# On Generalized Middle Level Problem*

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**Abstract**

Let $G_n^k$ be the subgraph of the hypercube $Q_n$ induced by levels between $k$ and $n-k$, where $n \geq 2k+1$ is odd. The well-known middle level conjecture asserts that $G_{2k+1}^k$ is Hamiltonian for all $k \geq 1$. We study this problem in $G_n^k$ for fixed $k$. It is known that $G_n^0$ and $G_n^1$ are Hamiltonian for all odd $n \geq 3$. In this paper we prove that also $G_n^2$ is Hamiltonian for all odd $n \geq 5$, and we conjecture that $G_n^k$ is Hamiltonian for every $k \geq 0$ and every odd $n \geq 2k+1$.

## 1 Introduction

Let $G_n^k$ be the subgraph of the $n$-dimensional hypercube $Q_n$ induced by the vertices in levels between $k$ and $n-k$, where $n \geq 2k+1$ is odd. The level $i$ consists of vertices with exactly $i$ 1's. Note that $n$ is required to be odd in order to have bipartite classes of equal size in $G_n^k$.

The well-know middle level conjecture, attributed to Havel [6], asserts that the graph $G_{2k+1}^k$ consisting of two middle levels $k$ and $k+1$ of $Q_{2k+1}$ is Hamiltonian for all $k \geq 1$. This graph is a notorious example of a connected vertex transitive graph, all of which were conjectured by Lovász [11] to have Hamiltonian paths.

Despite many attempts, the middle level problem remains open. The conjecture was verified for $k \leq 11$ by Moews and Reid in unpublished work

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in 1990. Then it was extended by Shields, Shields, and Savage [15, 16] also for $12 \leq k \leq 17$.

One possible relaxation of this problem is to show that $G^k_{2k+1}$ at least contains long cycles. Savage and Winkler [13] showed that $G^k_{2k+1}$ has a cycle of length at least $0.867|V(G^k_{2k+1})|$. The best lower bound by Johnson [9] shows that $G^k_{2k+1}$ is "asymptotically" Hamiltonian: it contains a cycle of length $(1-o(1))|V(G^k_{2k+1})|$. On the other hand, Horák, Kaiser, Rosenfeld, and Ryjáček [7] showed that $G^k_{2k+1}$ has a closed spanning walk in which every vertex appears at most twice, by proving that the prism over $G^k_{2k+1}$ is Hamiltonian.

Our approach is to study this problem for the graph $G^k_n$ where $0 \leq k \leq (n-1)/2$ is fixed. It is well-known that $G^0_n = Q_n$ is Hamiltonian for every $n \geq 2$. El-Hashash and Hassan [5], and independently (in a more general setting) Locke and Stong [12] proved that $G^1_n$ is Hamiltonian for all odd $n \geq 3$. As a first step towards the general problem, in this paper we prove that also $G^2_n$ is Hamiltonian for all odd $n \geq 5$. Now, it becomes naturally to conjecture:

**Conjecture 1.** *$G^k_n$ is Hamiltonian for every $k \geq 0$ and every odd $n \geq 2k+1$.*

A different approach to generalize the middle level problem was proposed, as far as we know, independently by Dejter, Cedeno, and Jaurequi [2] and by Hurlbert [8] who studied Hamiltonian cycles in the graph $H^k_n$ consisting of level $k$ and $n-k$ of $Q_n$, and edges joining a vertex from level $k$ with a vertex from level $n-k$ if their Hamming distance (distance in $Q_n$) is $n-2k$. In other words, $H^k_n$ is the cover graph of the ordered set consisting of level $k$ and $n-k$ of the Boolean lattice $B_n$ with the order inherited from $B_n$. For other results on ordered sets obtained by removing selected levels of the Boolean lattice, called Boolean layer cakes, we refer to a survey of Schmidt [14].

# 2 Preliminaries

Let $[n]$ denote the set $\{1, \ldots, n\}$. For a binary vector $v \in \{0,1\}^n$ and $i \in [n]$ we denote by $v[i]$ the $i$-th coordinate of $v$. For vectors $u, v \in \{0,1\}^n$ let $u \oplus v$ denote the vector obtained by the coordinate-wise addition modulo 2 of $u$ and $v$. The *$n$-dimensional hypercube* $Q_n$ is a (bipartite) graph with all binary vectors of length $n$ as vertices and with edges joining every two vertices that differ in exactly one coordinate, i.e.

$$V(Q_n) = \{0,1\}^n \text{ and } E(Q_n) = \{uv; |\Delta(u,v)| = 1\},$$

where $\Delta(u, v) = \{i \in [n]; u[i] \neq v[i]\}$. Thus the distance of vertices $u$ and $v$ is $d(u, v) = |\Delta(u, v)|$. The distance of two edges $uv$ and $xy$ is the minimum distance between a vertex of $uv$ to a vertex of $xy$. A vertex $v$ is said to be *even* (*odd*) if it has even (odd) weight. The *weight* of $v$ is the number of 1's in $v$. Note that vertices of each parity form bipartite classes of $Q_n$. Consequently, $u$ and $v$ have the same parity if and only if $d(u, v)$ is even.

Let $\mathbf{0}, \mathbf{1} \in V(Q_n)$ be the vertices of all 0's and 1's, respectively. For $i \in [n]$ we denote by $e_i$ the vertex containing 1 exactly in the $i$-th coordinate. Note that each vertex $e_i$ is adjacent to $\mathbf{0}$. An $i$-th *level* of $Q_n$ for $0 \leq i \leq n$ is the set of vertices of weight $i$. An *antipodal* vertex to a vertex $u \in V(Q_n)$ is the vertex denoted by $\overline{u}$ such that $\overline{u}[i] = \overline{u[i]}$ for all $i \in [n]$, that is $d(u, \overline{u}) = n$.

If adjacent vertices $u$ and $v$ of $Q_n$ differ in the $i$-th coordinate, then $u \oplus v = e_i$ and we say that the edge $uv \in E(Q_n)$ has *direction* $i$. By removing all edges of a fixed direction $i \in [n]$, the hypercube $Q_n$ is split into two (induced) subgraphs isomorphic to $Q_{n-1}$. We say that $Q_n$ is split *along the direction* $i$ into *subcubes* $Q^0_{n-1}$ and $Q^1_{n-1}$. For $a \in \{0, 1\}$ the subcube $Q^a_{n-1}$ is induced by all vertices $u \in V(Q_n)$ with $u[i] = a$. Furthermore, by splitting $Q^0_{n-1}$ and $Q^1_{n-1}$ along another direction $j \in [n] \setminus \{i\}$ we obtain four subcubes $Q^{00}_{n-2}$, $Q^{01}_{n-2}$, $Q^{10}_{n-2}$, and $Q^{11}_{n-2}$. Note that for $a, b \in \{0, 1\}$ the subcube $Q^{ab}_{n-2}$ is induced by all vertices $u \in V(Q_n)$ with $u[i] = a$ and $u[j] = b$.

We consider a *path* $P$ to be a nonempty sequence of distinct vertices such that every two consecutive vertices are adjacent. If $a$ and $b$ are the first and the last vertices of $P$, respectively, we say that $P$ is an *ab-path* and $a$, $b$ are its *endvertices*. Assume that an $ab$-path $P$ and an $cd$-path $R$ are (vertex) disjoint. If $b$ and $c$ are adjacent, then the concatenation of $P$ and $R$ is an $ad$-path. If $P$ contains consecutive vertices $x$ and $y$ such that both $x$, $c$ and $y$, $d$ are adjacent, then by inserting $R$ into $P$ between $x$ and $y$ we obtain an $ab$-path containing vertices $P \cup R$. A *reversed* path of an $ab$-path $P$ is the $ba$-path obtained by the reversed sequence.

It is well known that the hypercube $Q_n$ for every $n \geq 2$ is Hamiltonian and also Hamiltonian-laceble; that is, there is a Hamiltonian path between every two vertices of opposite parity. We will also need several simple results on Hamiltonian cycles and paths in the hypercube with some removed vertices. The case of one removed vertex was described by Lewinter and Widulski [10].

**Proposition 1.** *If distinct $u, v \in V(Q_n)$, where $n \geq 2$, have the same parity that is opposite to the parity of $x \in V(Q_n)$, then $Q_n - \{x\}$ has a Hamiltonian $uv$-path.*

A similar result holds, up to one exception, for the case of two removed vertices that are adjacent.

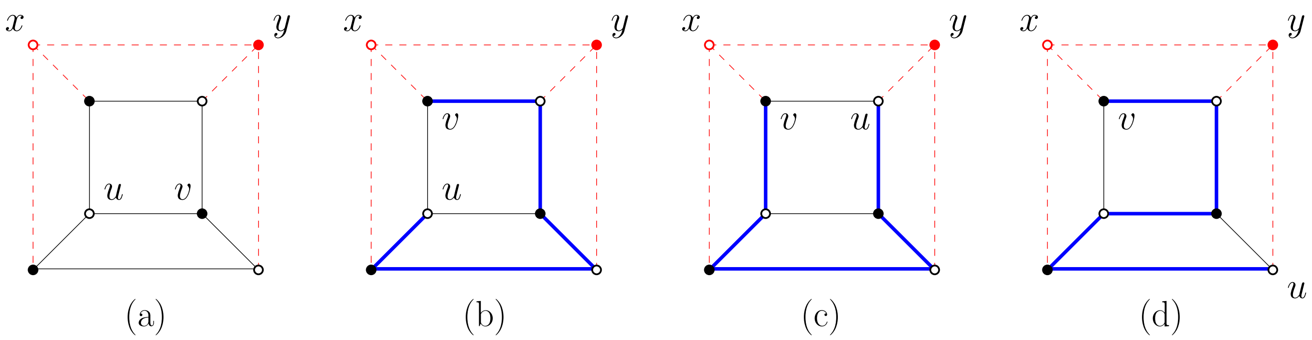


Figure 1: All configurations (up to isomorphism) in Proposition 2 for $n = 3$.

**Proposition 2.** *If $u, v \in V(Q_n) \setminus \{x, y\}$, where $xy \in E(Q_n)$ and $n \geq 2$, have the opposite parity, then $Q_n - \{x, y\}$ has a Hamiltonian $uv$-path unless:*

$$n = 3, \ u \oplus v = x \oplus y, \ \textit{and} \ \ d(uv, xy) = 2. \tag{1}$$

*Proof.* The exceptional configuration (1) is depicted on Figure 1(a). We proceed by induction on the dimension $n$. For $n = 2$ the statement trivially holds. For $n = 3$, aside from the exceptional configuration (1), we have (up to isomorphism) another three configurations depicted on Figure 1(b)-(d). Observe that the statement holds for each of them.

For $n \geq 4$ we split $Q_n$ into two subcubes $Q^0_{n-1}$ and $Q^1_{n-1}$ such that the edge $xy$ belongs to $Q^0_{n-1}$ or $Q^1_{n-1}$, and moreover, the vertex $v$ is in the other subcube than the edge $xy$. Assume without loss of generality that $xy \in E(Q^0_{n-1})$ and $v \in V(Q^1_{n-1})$. Considering the position of the vertex $u$, we distinguish two cases.

**Case 1**: $u \in V(Q^1_{n-1})$. Let $P_1$ be a Hamiltonian $uv$-path of $Q^1_{n-1}$. We claim that $P_1$ contains consecutive vertices $a$ and $b$ such that their neighbors $a'$ and $b'$ in $Q^0_{n-1}$ are distinct from both $x$ and $y$, and the edge $a'b' \in E(Q^0_{n-1})$ does not form the exceptional configuration (1) in $Q^0_{n-1}$. Since $n \geq 4$, the path $P_1$ contains at least 7 edges. At most 4 of them contain a vertex whose neighbor in $Q^0_{n-1}$ is $x$ or $y$. In addition, at most one of them contains vertices whose neighbors in $Q^0_{n-1}$ form the configuration (1). Hence $P_1$ contains at least 2 edges that satisfy the claim.

Applying induction we obtain a Hamiltonian $a'b'$-path $P_0$ of $Q^0_{n-1} - \{x, y\}$. By inserting $P_0$ into $P_1$ instead of the edge $ab$, we have the desired path.

**Case 2**: $u \in V(Q^0_{n-1})$. First we choose a neighbor $a \in V(Q^0_{n-1})$ of $u$ such that the edge $ua$ does not form the configuration (1). Amongst neighbors of $u$ in $Q^0_{n-1}$ at most one is $x$ or $y$, and at most one forms the configuration (1). Thus, such neighbor $a$ exists since $u$ has at least 3 neighbors in $Q^0_{n-1}$. Applying induction we obtain a Hamiltonian $ua$-path $P_0$ of $Q^0_{n-1}$. Let $a'$ be the neighbor of $a$ in $Q^1_{n-1}$, and let $P_1$ be a Hamiltonian $a'v$-path of $Q^1_{n-1}$. By concatenating $P_0$ and $P_1$, we are finished. $\square$

Let us denote by $N_G(u)$ the set of neighbors of a vertex $u$ in a subgraph $G$ of $Q_n$. If $G = Q_n$, the subscript $Q_n$ is omitted. Recall that $N(\mathbf{0}) = \{e_1, \dots, e_n\}$. For $n \geq 2$ and distinct $i, j \in [n]$, we define the set

$$A_{ij} = (N(\mathbf{0}) \setminus \{e_i, e_j\}) \cup (N(e_i) \setminus \{e_i \oplus e_j\}). \tag{2}$$

Note that $A_{ij}$ contains $n-2$ odd vertices and $n-1$ even vertices of $Q_n$ including the vertex $\mathbf{0}$. We continue with a result on Hamiltonicity of $Q_n$ in case of $2n-3$ removed vertices of this set $A_{ij}$.

**Proposition 3.** *If $z \in V(Q_n) \setminus A_{ij}$ is odd, $n \geq 2$, and $z \neq e_i$ where $i, j \in [n]$ are distinct, then $Q_n - A_{ij}$ has a Hamiltonian $e_i z$-path.*

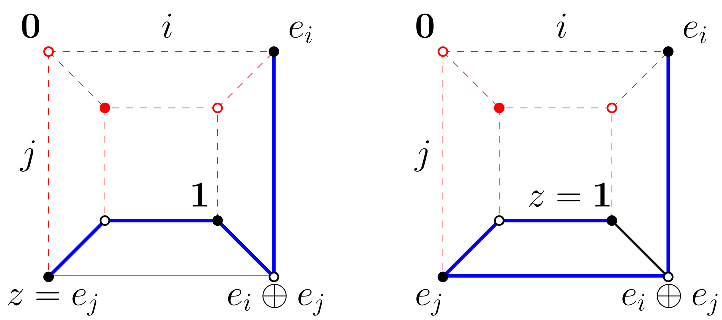


Figure 2: The Hamiltonian paths in Proposition 3 for $n = 3$.

*Proof.* We proceed by induction on the dimension $n$. For $n = 2$ the statement trivially holds. For $n = 3$ we have either $z = e_j$ or $z = \mathbf{1}$. The Hamiltonian paths for both cases are depicted on Figure 2.

For $n \geq 4$ we choose $k \in [n]$ distinct from $i$ and $j$, and we split $Q_n$ along the direction $k$ into subcubes $Q^0_{n-1}$ and $Q^1_{n-1}$. For $A_0 = A_{ij} \cap V(Q^0_{n-1})$ and $A_1 = A_{ij} \cap V(Q^1_{n-1})$ observe that $A_0$ restricted to $n-1$ directions of $Q^0_{n-1}$ satisfies (2), and $A_1 = \{e_k, e_i \oplus e_k\}$. The idea is to apply induction in $Q^0_{n-1}$ and Proposition 2 in $Q^1_{n-1}$. We distinguish the following two cases regarding $z$; see Figures 3 and 4 for an illustration.

**Case 1**: $z \in V(Q^0_{n-1})$. Applying induction we obtain a Hamiltonian $e_i z$-path $P_0$ of $Q^0_{n-1} - A_0$. Note that $P_0$ goes first from $e_i$ to $e_i \oplus e_j$ since $e_i$ has no other neighbors in $Q^0_{n-1} - A_0$. Let $a \neq e_i$ be the next vertex on $P_0$ after $e_i \oplus e_j$, and let $u$ and $v$ be the neighbors of $e_i \oplus e_j$ and $a$ in $Q^1_{n-1}$, i.e. $u = e_i \oplus e_j \oplus e_k$ and $v = a \oplus e_k$. By Proposition 2 for $x = e_i \oplus e_k$ and $y = e_k$, we obtain a Hamiltonian $uv$-path $P_1$ of $Q^1_{n-1} - A_1$. Note that we

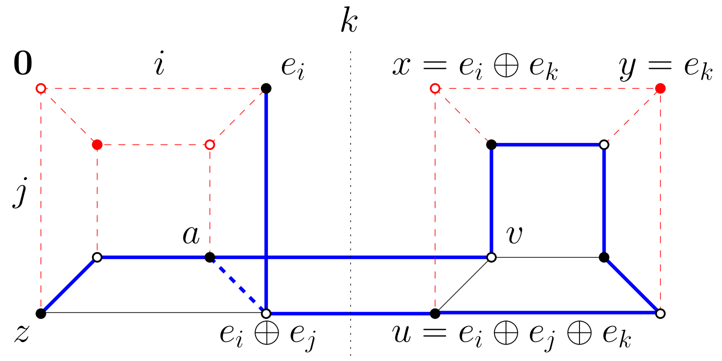


Figure 3: The case $z \in V(Q^0_{n-1})$ in Proposition 3 for $n = 4$.

avoid the exceptional configuration (1) since $d(u, x) = 1$. By inserting $P_1$ into $P_0$ instead of the edge between $e_i \oplus e_j$ and $a$, we construct the desired path.

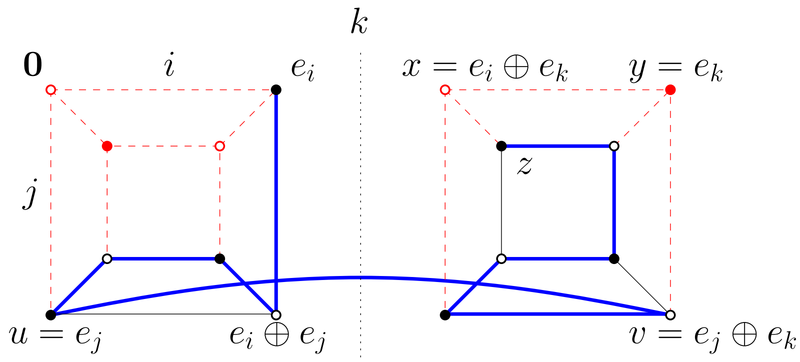


Figure 4: The case $z \in V(Q^1_{n-1})$ in Proposition 3 for $n = 4$.

**Case 2**: $z \in V(Q^1_{n-1})$. Applying induction we obtain a Hamiltonian path $P_0$ of $Q^0_{n-1} - A_0$ between $e_i$ and $u = e_j$. By Proposition 2 for $x = e_i \oplus e_k$ and $y = e_k$ we obtain a Hamiltonian path $P_1$ of $Q^1_{n-1} - A_1$ between $v = e_j \oplus e_k$ and $z$. Note that we avoid the exceptional configuration (1) since $d(v, y) = 1$. It remains to concatenate $P_0$ and $P_1$, and we are done. $\square$

For the sake of simplicity, the set $A_{ij}$ is defined and Proposition 3 is stated with respect to the vertex $\mathbf{0}$. However, note that by the automorphism of $Q_n$, Proposition 3 could be stated more generally for the set $A'_{ij} = A_{ij} \oplus w$ and the endvertices $e'_i = e_i \oplus w$, $z' = z \oplus w$ for any $w \in V(Q_n)$. Typically, we will apply Proposition 3 in Lemma 1 for $w = \mathbf{1}$.

## 3 Path partition of $Q_n - \{\mathbf{0}, \mathbf{1}\}$

Assume that we have $2k$ distinct vertices $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ of a subgraph $G$ of $Q_n$. We say that $G$ has an *$a_ib_i$-paths partition* if $V(G)$ can be partitioned into $k$ vertex-disjoint paths of $G$ between $a_i$ and $b_i$. Note that this notion generalizes the problem of Hamiltonian paths for more paths with prescribed endvertices, and it was previously studied for hypercubes by Caha and Koubek [1], and by Dvořák and Gregor [4] and also in a variation of faulty vertices [3].

We proceed with a technical, but useful lemma on $a_ib_i$-paths partition of $Q_n - \{\mathbf{0}, \mathbf{1}\}$.

**Lemma 1.** *Let $n \geq 3$ be odd, $k = n-1$, $\{a_1, \ldots, a_k\} \subseteq N(\mathbf{0})$, $\{b_1, \ldots, b_k\} \subseteq N(\mathbf{1})$ such that $a_1 = \overline{b_1}$ and $a_i \neq \overline{b_i}$ for every $1 < i \leq k$. Then $Q_n - \{\mathbf{0}, \mathbf{1}\}$ has an $a_ib_i$-paths partition.*

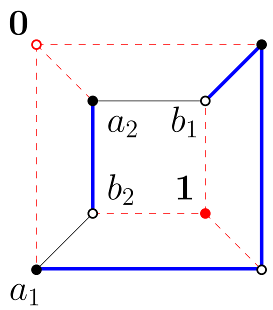


Figure 5: The only (up to isomorphism) configuration in Lemma 1 for $n = 3$.

*Proof.* For $n = 3$ there is only one (up to isomorphism) configuration of sets $\{a_1, a_2\} \subseteq N(\mathbf{0})$ and $\{b_1, b_2\} \subseteq N(\mathbf{1})$ such that $a_1 = \overline{b_1}$ and $a_2 \neq \overline{b_2}$. This configuration with the $a_ib_i$-paths partition of $Q_n - \{\mathbf{0}, \mathbf{1}\}$ is depicted on Figure 5.

Now we assume that $n \geq 5$. Let $a_0$ and $b_0$ denote the remaining neighbors of $\mathbf{0}$ and $\mathbf{1}$ that are not amongst $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$, respectively.

**Claim 1.** *The hypercube $Q_n$ can be split along two distinct directions $d_1, d_2 \in [n]$ into four subcubes $Q^{00}_{n-2}$, $Q^{01}_{n-2}$, $Q^{10}_{n-2}$, and $Q^{11}_{n-2}$ such that*

*(i) $\{a_0, a_1\} \subseteq V(Q^{00}_{n-2})$, $\{b_0, b_1\} \subseteq V(Q^{11}_{n-2})$, and*

*(ii) $\{a_i, b_i\} \subseteq V(Q^{01}_{n-2})$ or $\{a_i, b_i\} \subseteq V(Q^{10}_{n-2})$ for at most one $i \in [k]$;*

*unless $n = 5$ and $a_i$'s with $b_i$'s comprise the configuration depicted on Figure 6.*

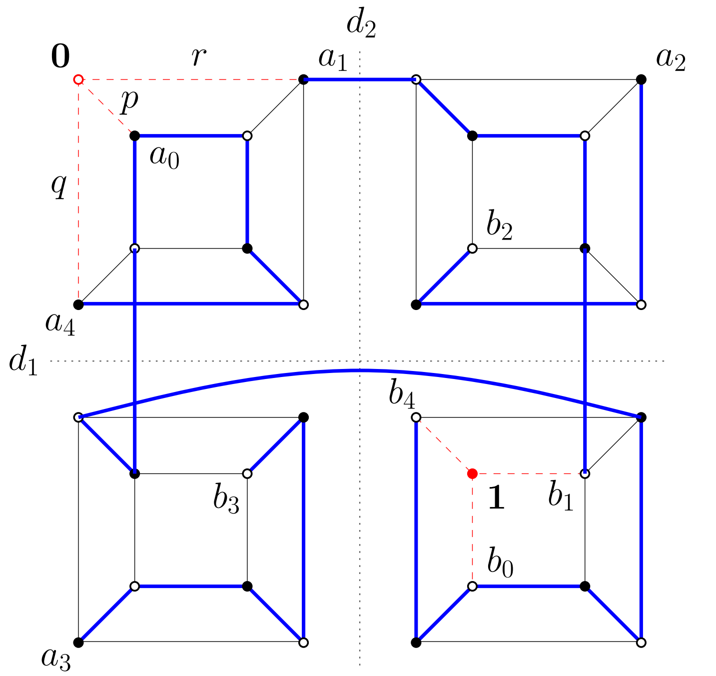


Figure 6: The only exceptional configuration which does not allow splitting satisfying conditions $(i)$ and $(ii)$ in Claim 1 for $n = 5$.

*Proof of Claim 1.* To satisfy the condition $(i)$, at most 3 directions from $[n]$ are forbidden for $d_1$ and $d_2$. More precisely, if $a_0 = e_p$, $\overline{b_0} = e_q$, and $a_1 = \overline{b_1} = e_r$, then we satisfy $(i)$ if and only if we choose $d_1$ and $d_2$ from the set $D = [n] \setminus \{p, q, r\}$. Note that $r$ is distinct from both $p$ and $q$, but we may have $p = q$ in general.

In the first step, we choose $d_1$ arbitrarily from $D$, and we split $Q_n$ into $Q^0_{n-1}$ and $Q^1_{n-1}$ along the direction $d_1$. Then we obtain $b_j \in V(Q^0_{n-1})$ and $a_l \in V(Q^1_{n-1})$ for exactly one $j$ and exactly one $l$ with $1 < j, l \leq k$. Observe that $j \neq l$ since $a_i \neq \overline{b_i}$ for every $1 < i \leq k$. By renaming the vertices we may assume that $j = 2$ and $l = 3$. Thus, we have

$$b_2[d_1] = a_2[d_1] = 0 \quad \text{and} \quad a_3[d_1] = b_3[d_1] = 1. \tag{3}$$

To satisfy also $(ii)$, it suffices to choose $d_2 \in D \setminus \{d_1\}$ such that $a_2[d_2] \neq b_2[d_2]$ or $b_3[d_2] \neq a_3[d_2]$. Since $a_i$ and $b_i$ differ in exactly $n - 2$ directions for every $1 < i \leq k$, and by (3), such $d_2 \in D \setminus \{d_1\}$ exists if $n \geq 7$ or $p = q$.

Now suppose that $n = 5$, $p \neq q$, and for the unique choice of $d_2 \in D \setminus \{d_1\}$ we have $a_2[d_2] = b_2[d_2]$ and $b_3[d_2] = a_3[d_2]$. Notice that it must be $a_2[d_2] = b_2[d_2] = 1$ and $b_3[d_2] = a_3[d_2] = 0$. If follows that

$$a_0 = \overline{b_4} = e_p, \ a_1 = \overline{b_1} = e_r, \ a_2 = \overline{b_3} = e_{d_2}, \ a_3 = \overline{b_2} = e_{d_1}, \text{ and } a_4 = \overline{b_0} = e_q.$$

This is exactly the configuration which is depicted on Figure 6. Therefore the claim holds. $\square$

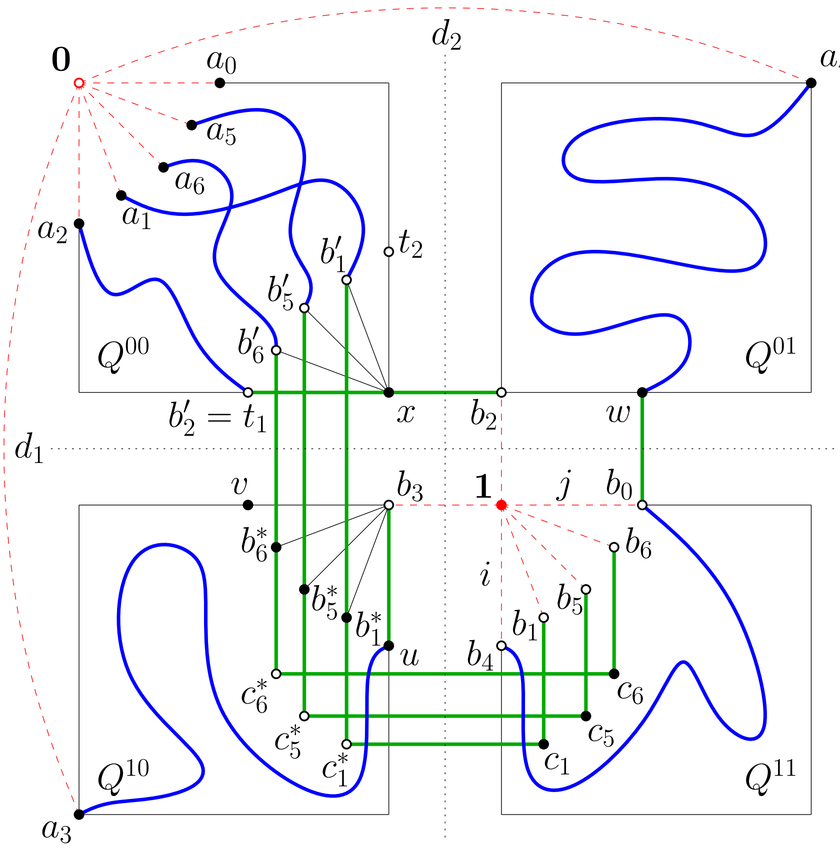


Figure 7: The construction of $a_ib_i$-path partition in Case 1 of Lemma 1.

The $a_ib_i$-paths partition for this exceptional configuration is also depicted on Figure 6. So, now we assume that we have splitting of $Q_n$ such that conditions $(i)$ and $(ii)$ hold. Furthermore, by renaming the vertices we may assume that $b_2 \in V(Q^{01}_{n-2})$ and $a_3 \in V(Q^{10}_{n-2})$. Moreover, by exchanging $d_1$ and $d_2$ we may assume that $\{a_i, b_i\} \subseteq V(Q^{01}_{n-2})$ for no $i \in [k]$, and therefore $a_2 \notin V(Q^{01}_{n-2})$. Thus, by renaming the vertices we have, say $a_4 \in V(Q^{01}_{n-2})$.

The idea of the rest of the proof is to apply induction in $Q^{00}_{n-2}$, Proposition 3 in $Q^{10}_{n-2}$ and in $Q^{11}_{n-2}$, and Proposition 1 in $Q^{01}_{n-2}$, and then glue all the paths together in order to obtain an $a_ib_i$-paths partition of $Q_n - \{\mathbf{0}, \mathbf{1}\}$. To this end, we distinguish two cases regarding whether $b_3$ is in $Q^{10}_{n-2}$. But before, to avoid ambiguity, let us mention that below we write simply $\{i, j, \dots k\}$ also for $k \leq j$ to denote the set $\{i\} \cup ([k] \setminus [j-1])$.

**Case 1**: $b_3 \in V(Q^{10}_{n-2})$. We start with the construction of an $a_4b_4$-path in $Q^{11}_{n-2}$ and $Q^{01}_{n-2}$. Note that most of the vertices of $Q^{11}_{n-2}$ and $Q^{01}$ are on this path. See Figure 7 for an illustration. Let $i, j \in [n] \setminus \{d_1, d_2\}$ be such that $b_4 = \overline{e_i}$ and $b_0 = \overline{e_j}$. Furthermore, let

$$B = \{b_l \mid l \in \{1, 5, \dots, k\}\} \quad \text{and} \quad C = \{c_l = b_l \oplus e_i \mid l \in \{1, 5, \dots, k\}\}.$$

Note that

$$N_{Q^{11}_{n-2}}(\mathbf{1}) = B \cup \{b_0, b_4\} \text{ and } N_{Q^{11}_{n-2}}(b_4) = C \cup \{\mathbf{1}, b_4 \oplus e_j\}.$$

Thus, applying Proposition 3 for the set $A_{ij} = B \cup C \cup \{\mathbf{1}\}$, we obtain a Hamiltonian $b_4b_0$-path $P_{11}$ of $Q^{11}_{n-2} - A_{ij}$.

Note that the vertex $w = b_0 \oplus e_{d_1}$ is adjacent to $b_2$ in $Q^{01}_{n-2}$, and therefore, $w$ is distinct from $a_4$ but has the same parity as $a_4$ which is opposite to the parity of $b_2$. So, we may apply Proposition 1 to construct a Hamiltonian $wa_4$-path $P_{01}$ of $Q^{01}_{n-2} - \{b_2\}$. By concatenating $P_{11}$ and $P_{01}$, we obtain the reversed $a_4b_4$-path.

Second, we construct an $a_3b_3$-path in $Q^{10}_{n-2}$. Note that most of the vertices of $Q^{10}_{n-2}$ are on this path. Let

$$B^* = \{b^*_l = b_l \oplus e_{d_2} \mid l \in \{1, 5, \dots, k\}\} \text{ and } C^* = \{c^*_l = b^*_l \oplus e_i \mid l \in \{1, 5, \dots, k\}\},$$

and let $u = b_3 \oplus e_i$ and $v = b_3 \oplus e_j$. Note that the vertices $b_3$, $u$, and $v$ in $Q^{10}_{n-2}$ correspond to the vertices $\mathbf{1}$, $b_4$, and $b_0$ in $Q^{11}_{n-2}$. Similarly as above, observe that

$$N_{Q^{10}_{n-2}}(b_3) = B^* \cup \{u, v\} \text{ and } N_{Q^{10}_{n-2}}(u) = C^* \cup \{b_3, u \oplus e_j\}.$$

Hence, applying Proposition 3 for the set $A^*_{ij} = B^* \cup C^* \cup \{b_3\}$, we obtain a Hamiltonian $a_3u$-path $P_{10}$ of $Q^{10}_{n-2} - A^*_{ij}$. By prolonging this path from $u$ to $b_3$, we have the $a_3b_3$-path.

Finally, we construct the remaining paths. Let $x = b_2 \oplus e_{d_2}$, $B' = \{b'_l = b^*_l \oplus e_{d_1} \mid l \in \{1, 5, \dots, k\}\}$. Notice that all these vertices are in $Q^{00}_{n-2}$, $x$ has the role of $\mathbf{1}$ in $Q^{00}_{n-2}$ and is adjacent to all vertices of $B'$. Furthermore, let $t_1, t_2$ be the remaining two neighbors of $x$ in $Q^{00}_{n-2}$ that are not in $B'$; that is, $t_1 = x \oplus e_j = x \oplus \overline{b_0}$ and $t_2 = x \oplus e_i = x \oplus \overline{b_4}$.

Observe that $d(a_1, b'_1) = n-2$ since $d(a_1, b_1) = n$. So the vertices $a_1$ and $b'_1$ are complementary in $Q^{00}_{n-2}$. Similarly, $d(a_l, b'_l) = n-4$ since $d(a_l, b_l) = n-2$ for every $l \in \{5, \dots, k\}$. We choose $b'_2 \in \{t_1, t_2\}$ such that also $d(a_2, b'_2) = n-4$. Applying induction we obtain an $a_lb'_l$-paths partition of $Q^{00}_{n-2} - \{\mathbf{0}, x\}$ where $l \in \{1, 2, 5, \dots, k\}$. Then, we prolong the $a_2b'_2$-path through $x$ to $b_2$, and each $a_lb'_l$-path through $b^*_l$, $c^*_l$, and $c_l$ to $b_l$ for $l \in \{1, 5, \dots, k\}$. Thus, we obtain remaining $a_lb_l$-paths.

To conclude Case 1, observe (on Figure 7) that all $a_ib_i$-paths for $i \in [k]$ are vertex-disjoint and they cover all vertices of $Q_n - \{\mathbf{0}, \mathbf{1}\}$.

**Case 2**: $b_3 \notin V(Q^{10}_{n-2})$. The constructions in this case differ only in small details to the construction in the previous case. However, for the sake of completeness, we present here the entire argument. First, recall that $b_0$,

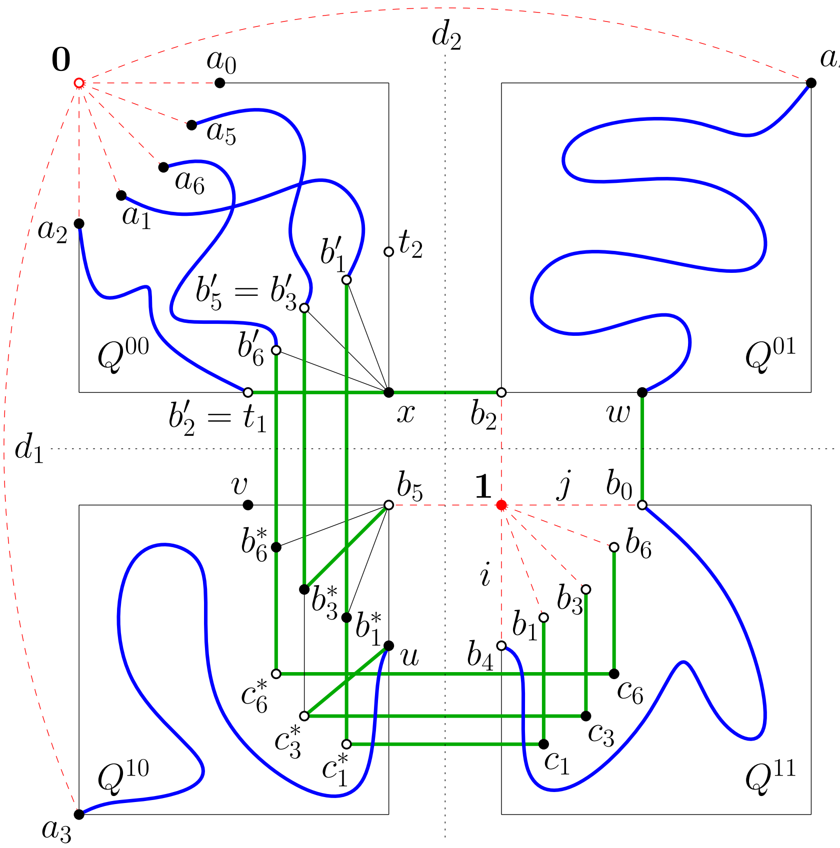


Figure 8: The construction of $a_ib_i$-path partition in Case 2.A of Lemma 1.

$b_1$, $b_2$, and $b_3$ are not in $Q^{10}_{n-2}$. Moreover, $b_4$ cannot be in $Q^{10}_{n-2}$ since $a_4$ is $Q^{01}_{n-2}$ and $a_4 \neq \overline{b_4}$. Thus, by renaming the vertices we may assume that $b_5 \in V(Q^{10}_{n-2})$. Note that it follows that $n \geq 7$ in Case 2.

We start with the construction of an $a_4b_4$-path in $Q^{11}_{n-2}$ and $Q^{01}_{n-2}$ which is completely the same as above. Let $i, j \in [n] \setminus \{d_1, d_2\}$ be such that $b_4 = \overline{e_i}$ and $b_0 = \overline{e_j}$. Furthermore, let

$$B = \{b_l \mid l \in \{1, 3, 6, \ldots, k\}\} \text{ and } C = \{c_l = b_l \oplus e_i \mid l \in \{1, 3, 6, \ldots, k\}.$$

Note that

$$N_{Q^{11}_{n-2}}(\mathbf{1}) = B \cup \{b_0, b_4\}$$

and

$$N_{Q^{11}_{n-2}}(b_4) = C \cup \{\mathbf{1}, b_4 \oplus e_j\}.$$

Thus, applying Proposition 3 for the set $A_{ij} = B \cup C \cup \{\mathbf{1}\}$, we obtain a Hamiltonian $b_4b_0$-path $P_{11}$ of $Q^{11}_{n-2} - A_{ij}$. Second, we apply Proposition 1 to construct a Hamiltonian path $P_{01}$ between vertices $w = b_0 \oplus e_{d_1}$ and $a_4$ in $Q^{01}_{n-2} - \{b_2\}$. By concatenating $P_{11}$ and $P_{01}$, we obtain the reversed $a_4b_4$-path. Now we distinguish two subcases regarding $d(a_5, b_3)$.

**Subcase 2.A**: $d(a_5, b_3) = n-2$. We construct an $a_3b_3$-path in $Q^{10}_{n-2}$ and $Q^{11}_{n-2}$. See Figure 8 for an illustration. Again, let

$B^* = \{b^*_l = b_l \oplus e_{d_2} \mid l \in \{1, 3, 6, \ldots, k\}\}$ and $C^* = \{c^*_l = b^*_l \oplus e_i \mid l \in \{1, 3, 6, \ldots, k\}\}$,

and let $u = b_5 \oplus e_i$ and $v = b_5 \oplus e_j$. Similarly as above, observe that

$$N_{Q^{10}_{n-2}}(b_5) = B^* \cup \{u, v\} \quad \text{and} \quad N_{Q^{10}_{n-2}}(u) = C^* \cup \{b_5, u \oplus e_j\}.$$

Hence, applying Proposition 3 for the set $A^*_{ij} = B^* \cup C^* \cup \{b_5\}$, we obtain a Hamiltonian $a_3u$-path $P_{10}$ of $Q^{10}_{n-2} - A^*_{ij}$. By prolonging this path from $u$ through $c^*_3$ and $c_3$ to $b_3$, we have the $a_3b_3$-path.

Finally, we construct the remaining paths. Let $x = b_2 \oplus e_{d_2}$, $B' = \{b'_l = b^*_l \oplus e_{d_1} \mid l \in \{1, 3, 6, \ldots, k\}\}$. Furthermore, let $t_1, t_2$ be the remaining two neighbors of $x$ in $Q^{00}_{n-2}$ that are not in $B'$; that is, $t_1 = x \oplus e_j = x \oplus \overline{b_0}$ and $t_2 = x \oplus e_i = x \oplus \overline{b_4}$.

Observe that $d(a_1, b'_1) = n-2$ since $d(a_1, b_1) = n$. So the vertices $a_1$ and $b'_1$ are complementary in $Q^{00}_{n-2}$. Similarly, $d(a_l, b'_l) = n-4$ since $d(a_l, b_l) = n-2$ for every $5 < l \leq k$. We put $b'_5 = b'_3$. Since $d(a_5, b_3) = n-2$ in this subcase, we have that $d(a_5, b'_5) = n-4$. Furthermore, we choose $b'_2 \in \{t_1, t_2\}$ such that also $d(a_2, b'_2) = n-4$. Applying induction we obtain an $a_lb'_l$-paths partition of $Q^{00}_{n-2} - \{\mathbf{0}, x\}$ where $l \in \{1, 2\} \cup \{5, \ldots, k\}$. Then, we prolong the $a_2b'_2$-path through $x$ to $b_2$, the $a_5b'_5$-path through $b^*_3$ to $b_5$, and the $a_lb'_l$-path through $b^*_l$, $c^*_l$, and $c_l$ to $b_l$ for every $l \in \{1, 6, \ldots, k\}$. Thus, we obtain the remaining $a_lb_l$-paths.

To conclude Subcase 2.$B$, observe (on Figure 8) that all $a_ib_i$-paths for $i \in [k]$ are vertex-disjoint and they cover all vertices of $Q_n - \{\mathbf{0}, \mathbf{1}\}$.

**Subcase 2.B**: $d(a_5, b_3) = n$. In this subcase we have to apply induction in $Q^{00}_{n-2}$ in a different way than in the previous subcase. The reason is that we need $d(a_5, b'_5) = n-4$, so now we cannot put $b'_5 = b'_3$ as we did before. As a consequence, we have to construct also an $a_3b_3$-path in $Q^{10}_{n-2}$ and $Q^{11}_{n-2}$ in a different way. See Figure 9 for an illustration.

Again, let $B^* = \{b^*_l = b_l \oplus e_{d_2} \mid l \in \{0, 1, 6, \ldots, k\}\}$ and $C^* = \{c^*_l = b^*_l \oplus e_i \mid l \in \{0, 1, 6, \ldots, k\}\}$, and let $u = b_5 \oplus e_i$, $b^*_3 = b_3 \oplus e_{d_2}$, and $c^*_3 = b^*_3 \oplus e_i$. Similarly as above, observe that

$$N_{Q^{10}_{n-2}}(b_5) = B^* \cup \{u, b^*_3\} \quad \text{and} \quad N_{Q^{10}_{n-2}}(u) = C^* \cup \{b_5, c^*_3\}.$$

Hence, applying Proposition 3 for the set $A^*_{ij} = B^* \cup C^* \cup \{b_5\}$, we obtain a Hamiltonian $a_3u$-path $P_{10}$ of $Q^{10}_{n-2} - A^*_{ij}$. Note that $c^*_3$ is the vertex previous to $u$ on $P_{10}$ since $u$ has no other neighbor in $Q^{10}_{n-2} - A^*_{ij}$ than $c^*_3$. By deleting

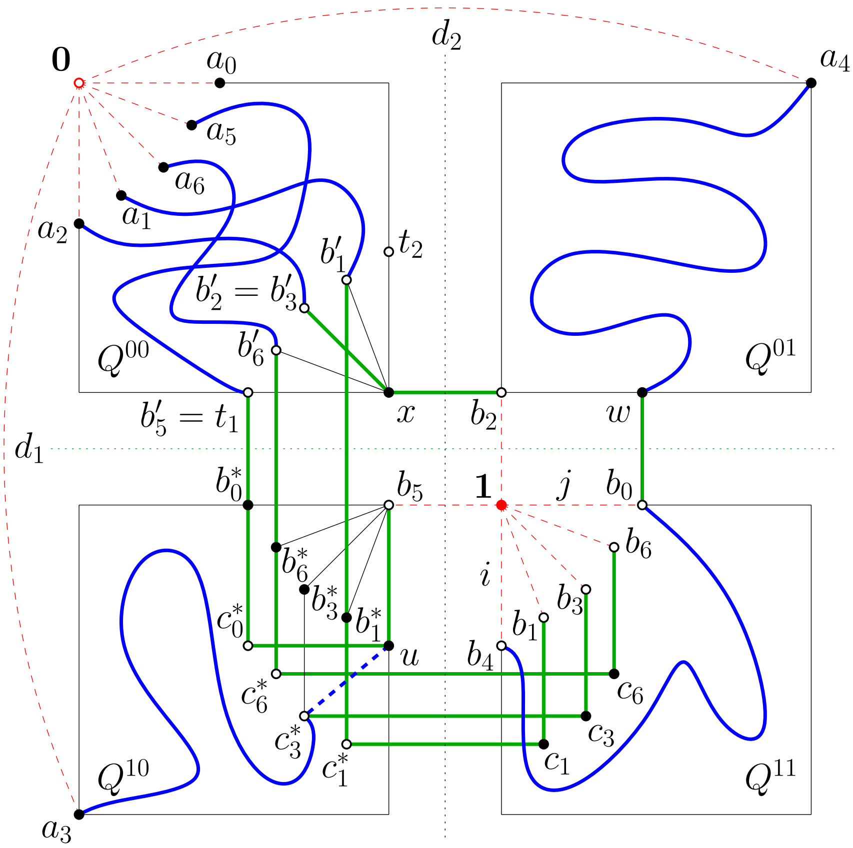


Figure 9: The construction of $a_ib_i$-path partition in Case 2.B of Lemma 1.

the vertex $u$ from $P_{10}$ and prolonging this path from $c^*_3$ through $c_3$ to $b_3$, we have the $a_3b_3$-path.

Finally, we construct the remaining paths. Let $x = b_2 \oplus e_{d_2}$, $B' = \{b'_l = b^*_l \oplus e_{d_1} \mid l \in \{1, 3, 6, \dots, k\}\}$. Furthermore, let $t_1, t_2$ be the remaining two neighbors of $x$ in $Q^{00}_{n-2}$ that are $t_1 = x \oplus e_j = x \oplus \overline{b_0}$ and $t_2 = x \oplus e_i = x \oplus \overline{b_4}$. We put $b'_5 = t_1$, so $d(a_5, b'_5) = n - 4$ since $d(a_5, b'_3) = n - 2$. We also choose $b'_2 \in \{b'_3, t_2\}$ such that $d(a_2, b'_2) = n - 4$.

Again, observe that $d(a_1, b'_1) = n - 2$ since $d(a_1, b_1) = n$, and $d(a_l, b'_l) = n - 4$ for every $l \in \{1, 2, 5, \dots, k\}$. Hence, applying induction we obtain an $a_lb'_l$-paths partition of $Q^{00}_{n-2} - \{\mathbf{0}, x\}$ where $l \in \{1, 2, 5, \dots, k\}$. Then, we prolong the $a_2b'_2$-path through $x$ to $b_2$, the $a_5b'_5$-path through $b^*_0$, $c^*_0$, and $u$ to $b_5$, and each $a_lb'_l$-path through $b^*_l$, $c^*_l$, and $c_l$ to $b_l$ for $l \in \{1, 6, \dots, k\}$. Therefore, we obtain the remaining $a_lb_l$-paths.

To conclude Subcase 2.$B$, observe (on Figure 9) that all $a_ib_i$-paths for $i \in [k]$ are vertex-disjoint and they cover all vertices of $Q_n - \{\mathbf{0}, \mathbf{1}\}$. $\square$

# 4 Hamiltonicity of $G_n^2$

Recall that $G_n^2$ is the graph induced on $Q_n$ by levels from 2 to $n-2$; that is,

$$G_n^2 = Q_n - (\{\mathbf{0}, \mathbf{1}\} \cup N(\mathbf{0}) \cup N(\mathbf{1})).$$

For our convenience, we use the following notation of vertices in subcubes of $Q_n$. Assume that $Q_n$ is split along two fixed directions $d_1, d_2 \in [n]$ into four subcubes $Q_{n-2}^{00}$, $Q_{n-2}^{01}$, $Q_{n-2}^{10}$, and $Q_{n-2}^{11}$, which are isomorphic to $Q_{n-2}$. For $w \in \{00, 01, 10, 11\}$ and $x \in V(Q_{n-2})$ we denote by $x^w$ the copy of the vertex $x$ in the subcube $Q_{n-2}^w$.

**Theorem 1.** *$G_n^2$ is Hamiltonian for every odd $n \geq 5$.*

*Proof.* We proceed by induction on $n$. The statement holds for $n=5$ since $G_5^2$ is the middle level graph which is known to be Hamiltonian [15]. For $n \geq 7$ we split $Q_n$ along two arbitrary directions into four subcubes $Q_{n-2}^{00}$, $Q_{n-2}^{01}$, $Q_{n-2}^{10}$, and $Q_{n-2}^{11}$. For $w \in \{00, 11\}$ let $H^w$ denote a copy of $G_{n-2}^2$ in $Q_{n-2}^w$, that is

$$H^w = Q_{n-2}^w - (\{\mathbf{0}^w, \mathbf{1}^w\} \cup N_{Q_{n-2}^w}(\mathbf{0}^w) \cup N_{Q_{n-2}^w}(\mathbf{1}^w)).$$

Initially, applying induction we obtain Hamiltonian cycles $C_1$ and $C_2$ of $H^{00}$ and $H^{11}$, respectively. Next, we construct a Hamiltonian cycle $C_3$ of $G_n^2 \setminus (H^{00} \cup H^{11})$ and finally, we interconnect $C_3$ with copies of $C_1$ and $C_2$ mapped by properly chosen automorphisms of $H^{00}$ and $H^{11}$.

The construction of $C_3$ is as follows. First, we label the neighbors of $\mathbf{0}$ and $\mathbf{1}$ in $Q_{n-2}$ in the following way. Let $p_i = e_i$ for every $i \in [n-2]$, and let

$$q_1 = \overline{e_1},\ q_2 = \overline{e_4},\ q_3 = \overline{e_2},\ q_{n-2} = \overline{e_3},\ \text{and}\ q_i = \overline{e_{i+1}} \quad \text{for every } 4 \leq i \leq n-3.$$

Note that $p_1 = \overline{q_1}$ and $p_i \neq \overline{q_i}$ for every $1 < i \leq n-3$. We use corresponding labelings $p_i^w$ and $q_i^w$ in $Q_{n-2}^w$ for each $w \in \{00, 01, 10, 11\}$.

Hence, by Lemma 1, $Q_{n-2}^{01} - \{\mathbf{0}^{01}, \mathbf{1}^{01}\}$ can be partitioned into $n-3$ vertex-disjoint paths $P_i$ between $p_i^{01}$ and $q_i^{01}$ for $i \in [n-3]$. Similarly, since $p_2 = \overline{q_3}$, $p_{n-2} \neq \overline{q_2}$ and $p_i \neq \overline{q_{i+1}}$ for $2 < i \leq n-3$, $Q_{n-2}^{10} - \{\mathbf{0}^{10}, \mathbf{1}^{10}\}$ can be partitioned into $n-3$ vertex-disjoint paths $R_i$ between $q_{i+1}^{10}$ and $p_i^{10}$ for $2 \leq i \leq n-3$, and $R_1$ between $q_2^{10}$ and $p_{n-2}^{10}$. See Figure 10 for an illustration.

To construct $C_3$, we interconnect paths $R_i$ and $P_i$ at $Q_{n-2}^{11}$, and paths $P_i$ and $R_{i-1}$ at $Q_{n-2}^{00}$ as follows (here $R_0$ means $R_{n-3}$). The path $R_1$ continues from $p_{n-2}^{10}$ through $p_{n-2}^{11}$, $\mathbf{0}^{11}$, $p_1^{11}$ to the vertex $p_1^{01}$ of $P_1$. The path $R_i$ for $2 \leq i \leq n-3$ continues from $p_i^{10}$ through $p_i^{11}$ to the vertex $p_i^{01}$ of $P_i$. The

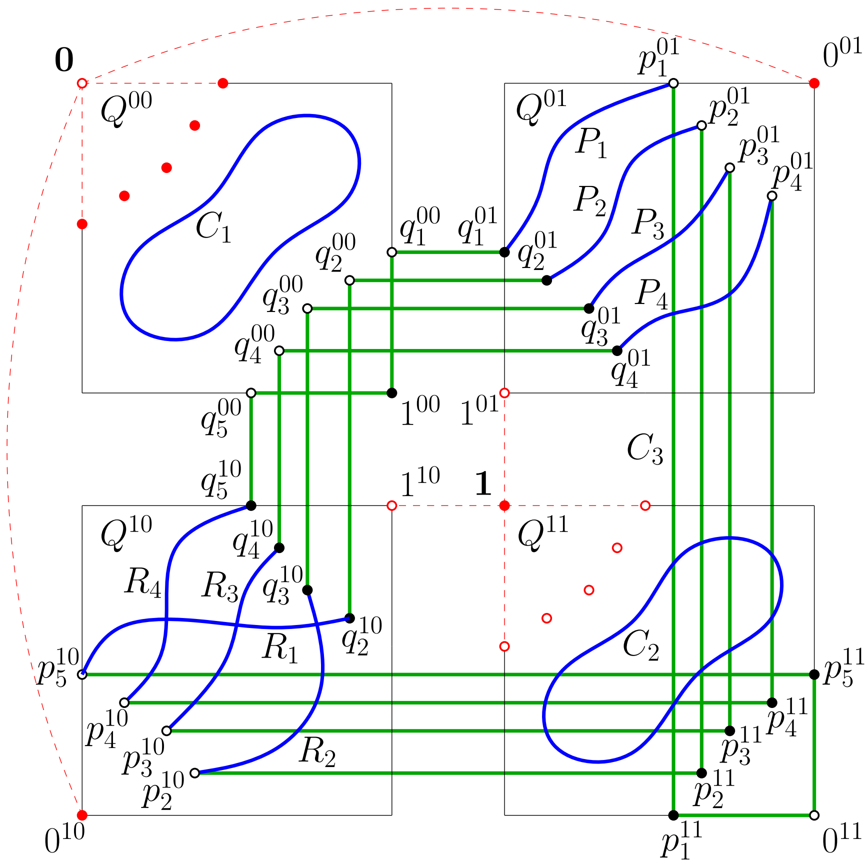


Figure 10: The construction in Theorem 1 with $n = 7$.

path $P_1$ continues from $q_1^{01}$ through $q_1^{00}$, $\mathbf{1}^{00}$, $q_{n-2}^{00}$ to the vertex $q_{n-2}^{10}$ of $R_{n-3}$. The path $P_i$ for $2 \leq i \leq n-3$ continues from $q_i^{01}$ through $q_i^{00}$ to the vertex $q_i^{10}$ of $R_{i-1}$. These connections are presented with green color on Figure 10. The choices of endvertices of paths $P_i$'s and $R_i$'s allow these connections which assures that $C_3$ is a Hamiltonian cycle of $G_n^2 \setminus (H^{00} \cup H^{11})$.

To conclude the proof, we interconnect $C_3$ with copies of $C_1$ and $C_2$. For two adjacent vertices $u$ and $v$ of $Q_n$ we say that $u$ is a *light* neighbor of $v$ if $w(u) < w(v)$, otherwise $u$ is a *heavy* neighbor of $v$. Since $w(p_1) = 1$ and $w(q_1) = n - 3 \geq 4$, the $p_1^{01}q_1^{01}$-path $P_1 \subset C_3$ contains a vertex $x^{01}$ such that $w(x) = 2$ and $x^{01}$ has a heavy neighbor on $P_1$. Note that the neighbor $x^{00}$ of $x^{01}$ in $Q_{n-2}^{00}$ belongs to $C_1$.

Let $y^{00}$ be one of two neighbors of $x^{00}$ on $C_1$. Observe that $y^{00}$ is a heavy neighbor of $x^{00}$ since $w(x^{00}) = w(x) = 2$ and $C_1$ does not visit vertices of weight 1. Then, let $z^{01}$ be a heavy neighbor of $x^{01}$ on $P_1$. Let $i$ and $j$ be the directions of edges $x^{00}y^{00} \in E(C_1)$, $x^{01}z^{01} \in E(C_3)$, respectively. If $i = j$, then $y^{00}$ and $z^{01}$ are adjacent in $G_n^2$, so we can interconnect $C_1$ with $C_3$ directly by replacing the edges $x^{00}z^{00} \in E(C_1)$ and $x^{01}z^{01} \in E(C_3)$ with $x^{00}x^{01}, z^{00}z^{01} \in E(G_n^2)$.

Now we assume $i \neq j$. Thus, we have

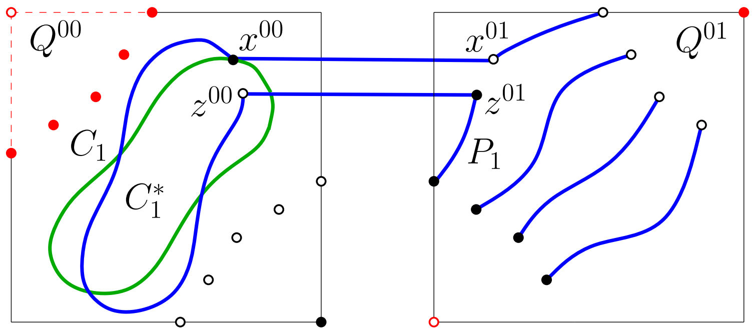


Figure 11: The interconnection of the path $P_1$ and a copy $C_1^*$ of the cycle $C_1$.

- $y^{00}[i] = z^{01}[j] = 1$, and
- $x^{00}[i] = x^{01}[i] = z^{01}[i] = x^{01}[j] = x^{00}[j] = y^{00}[j] = 0$.

Hence, by switching the directions $i$ and $j$ we obtain a bijection $\pi$ of $V(H^{00})$ such that $\pi(x^{00}) = x^{00}$ and $\pi(y^{00}) = z^{00}$ where $z^{00}$ is the neighbor of $z^{01}$ in $Q^{00}_{n-2}$. Moreover, $\pi$ is an automorphism of $H^{00}$, and therefore, $C_1^* = \pi(C_1)$ is a Hamiltonian cycle of $H^{00}$ containing the edge $x^{00}z^{00}$. Therefore, we can interconnect $C_1^*$ with $C_3$ by replacing the edges $x^{00}z^{00} \in E(C_1^*)$ and $x^{01}z^{01} \in E(C_3)$ with $x^{00}x^{01}, z^{00}z^{01} \in E(G_n^2)$. See Figure 11 for an illustration, where $C_1$ is represented by green and $C_1^*$ by blue color.

To interconnect $C_3$ also with a copy of $C_2$, we proceed similarly with the path $R_1 \subset C_3$ in $Q^{10}$ as above. By a proper choice of a vertex $x^{10}$ (with his neighbor $z^{10}$) on $R_1$ and an automorphism of $H^{11}$ which maps an edge of $C_2$ to $x^{11}z^{11}$ we easily connect $C_3$ and a copy of $C_2$. Therefore, we obtain a Hamiltonian cycle of $G_n^2$. $\square$

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