of their inverses. The proof in [10] is more ad-hoc, using some non-obvious distinction in a number of different cases, depending on the size of W and the degrees of some vertices in W.

5.2 Further Work

We feel that our work is just the beginning of the study of general (A, B)-colouring. It should be possible to obtain deeper results taking into account the structure of the two sets A and B, and not just the degrees of the vertices. The following easy result is an example of this.

Theorem 5.1
Let $G = (V, E)$ be a planar graph and $A, B \subseteq V$ so that $\Delta(G; A, B) \neq 0$. Suppose that for every two distinct vertices in A we have that their distance in G is at least three. Then $ch(G; A, B) \leq \Delta(G; A, B) + 5$.

Proof Since G is planar, there exists a ordering $v_1, \ldots, v_n$ of the vertices so that each v_i has at most five neighbours in $\{v_1, \ldots, v_{i-1}\}$. We greedily colour the vertices $v_1, \ldots, v_n$ that are in B in that order. Note that each vertex has at most one neighbour from A.

When colouring the vertex v_i, we need to take into account its neighbours in $\{v_1, \ldots, v_{i-1}\}$, plus the neighbours in $\{v_1, \ldots, v_{i-1}\}$ of a vertex $a \in A$ adjacent to v_i (where that vertex a can be in $\{v_{1+1}, \ldots, v_n\}$). By construction of the ordering, there are at most five neighbours of v_i in $\{v_1, \ldots, v_{i-1}\}$. And a neighbour $a \in A$ has at most $\max\{\, 0, d_B(a) - 1\,\} \leq \Delta(G; A, B) - 1$ neighbours in $\{v_1, \ldots, v_{i-1}\}$ different from v_i. So the total number of forbidden colours when colouring v_i is at most $\Delta(G; A, B) + 4$. Since each vertex has $\Delta(G; A, B) + 5$ colours available, the greedy algorithm will always find a free colour. ■

Note that saying that the vertices in A have distance at least three is the same as saying that two different vertices in A have no common neighbour. We think that it is possible to generalise our main theorem and the theorem above in the following way. For $A, B \subseteq V$, let $k(G; A, B)$ be the maximum of $|N_B(a_1) \cap N_B(a_2)|$ over all $a_1, a_2 \in A$, $a_1 \neq a_2$.

Conjecture 5.2
Let S be a fixed surface. Then there exists a constant c_S so that for all graphs G embeddable on S, and $A, B \subseteq V$, we have

$$ch(G; A, B) \leq \Delta(G; A, B) + k(G; A, B) + c_S.$$

This conjecture would fit with our current proof of Theorem 1.4, the main part of which is a reduction of the original problem to a list edge-colouring problem. For this approach, Shannon's Theorem [29] that a multigraph with maximum degree Δ has an edge-colouring using at most $\left\lfloor \frac{3}{2}\,\Delta(G) \right\rfloor$ colours, forms a natural base for the bounds conjectured in Conjecture 1.1. If the relation between colouring the square of graphs embeddable on a fixed surface and edge-colouring multigraphs holds in a stronger sense, then Conjecture 5.2 forms a logical extension of Vizing's Theorem [30] that a multigraph with maximum degree Δ and maximum edge-multiplicity μ has an edge-colouring with at most $\Delta + \mu$ colours.

In Borodin *et al.* [5], a weaker version of Conjecture 5.2 for cyclic colouring of plane graphs was proved. Recall that if G is a plane graph, then Δ^* is the maximum number of vertices in a face. Let k^* denote the maximum number of vertices that two faces of G have in common.

Theorem 5.3 (Borodin, Broersma, Glebov & van den Heuvel [5])
For a plane graph G with $\Delta^* \geq 4$ and $k^* \geq 4$ we have $\chi^*(G) \leq \Delta^* + 3\,k^* + 2$.

6 Kahn's Work on List Edge-Colourings

As mentioned earlier, Theorem 2.8 is not explicitly stated in [17], but is implicit in the proof of the main result of that paper. In this final section, we give an overview of how the theorem can be obtained from the ideas in Kahn's paper.

The main result in [17] is that the list chromatic index is asymptotically equal to the fractional chromatic index of a multigraph :

Theorem 6.1 (Kahn [17])
For all $\varepsilon > 0$, there exists a Δ_ε so that for all $\Delta \geq \Delta_\varepsilon$ the following holds. If H is a multigraph with maximum degree at most Δ, then

$$\chi'_f(H) \leq \chi'(H) \leq ch'(H) \leq (1+\varepsilon)\,\chi'_f(H).$$

Here $\chi'(H)$ is the normal chromatic index (or edge-chromatic number) of H, $\chi'_f(H)$ is the fractional chromatic index of H, and $ch'(H)$ is the list chromatic index of H. The crucial step to relate this result to the matching polytope $\mathcal{MP}(H)$ is the following well-known characterisation of the fractional chromatic index:

$$\chi'_f(H) \;=\; \min\{\, \gamma > 0 \mid \text{the vector } (x_e)_{e\in E(H)} \text{ with } x_e = \gamma^{-1} \text{ is in } \mathcal{MP}(H)\,\}.$$

So Theorem 6.1 is just a special case of Theorem 2.8 if we set $|L(e)| \geq \dfrac{\chi'_f(H)}{1-\delta}$ for all edges e. (The second condition of Theorem 2.8 is automatically satisfied in that case, since trivially $\chi'_f(H) \geq \Delta(H)$.)

In order to prove Theorem 6.1, Kahn describes a randomised iterative procedure that colours the edges of H in a number of stages. During this procedure, the lists of available colours for each edge will change, and the lists will not be the same size for the uncoloured edges. This is why, roughly speaking, Kahn's actual proof deals with the more general case, as described in Theorem 2.8.

In order to give the reader a better understanding of the background of Kahn's approach, we give an overview of the crucial elements in the following subsections.

6.1 Hardcore Distributions

Hardcore distributions are distributions that originally arose in Statistical Physics, and that satisfy very natural conditions and generally provide strong independence properties allowing good sampling from a given family. Given a family of subsets $\mathcal{F}$ of a given set $\mathcal{E}$, a natural way of picking at random an element of $\mathcal{F}$ (or, in an other words, a probability distribution on $\mathcal{F}$) is as follows.

Let us suppose that each element e of $\mathcal{E}$ has been assigned a positive weight $\lambda_e > 0$. Then we pick each element $M \in \mathcal{F}$ with probability proportional to $\prod\limits_{e\in M} \lambda_e$. More precisely, the probability P_M of picking $M \in \mathcal{F}$ at random is given by

$$P_M \;=\; \frac{\prod\limits_{e\in M}\lambda_e}{\sum\limits_{M'\in\mathcal{F}}\;\prod\limits_{e\in M'}\lambda_e}.$$

We define the vector $\vec{x} = (x_e)_{e\in\mathcal{E}}$ by setting $x_e = \sum\limits_{M\in\mathcal{F},\, e\in M} P_M$. It is clear that x_e is the probability that a given random element of $\mathcal{F}$ contains the element e. The probability distribution $\{P_M\}$ is called a *hardcore* distribution with *activities* $\{\lambda_e\}$ and *marginals* $\{x_e\}$. The vector $\vec{x}$ is called the *marginal vector* associated with the hardcore distribution $\{P_M\}$.

Given a vector $\vec{x}$, it is not always true that $\vec{x}$ is the marginal vector of some hardcore distribution. Indeed if $\mathcal{P}(\mathcal{F})$ denotes the polytope defined by taking the convex hull of the characteristic vectors of the elements of $\mathcal{F}$[1], then the marginal vector $\vec{x}$ of a hardcore distribution is in $\mathcal{P}(\mathcal{F})$:

$$\vec{x} \;=\; \sum_{M\in\mathcal{F}} P_M\,\mathbf{1}_M.$$

This provides a necessary condition for a vector to be the marginal vector of a hardcore distribution. It is not difficult to prove that the activities λ_e corresponding to $\vec{x}$, if they exist, are unique.

From now on, let H be a given multigraph. We recall that $\mathcal{M}(H)$ and $\mathcal{MP}(H)$ are the family of matchings and the matching polytope of H, respectively. (So $\mathcal{M}(H)$ will play the role of the family $\mathcal{F}$ from above. And using the notation from above means $\mathcal{MP}(H) = \mathcal{P}(\mathcal{M}(H))$.)

We have the following theorem relating the matching polytope and hardcore distributions.

Theorem 6.2 (Lee [20], Rabinovich *et al.* **[26])**
For a given real number $0 < \delta < 1$, suppose $\vec{x}$ is a vector in $(1-\delta)\,\mathcal{MP}(H)$, for some multigraph H. Then there exists a unique family of activities λ_e such that $\vec{x}$ is the marginal vector of the hardcore distribution defined by the λ's. The hardcore distribution $\{P_M\}_{M\in\mathcal{M}(H)}$ is the unique distribution maximising the entropy function

$$\mathcal{H}(Q_M) \;=\; -\sum_{M\in\mathcal{M}(H)} Q_M\,\log(Q_M)$$

among all the distributions $\{Q_M\}_{M\in\mathcal{M}(H)}$ satisfying $\vec{x} = \sum\limits_{M\in\mathcal{M}(H)} Q_M\,\mathbf{1}_M$.

Kahn and Kayll proved in [18] a family of results, resulting in a long-range independence property for the hardcore distributions defined by a marginal vector $\vec{x}$ inside $(1-\delta)\,\mathcal{MP}(H)$, see [17]. We refer to the original papers of Kahn [16, 17] and Kahn and Kayll [18], and the book by Molloy and Reed [23] for more on these issues. We settle here for citing the following lemma.

Lemma 6.3 ([18, Lemma 4.1])
For every δ, $0 < \delta < 1$, there is a $\rho_\delta > 0$ such that if $\{P_M\}$ is a hardcore distribution with marginal vector $\vec{x} \in (1-\delta)\,\mathcal{MP}(H)$, then for all $u, v \in V(H)$:

$$\mathbf{Pr}(M \textit{ does not touch } u \textit{ and } v) \;>\; \rho_\delta.$$

[1] Recall that the characteristic vector, $\mathbf{1}_M$, of a given element $M\in\mathcal{F}$ is the $|\mathcal{E}|$-dimensional vector $(y_e)_{e\in\mathcal{E}}$ such that $y_e = 1$ if $e\in M$ and $y_e = 0$ otherwise.

6.2 Hardcore Distributions and Edge-Colouring

We present here Kahn's algorithm for list edge-colouring of multigraphs first introduced and analysed in [17]. We continue to use the notations of the previous subsection. In particular, we suppose that H is a multigraph and L a list assignment of colours to the edges of H so that the conditions of Theorem 2.8 are satisfied. By Lemma 6.2 there exists a hardcore distribution $\{P_M\}$ with marginals $\{|L(e)|^{-1}\}_{e\in E(H)}$ which in addition satisfies the property of Lemma 6.3. Let $\{\lambda_e\}$ be the activities on the edges (which are unique by Theorem 6.2) corresponding to this distribution. An extra condition is indeed true: for every subgraph H^* of H it is possible to find a hardcore distribution $\{P^*_M\}$ with corresponding marginals $|L(e)|^{-1}$ for $e \in E(H^*)$. The corresponding activities λ^*_e will in general be different from the λ_e's.

The algorithm works as follows: Let $\mathcal{L} = \bigcup\limits_{e\in E(H)} L(e)$ be the union of the colours in the lists. For each colour α, let us define the *colour graph* H_α to be the graph containing all the edges whose lists contain the colour α. And denote by $\{\lambda_{\alpha,e}\}$ the activities producing the hardcore distribution with marginals $|L(e)|^{-1}$ for $e \in E(H_\alpha)$. The colouring procedure consists in a finite number of iterations of a procedure that we may call *naive colouring*. At step i of the iteration, we are left with subgraphs H^i_α containing some uncoloured edges whose lists contain the colour α. Of course we have $H^i_\alpha \subseteq H^{i-1}_\alpha \subseteq \cdots \subseteq H^0_\alpha = H_\alpha$.

The naive colouring procedure at step $i+1$ consists of the following substeps.

(a) For each colour $\alpha \in \mathcal{L}$, choose independently of the other colours a random matching $M^{i+1}_\alpha \subseteq E(H^i_\alpha)$. The distribution of the matchings is the hardcore distribution defined by the activities $\lambda_{\alpha,e}$ on the edges $e \in E(H^i_\alpha)$.

(b) If an edge e is in one or more of the M^{i+1}_α's, then choose one of the colours from those, chosen uniformly at random, and colour e with that colour.

(c) For each colour α, form H^{i+1}_α by removing from H^i_α all the edges that received some colour at this stage, and all vertices that are incident to one of the edges coloured with α. (While removing a vertex, all the edges incident to it are of course removed as well.)

Note that the process above can also be described in terms of subgraphs H^i of the original multigraph H, where the edges of H^i are the edges that are still uncoloured after step i, and each edge e in H^i has a list of colours $L^i(e)$ formed by all colours α for which $e \in E(H^i_\alpha)$. Also note that the activities $\lambda_{\alpha,e}$ remain unchanged all through the process (but the edge sets on which they are applied change).

A sufficient number of iterations of the naive colouring procedure results in a graph H^I, consisting of all the uncoloured edges at this step, such that H^I has maximum degree T, for some integer T, and that the list sizes are at least $2T$ (i.e., each uncoloured edge is in at least $2T$ of the H^I_α's). Remember that the conditions of Theorem 2.8 imply that the lists are quite large at the beginning. At this stage it is easy to finish the procedure by a simple greedy algorithm.

The heart of the analysis of the above algorithm in Kahn's approach is the following strong lemma, the proof of which can be found in [17].

Lemma 6.4 (Kahn [17, Lemma 3.1])
For each $K > 0$ and $0 < \eta < 1$, there are constants $0 < \xi_{K,\eta} \leq \eta$ and $\Delta_{K,\eta}$ such that the following holds for all $\Delta \geq \Delta_{K,\eta}$. Let H be a multigraph with lists $L(e)$ of colours for each edge e. For each colour α, define the colour graph H_α as above. Finally, for each colour α we are given a hardcore distribution with activities $\{\lambda_{\alpha,e}\}_{e\in E(H_\alpha)}$ and marginals $\{x_{\alpha,e}\}_{e\in E(H_\alpha)}$. Suppose the following conditions are satisfied:

- *for every vertex v: $d_H(v) \leq \Delta$;*
- *for every colour α and edge $e \in E(H_\alpha)$: $\lambda_{\alpha,e} \leq \dfrac{K}{\Delta}$; and*
- *for every edge e: $1 - \xi_{K,\eta} \leq \sum\limits_{\alpha\in L(e)} x_{\alpha,e} \leq 1 + \xi_{K,\eta}$.*

Then with positive probability the naive colouring procedure described above gives matchings $M_\alpha \subseteq E(H_\alpha)$ for all colours α, so that if we set $H^* = H - \bigcup_{\alpha'} M_{\alpha'}$, $H^*_\alpha = H_\alpha - V(M_\alpha) - \bigcup_{\alpha'} M_{\alpha'}$, and form lists $L^*(e)$ for all edges $e \in E(H^*)$ by removing no longer allowed colours from $L(e)$, we have:

- *for every vertex v: $d_{H^*}(v) \leq \dfrac{1+\eta}{1+\xi_{K,\eta}}\, \mathrm{e}^{-1}\,\Delta$;[2] and*
- *for every edge e in H^*: $1 - \eta \leq \sum\limits_{\alpha\in L^*(e)} x^*_{\alpha,e} \leq 1 + \eta$.*

Here $\{x^*_{\alpha,e}\}_{e\in E(H^*_\alpha)}$ are the marginals associated to $\lambda_{\alpha,e}$ in H^*_α.

In other words, the lemma guarantees that after one iteration of the naive colouring procedure, with positive probability the multigraph formed by the uncoloured edges has maximum degrees bounded by $\dfrac{1+\eta}{1+\xi_{K,\eta}}\, \mathrm{e}^{-1}\,\Delta$, while the sum of the marginal probabilities $x^*_{\alpha,e}$ for every edge e will be close to 1.

In the next subsection we will combine all the strands and use the lemma above to conclude the proof of Theorem 2.8.

[2] To avoid confusion between an edge 'e' and the base of the natural logarithms 2.718..., we will use the roman letter 'e' for the latter one.

6.3 Completing the Proof of Theorem 2.8 — after Kahn

Let $0 < \delta < 1$ and $\mu > 0$. Then we should prove the existence of a $\Delta_{\delta,\mu}$ such that for $\Delta \geq \Delta_{\delta,\mu}$ the following holds. Let H be a multigraph and L a list assignment of colours to the edges of H so that

- for every vertex v: $d_H(v) \leq \Delta$;
- for all edges $e \in E(H)$: $|L(e)| \geq \mu\,\Delta$;
- the vector $\vec{x} = (x_e)$ with $x_e = \dfrac{1}{|L(e)|}$ for all $e \in E(H)$ is an element of $(1-\delta)\,\mathcal{MP}(H)$.

Then there should exist a proper edge-colouring of H, where each edge receives a colour from its own list.

For each colour α, define the colour graph H_α as in the previous subsection. For each colour α and edge e, set $x_{\alpha,e} = x_e = \dfrac{1}{|L(e)|}$, and let $\{\lambda_{\alpha,e}\}_{e\in E(H_\alpha)}$ be the activities associated with the marginals $x_{\alpha,e}$ on H_α.

Since for every edge e we have $\sum\limits_{\alpha\in L(e)} x_{\alpha,e} = \sum\limits_{\alpha\in L(e)} |L(e)|^{-1} = 1$, we certainly know that

- for every edge e and $\xi > 0$: $1-\xi \leq \sum\limits_{\alpha\in L(e)} x_{\alpha,e} \leq 1+\xi$.

We next bound the activities $\lambda_{\alpha,e}$, using Lemma 6.3. First observe that for all α the vector $(x_{\alpha,e})_{e\in E(H_\alpha)}$ is in $(1-\delta)\,\mathcal{MP}(H_\alpha)$. So by Lemma 6.3 there is a constant ρ_δ such that if M_α is chosen according to the hardcore distribution with marginals $\{x_{\alpha,e}\}$ on H_α, then for all $u,v \in E(H_\alpha)$ we have $\mathbf{Pr}(M_\alpha \text{ does not touch } u \text{ and } v) > \rho_\delta$. Let $e = uv$ be an edge of H_α. Then we have

$$x_{\alpha,e} = \mathbf{Pr}(M_\alpha \text{ contains } e) = \lambda_{\alpha,e}\cdot\mathbf{Pr}(M_\alpha \text{ does not touch } u \text{ and } v) > \lambda_{\alpha,e}\cdot\rho_\delta.$$

Given the fact that $x_{\alpha,e} = \dfrac{1}{|L(e)|}$ and $|L(e)| \geq \mu\,\Delta$, and setting $K_{\delta,\mu} = \dfrac{1}{\rho_\delta\,\mu}$, we infer that $\lambda_{\alpha,e} < \dfrac{x_{\alpha,e}}{\rho_\delta} \leq \dfrac{1}{\rho_\delta\,\mu\,\Delta} = \dfrac{K_{\delta,\mu}}{\Delta}$. We have show that there exists a $K_{\delta,\mu} > 0$ so that

- for every colour α and edge $e \in E(H_\alpha)$: $\lambda_{\alpha,e} \leq \dfrac{K_{\delta,\mu}}{\Delta}$.

Suppose we repeat the naive colouring procedure from the previous subsection $s = s_{K_{\delta,\mu}}$ times (where $s_{K_{\delta,\mu}}$ is a fixed constant to be made more precise later). Let H^i be the subgraph of H formed by the edges that are as yet uncoloured at step i, and for each $e \in E(H^i)$ let $L^i(e)$ be the list of colours from $L(e)$ that are still allowed for e at that stage.

Set $\eta_s = 1 - \mathrm{e}^{-1}$ and recursively for $i = s-1, \ldots, 1$, set $\eta_i = \xi_{K_{\delta,\mu},\eta_{i+1}}$, where $\xi_{K_{\delta,\mu},\eta_{i+1}}$ is the function given by Lemma 6.4. Let $\Delta_{\delta,\mu} = \max\limits_{i=1,\ldots,s} \Delta_{K_{\delta,\mu},\eta_i}$ ($\Delta_{K_{\delta,\mu},\eta_i}$ according to Lemma 6.4 again), and $\eta_0 = 0$. By applying Lemma 6.4 and the observations above, we can ensure inductively, starting from $i = 0$, that for $\Delta \geq \Delta_{\mu,\delta}$ the following conditions are satisfied for all $i = 0, \ldots, s$, with positive probability :

- for all vertices v : $d_{H^i}(v) \leq T_i$, where $T_0 = \Delta$ and $T_i = \dfrac{1+\eta_i}{1+\eta_{i-1}}\,\mathrm{e}^{-1}\,T_{i-1}$ for $i \geq 1$; and
- for all edges $e \in E(H^i)$: $1 - \eta_i \leq \sum\limits_{\alpha \in L^i(e)} x^i_{\alpha,e} \leq 1 + \eta_i$, where $\{x^i_{\alpha,e}\}_{e \in E(H^i)}$ are the marginals associated to the hardcore distribution with activities $\lambda_{\alpha,e}$ in H^i_α.

It follows that with positive probability after s steps we have

- for all vertices v : $d_{H^s}(v) \leq (2 - \mathrm{e}^{-1})\,\mathrm{e}^{-s}\,\Delta$; and
- for all edges $e \in E(H^s)$: $\mathrm{e}^{-1} \leq \sum\limits_{\alpha \in L^s(e)} x^s_{\alpha,e} \leq 2 - \mathrm{e}^{-1}$.

We note that for an edge $e = uv$,

$$x^s_{\alpha,e} \;=\; \lambda_{\alpha,e} \cdot \mathbf{Pr}(M^s_\alpha \text{ does not touch } u \text{ and } v) \;\leq\; \lambda_{\alpha,e},$$

which implies that $x^s_{\alpha,e} \leq \lambda_{\alpha,e} \leq \dfrac{K_{\delta,\mu}}{\Delta}$. We infer that for all $e \in E(H^s)$,

$$|L^s(e)| \;=\; |\{\,\alpha \mid e \in E(H^s_\alpha)\,\}| \;\geq\; \frac{\Delta}{\mathrm{e}\,K_{\delta,\mu}}.$$

Let $T = \dfrac{\Delta}{2\,\mathrm{e}\,K_{\delta,\mu}}$. It is now clear that if we choose s so that $2\,\mathrm{e}^{-s} \leq \dfrac{1}{2\,\mathrm{e}\,K_{\delta,\mu}}$ (i.e., set $s = s_{K_{\delta,\mu}} \geq \ln(4\,K_{\delta,\mu}) + 1$), we can ensure with positive probability that

$$d_{H^s}(v) \;\leq\; T \quad \text{for all } v \in V(H^s),$$

and

$$|L^s(e)| \;\geq\; 2\,T \quad \text{for each } e \in E(H^s).$$

Hence we can apply a greedy algorithm to H^s to extend the colouring obtained by the naive colouring procedure so far to a colouring of the whole graph.

References

[1] G. Agnarsson and M.M. Halldórsson, *Coloring powers of planar graphs.* SIAM J. Discrete Math. **16** (2003), 651–662.

[2] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*. Macmillan, London and Elsevier, New York, 1976.

[3] O.V. Borodin, *Solution of the Ringel problem on vertex-face coloring of planar graphs and coloring of 1-planar graphs* (in Russian). Metody Diskret. Analyz. **41** (1984), 12–26.

[4] O.V. Borodin, H.J. Broersma, A. Glebov, and J. van den Heuvel, *Minimal degrees and chromatic numbers of squares of planar graphs* (in Russian). Diskretn. Anal. Issled. Oper. Ser. 1 **8**, no. 4 (2001), 9–33.

[5] O.V. Borodin, H.J. Broersma, A. Glebov, and J. van den Heuvel, *A new upper bound on the cyclic chromatic number*. J. Graph Theory **54** (2007), 58–72.

[6] O.V. Borodin, D.P. Sanders, and Y. Zhao, *On cyclic colorings and their generalizations*. Discrete Math. **203** (1999), 23–40.

[7] N. Cohen and J. van den Heuvel, *An exact bound on the clique number of the square of a planar graph*. In preparation.

[8] R. Diestel, *Graph Theory*, 3rd edition. Springer-Verlag, Berlin, 2005.

[9] J. Edmonds, *Maximum matching and a polyhedron with* $0,1$*-vertices*. J. Res. Nat. Bur. Standards Sect. B **69B** (1965), 125–130.

[10] F. Havet, J. van den Heuvel, C. McDiarmid, and B. Reed, *List colouring squares of planar graphs*. Submitted (2008). Preprint available at `www.cdam.lse.ac.uk/Reports/Files/cdam-2008-09.pdf`.

[11] P. Hell and K. Seyffarth, *Largest planar graphs of diameter two and fixed maximum degree*. Discrete Math. **111** (1993), 313–322.

[12] T.J. Hetherington and D.R. Woodall, *List-colouring the square of a* K_4*-minor-free graph*. Discrete Math. **308** (2008), 4037–4043.

[13] J. van den Heuvel and S. McGuinness, *Coloring the square of a planar graph*. J. Graph Theory **42** (2003), 110–124.

[14] T.R. Jensen and B. Toft, *Graph Coloring Problems*. John-Wiley & Sons, New York, 1995.

[15] T.K. Jonas, *Graph coloring analogues with a condition at distance two :* $L(2,1)$*-labelings and list* λ*-labelings*. Ph.D. Thesis, University of South Carolina, 1993.

[16] J. Kahn, *Asymptotics of the chromatic index for multigraphs*. J. Combin. Theory Ser. B **6**8 (1996), 233–254.

[17] J. Kahn, *Asymptotics of the list-chromatic index for multigraphs*. Random Structures Algorithms **17** (2000), 117–156.

[18] J. Kahn and P.M. Kayll, *On the stochastic independence properties of hard-core distributions*. Combinatorica **17** (1997), 369–391.

[19] A.V. Kostochka and D.R. Woodall, *Choosability conjectures and multicircuits*. Discrete Math. **240** (2001), 123–143.

[20] C.W. Lee, *Some recent results on convex polytopes*. In: J.C. Lagarias and M.J. Todd, eds., *Mathematical Developments Arising from Linear Programming*. Contemp. Math. **114** (1990), 3–19.

[21] K.-W. Lih, W.F. Wang and X. Zhu, *Coloring the square of a K_4-minor free graph*. Discrete Math. **269** (2003), 303–309.

[22] B. Mohar and C. Thomassen, *Graphs on Surfaces*. Johns Hopkins University Press, Baltimore, 2001.

[23] M. Molloy and B. Reed, *Graph Colouring and the Probabilistic Method*. Springer-Verlag, Berlin, 2002.

[24] M. Molloy and M.R. Salavatipour, *A bound on the chromatic number of the square of a planar graph*. J. Combin. Theory Ser. B **94** (2005), 189–213.

[25] O. Ore and M.D. Plummer, *Cyclic coloration of plane graphs*. In: *Recent Progress in Combinatorics; Proceedings of the Third Waterloo Conference on Combinatorics*. Academic Press, San Diego (1969) 287–293.

[26] Y. Rabinovich, A. Sinclair and A. Wigderson, *Quadratic dynamical systems*. In: *Proceedings of the 33rd Annual Conference on Foundations of Computer Science (FOCS)*, (1992), 304–313.

[27] D.P. Sanders and Y. Zhao, *A new bound on the cyclic chromatic number*. J. Combin. Theory Ser. B **83** (2001), 102–111.

[28] A. Schrijver, *Combinatorial Optimization; Polyhedra and Efficiency*. Springer-Verlag, Berlin, 2003.

[29] C.E. Shannon, *A theorem on colouring lines of a network*. J. Math. Physics **28** (1949), 148–151.

[30] V.G. Vizing, *On an estimate of the chromatic class of a p-graph* (in Russian). Metody Diskret. Analiz. **3** (1964), 25–30.

[31] G. Wegner, *Graphs with given diameter and a coloring problem*. Technical Report, University of Dortmund, 1977.

[32] S.A. Wong, *Colouring graphs with respect to distance*. M.Sc. Thesis, Department of Combinatorics and Optimization, University of Waterloo, 1996.