Forbidden Minors for Projective Plane are Free-Toroidal or Non-Toroidal

Dainis ZEPS * †

Abstract

Forbidden graphs for projective plane are examined. Using mostly computer, we test how many edges may be added so that these graphs remain free-toroidal. Using the method, some forbidden graphs on torus are obtained by the way.

Most of the definitions of the topological graph theory are the same as in [3]. The definitions concerning free minor closed classes are as in [5, 6, 7]. The definitions and denotations concerning forbidden subgraphs on projective plane are as in [3, 4].

Let us denote by A_{Pl}, A_P, A_T the classes of graphs on plane, projective plane and torus respectively.

We say that a graph is free-toroidal if it is free planar on torus in the same sense as in [7]. More precisely, graph is free-toroidal or free planar on torus if it remains toroidal after adding arbitrary edge to it. More specifically, we say that a graph is free k toroidal if it remains toroidal after adding arbitrary k edges to it. Then, k-freeness in our context is the graph property of being free k toroidal.

We examine forbidden minors (35 in number), and forbidden subgraphs (103 in number, in [4] called irreducible), of projective plane for embeddability on torus. Let us call the subset of forbidden subgraphs and minors of projective plane that are not biconnected correspondingly *heavy forbidden subgraphs* and *heavy forbidden minors*. All heavy forbidden minors are $A_1, A_5, C_1, C_{11}, E_1, E_{42}$ in [4]. It is well known that just heavy forbidden subgraphs are non-toroidal, others being toroidal [3].

Using computer we have tested that all not heavy minimal minors on projective plane are free-toroidal. Taking into account that heavy minors are forbidden graphs for torus (by trivial reason, see [3]), we may state that all minimal minors on projective plane are either free-toroidal or forbidden minors for torus.

Let us formulate this as theorem.

Theorem 1. *Minimal forbidden minor for projective plane is either biconnected free-toroidal or non-biconnected forbidden minor for torus.*

Shorter:

Corollary 2. *Non-heavy forbidden minors on projective plane are free-toroidal.*

Further, it may be useful to examine all other forbidden subgraphs of projective plane for free-toroidality. It turns out that some subgraphs are not free-toroidal. For example, some children (subgraphs contained as minors) of B_3, C_2, E_6, F_6 are such. They give birth to some non-free-toroidals: $C_5, E_{13}, E_{36}, F_7, F_{11}, ...$

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Number of vertices / edges	7	8	9	10	11
15		$E_3;6;9$ $E_{18};8;9$	$F_1;5;12$	$G;3;15$	
16		$D_3;5;8$ $D_{17};6;8$	$E_{22};2;11$ $E_5;2;11$ $E_{20};3;11$ $E_{19};3;11$	$F_4;3;14$ $F_6;2;14$	
17	$B_1;4;4$	$C_7;4;7$	$D_{12};3;10$ $D_4;2;10$	$E_{27};2;13$ $E_{11};2;13$ $E_6;1;13$	
18	$A_2;3;3$	$B_3;3;6$ $B_7;5;6$	$C_4;3;9$ $C_3;3;9$ $C_2;2;9$	$D_2;2;12$ $D_9;2;12$ $D_1;2;12$	$E_2;2;15$

Table 1: Results of computation as triplet (forbidden minor on projective plane; k; K) where k means k-freeness, and $K = 3 \cdot n - m$, i.e., additional edge number limit for torus embedding. Essential results of computations are just the numbers k.

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We use the nice property that $3 \cdot n$ edges triangulate torus (Euler formula [3]). If we add edges to some forbidden graph for projective plane, then, reaching additional edge number $3n - m$, we have passed already over critical forbidden graph for torus, except in case of K_7. Thus, in case G is a forbidden graph for projective plane and free k toroidal, then, augmenting it with adding edges, between limits of additional edge number k and $3n - m$ there live critical graphs that are eventual forbidden graphs for torus.

All forbidden minors on torus with less than 12 vertices except heavy forbidden minors may be obtained augmenting forbidden minors on projective plane (compare [3], page 202). In appendix below some examples of forbidden minors on torus are shown which were obtained with the program that was used in this research.

Authors of [3], referencing to [1], give only one example of minimal forbidden graph on torus with 10 vertices and 19 edges. Using Table 1, it may be seen how we could find all these minimal graphs with 19 edges which turned out to be two with 9 vertices and four with 10 vertices.

To get forbidden graph for torus with 9 vertices and 19 edges, according to Table 1, we may start only from E_5 and E_{22} adding three edges. In this way we get two non-isomorphic graphs. The case with 10 vertices and 19 edges, starting from graphs E_6 (adding 2 edges), F_6 (adding 3 edges) and G (adding 4 edges), gives us four non-isomorphic graphs.

Let us discuss shortly the program that was used to come to the results of this article. We used exhaustive search algorithm to find genus of the graph, or, more specifically, we performed test whether graph G may be embedded in the orientable surface of genus k. We found that exponential algorithm for our purpose would be better than good polynomial algorithm, say such as in [2], not only because of time consuming aspects, but because of possibility to check its correctness or at least to follow in some accessible limits its credibility of the correctness of the results. Of course, program errors are hard to escape. Nevertheless or at least, we have some proof by the result itself. All obtained graphs are forbidden graphs for torus, that may be checked by hand. As to the credibility of cyphers of Table 1, number k in the worst case may be lower, not higher,thus, providing estimation from above. In the whole, the result in its qualitative aspect remains as it is stated in the title of this article.

References

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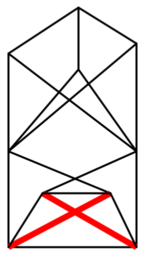
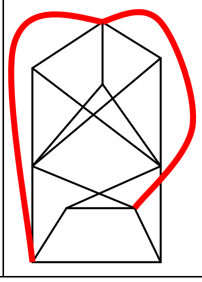
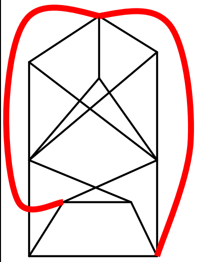
Appendix

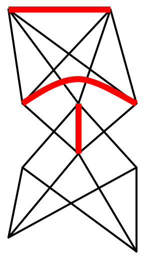
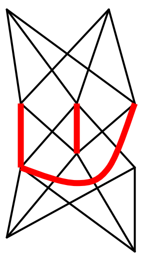
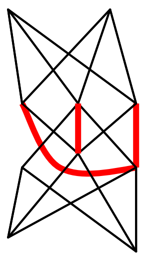
The forbidden subgraphs for torus that were generated by our computer program are demonstrated below. Besides the subgraphs in the tables below we give adjacency lists of vertices of graphs in numerical form as obtained by program to elucidate the process how the graphs were found by the program. (Adjacency lists use three colors for different degrees of vertices to allow easier observation of isomorphic graphs which wouldn't be seen in grey print without loss of information.) Program found subgraphs are numbered according to their isomorphic equivalence, e.g., we have two isomorphic descendants $E_{6,2}$ from E_6 adding to it two edges.

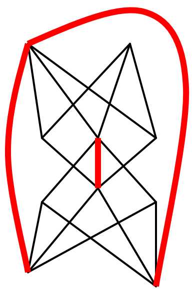
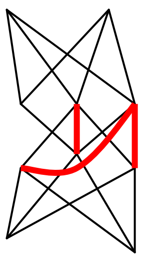
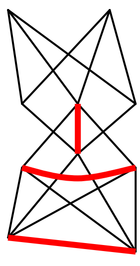
10 vertices and 19 edges:

E_6+ 2 edges gives three (two non-isomorphic) minimal forbidden graphs on torus:

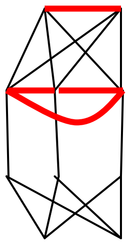
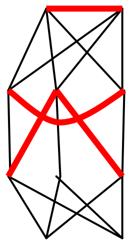
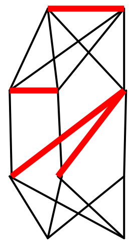
1: 2 3 4 6 2: 1 3 4 5 3: 1 2 4 6 4: 1 2 3 5 5: 2 4 7 8 9 6: 1 3 7 8 9 7: 5 6 10 8: 5 6 10 9: 5 6 10 10: 7 8 9	$E_{6,1}$ 	1: 2 4 6 10 2: 1 3 5 3: 2 4 6 10 4: 1 3 5 5: 2 4 7 8 9 6: 1 3 7 8 9 7: 5 6 10 8: 5 6 10 9: 5 6 10 10: 3 1 7 8 9	$E_{6,2}$ 	1: 2 4 6 2: 1 3 5 10 3: 2 4 6 4: 1 3 5 10 5: 2 4 7 8 9 6: 1 3 7 8 9 7: 5 6 10 8: 5 6 10 9: 5 6 10 10: 2 4 7 8 9	$E_{6,2}$ 

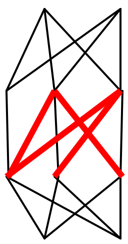
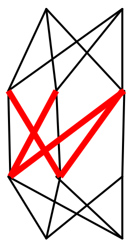
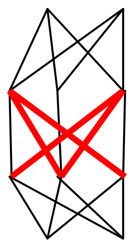
F_6 + 3 edges gives six (two non-isomorphic) forbidden graphs on torus:

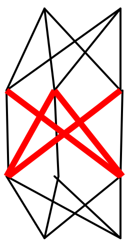
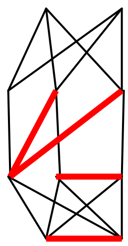
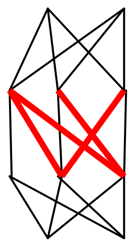
1: 2 3 4 6 2: 1 3 4 5 3: 1 2 4 6 4: 1 2 3 5 5: 2 4 6 8 10 6: 1 3 5 7 9 7: 6 8 10 8: 5 7 9 9: 6 8 10 10: 5 7 9	$F_{6,1}$ 	1: 2 4 6 8 2: 1 3 5 3: 2 4 6 8 4: 1 3 5 5: 2 4 6 8 10 6: 1 3 5 7 9 7: 6 8 10 8: 1 3 5 7 9 9: 6 8 10 10: 5 7 9	$F_{6,2}$ 	1: 2 4 6 10 2: 1 3 5 3: 2 4 6 10 4: 1 3 5 5: 2 4 6 8 10 6: 1 3 5 7 9 7: 6 8 10 8: 5 7 9 9: 6 8 10 10: 1 3 5 7 9	$F_{6,2}$ 

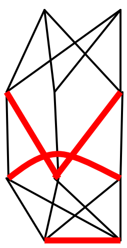
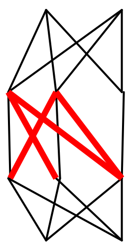
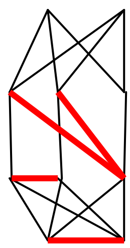
1: 2 4 6 2: 1 3 5 7 9 3: 2 4 6 4: 1 3 5 5: 2 4 6 8 10 6: 1 3 5 7 9 7: 2 6 8 10 8: 5 7 9 9: 2 6 8 10 10: 5 7 9	$F_{6,2}$ 	1: 2 4 6 2: 1 3 5 3: 2 4 6 8 10 4: 1 3 5 5: 2 4 6 8 10 6: 1 3 5 7 9 7: 6 8 10 8: 3 5 7 9 9: 6 8 10 10:3 5 7 9	$F_{6,2}$ 	1: 2 4 6 2: 1 3 5 3: 2 4 6 4: 1 3 5 5: 2 4 6 8 10 6: 1 3 5 7 9 7: 6 8 9 10 8: 5 7 9 10 9: 6 7 8 10 10: 5 7 8 9	$F_{6,1}$ 

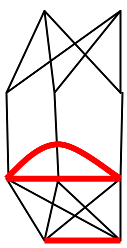
G + 4 edges gives 13 (three non-isomorphic) forbidden graphs on torus:

1: 2 3 4 5 9 2: 1 3 4 5 3: 1 2 4 5 10 4: 1 2 3 5 5: 1 2 3 4 7 6: 7 9 10 7: 5 6 8 8: 7 9 10 9: 1 6 8 10: 3 6 8	G_1 	1: 2 4 5 9 2: 1 3 4 5 3: 2 4 7 9 10 4: 1 2 3 5 5: 1 2 4 7 6: 7 9 10 7: 3 5 6 8 8: 7 9 10 9: 1 3 6 8 10: 3 6 8	G_2 	1: 2 4 7 9 10 2: 1 3 4 5 3: 2 4 5 10 4: 1 2 3 5 5: 2 3 4 7 6: 7 9 10 7: 1 5 6 8 8: 7 9 10 9: 1 6 8 10: 1 3 6 8	G_2 

1: 2 4 7 9 10 2: 1 3 5 3: 2 4 7 9 10 4: 1 3 5 5: 2 4 7 6: 7 9 10 7: 1 3 5 6 8 8: 7 9 10 9: 1 3 6 8 10: 1 3 6 8	G_3 	1: 2 4 7 9 10 2: 1 3 5 3: 2 4 7 10 4: 1 3 5 5: 2 4 7 10 6: 7 9 10 7: 1 3 5 6 8 8: 7 9 10 9: 1 6 8 10: 1 3 5 6 8	G_3 	1: 2 4 7 9 10 2: 1 3 5 3: 2 4 10 4: 1 3 5 5: 2 4 7 9 10 6: 7 9 10 7: 1 5 6 8 8: 7 9 10 9: 1 5 6 8 10: 1 3 5 6 8	G_3 

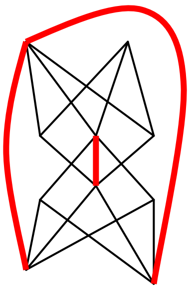
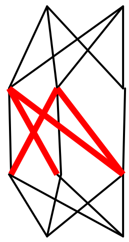
1: 2 4 7 9 2: 1 3 5 3: 2 4 7 9 10 4: 1 3 5 5: 2 4 7 9 6: 7 9 10 7: 1 3 5 6 8 8: 7 9 10 9: 1 3 5 6 8 10: 3 6 8	G_3 	1: 2 4 7 9 2: 1 3 5 3: 2 4 7 10 4: 1 3 5 5: 2 4 7 6: 7 8 9 10 7: 1 3 5 6 8 8: 6 7 9 10 9: 1 6 8 10 10: 3 6 8 9	G_2 	1: 2 4 9 10 2: 1 3 5 3: 2 4 9 10 4: 1 3 5 5: 2 4 7 9 10 6: 7 9 10 7: 5 6 8 8: 7 9 10 9: 1 3 5 6 8 10: 1 3 5 6 8	G_3 

1: 2 4 9 10 2: 1 3 5 3: 2 4 10 4: 1 3 5 5: 2 4 7 10 6: 7 8 9 10 7: 5 6 8 9 8: 6 7 9 10 9: 1 6 7 8 10: 1 3 5 6 8	G_2 	1: 2 4 9 2: 1 3 5 3: 2 4 7 9 10 4: 1 3 5 5: 2 4 7 9 10 6: 7 9 10 7: 3 5 6 8 8: 7 9 10 9: 1 3 5 6 8 10: 3 5 6 8	G_3 	1: 2 4 9 2: 1 3 5 3: 2 4 9 10 4: 1 3 5 5: 2 4 7 9 6: 7 8 9 10 7: 5 6 8 10 8: 6 7 9 10 9: 1 3 5 6 8 10: 3 6 7 8	G_2 

1: 2 4 9 2: 1 3 5 3: 2 4 10 4: 1 3 5 5: 2 4 7 6: 7 8 9 10 7: 5 6 8 9 10 8: 6 7 9 10 9: 1 6 7 8 10 10: 3 6 7 8 9	G_1 

$E_{6,1}$ and $F_{6,1}$ are not isomorphic, because vertices of maximal degree in the second one are adjacent, but in the first one they are not.

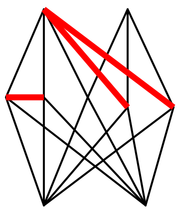
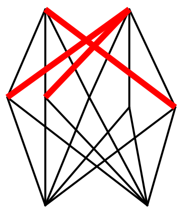
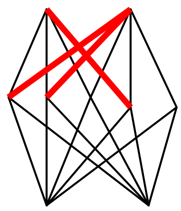
Graphs $F_{6,2}$ and G_3 are isomorfic [with substitution (1 2 3 4) (5 9 10 8 6) (7) as below]:

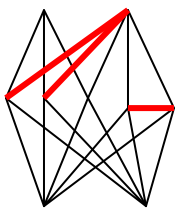
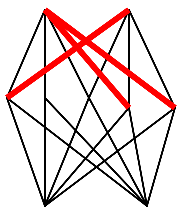
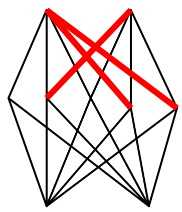
1: 2 4 6 2: 1 3 5 7 9 3: 2 4 6 4: 1 3 5 5: 2 4 6 8 10 6: 1 3 5 7 9 7: 2 6 8 10 8: 5 7 9 9: 2 6 8 10 10: 5 7 9	$F_{6,2}$ 	1: 2 4 9 2: 1 3 5 3: 2 4 7 9 10 4: 1 3 5 5: 2 4 7 9 10 6: 7 9 10 7: 3 5 6 8 8: 7 9 10 9: 1 3 5 6 8 10: 3 5 6 8	G_3 

Thus we have 6 non-isomorphic graphs with 10 vertices and 19 edges that are minimal forbidden minors for torus.

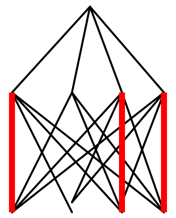
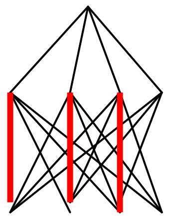
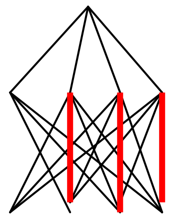
9 vertices case

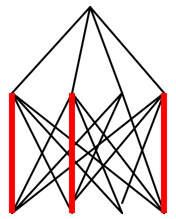
E_5 + 3 edges gives six (two non-isomorphic) forbidden graphs on torus:

1: 2 4 6 8 9 2: 1 3 4 5 3: 2 4 6 8 9 4: 1 2 3 5 5: 2 4 6 8 9 6: 1 3 5 7 7: 6 8 9 8: 1 3 5 7 9: 1 3 5 7		1: 2 4 6 8 9 2: 1 3 5 7 3: 2 4 6 8 9 4: 1 3 5 7 5: 2 4 6 9 6: 1 3 5 7 7: 2 4 6 8 9 8: 1 3 7 9: 1 3 5 7		1: 2 4 6 8 9 2: 1 3 5 7 3: 2 4 6 8 9 4: 1 3 5 7 5: 2 4 8 9 6: 1 3 7 7: 2 4 6 8 9 8: 1 3 5 7 9: 1 3 5 7	

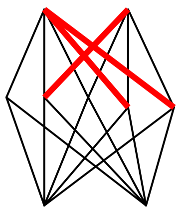
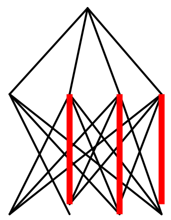
1: 2 4 6 8 9 2: 1 3 5 7 3: 2 4 6 8 9 4: 1 3 5 7 5: 2 4 9 6: 1 3 7 8 7: 2 4 6 8 9 8: 1 3 6 7 9: 1 3 5 7		1: 2 4 6 8 9 2: 1 3 5 7 3: 2 4 6 8 9 4: 1 3 5 5: 2 4 6 8 9 6: 1 3 5 7 7: 2 6 8 9 8: 1 3 5 7 9: 1 3 5 7		1: 2 4 6 8 9 2: 1 3 5 3: 2 4 6 8 9 4: 1 3 5 7 5: 2 4 6 8 9 6: 1 3 5 7 7: 4 6 8 9 8: 1 3 5 7 9: 1 3 5 7	

E_{22} + 3 edges gives four (one non-isomorphic) forbidden graphs on torus:

1: 2 4 6 8 9 2: 1 3 5 7 3: 2 4 6 8 9 4: 1 3 7 5: 2 6 8 9 6: 1 3 5 7 7: 2 4 6 8 9 8: 1 3 5 7 9: 1 3 5 7		1: 2 4 6 8 9 2: 1 3 5 7 3: 2 4 6 9 4: 1 3 5 7 5: 2 4 6 8 9 6: 1 3 5 7 7: 2 4 6 8 9 8: 1 5 7 9: 1 3 5 7		1: 2 4 6 8 9 2: 1 3 5 3: 2 4 6 8 9 4: 1 3 5 7 5: 2 4 6 8 9 6: 1 3 5 7 7: 4 6 8 9 8: 1 3 5 7 9: 1 3 5 7	

1: 2 4 8 9 2: 1 3 5 7 3: 2 4 6 8 9 4: 1 3 5 7 5: 2 4 6 8 9 6: 3 5 7 7: 2 4 6 8 9 8: 1 3 5 7 9: 1 3 5 7	

These graphs, arising from different forbidden graphs for projective plane, are identical:

1: 2 4 6 8 9 2: 1 3 5 3: 2 4 6 8 9 4: 1 3 5 7 5: 2 4 6 8 9 6: 1 3 5 7 7: 4 6 8 9 8: 1 3 5 7 9: 1 3 5 7		1: 2 4 6 8 9 2: 1 3 5 3: 2 4 6 8 9 4: 1 3 5 7 5: 2 4 6 8 9 6: 1 3 5 7 7: 4 6 8 9 8: 1 3 5 7 9: 1 3 5 7	

Thus, with 9 vertices and 19 edges, there are two forbidden minimal minors for torus.