L(p,1)-labelling of graphs*

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Abstract

An $L(p,1)$-labelling of a graph is a function f from the vertex set to the positive integers such that $|f(x)-f(y)| \geq p$ if $\mathrm{dist}(x,y)=1$ and $|f(x)-f(y)| \geq 1$ if $\mathrm{dist}(x,y)=2$, where $\mathrm{dist}(x,y)$ is the distance between the two vertices x and y in the graph. The *span* of an $L(p,1)$-labelling f is the difference between the largest and the smallest labels used by f plus 1. In 1992, Griggs and Yeh conjectured that every graph with maximum degree $\Delta \geq 2$ has an $L(2,1)$-labelling with span

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at most Δ^2+1. We settle this conjecture for Δ sufficiently large. More generally, we show that for any positive integer p there exists a constant Δ_p such that every graph with maximum degree $\Delta \ge \Delta_p$ has an $L(p,1)$-labelling with span at most Δ^2+1. This yields that, for each positive integer p, there is an integer C_p such that every graph with maximum degree Δ has an $L(p,1)$-labelling with span at most Δ^2+C_p.

1 Introduction

In the channel assignment problem, transmitters at various nodes within a geographic territory must be assigned channels or frequencies in such a way as to avoid interferences. A model for the channel assignment problem developed wherein channels or frequencies are represented with integers, "close" transmitters must be assigned different integers and "very close" transmitters must be assigned integers that differ by at least 2. This quantification led to the definition of an $L(p,q)$*-labelling* of a graph $G=(V,E)$ as a function f from the vertex set to the positive integers such that $|f(x)-f(y)| \ge p$ if $\text{dist}(x,y)=1$ and $|f(x)-f(y)| \ge q$ if $\text{dist}(x,y)=2$, where $\text{dist}(x,y)$ is the distance between the two vertices x and y in the graph G. The notion of $L(2,1)$-labelling first appeared in 1992 [13]. Since then, a large number of articles has been published devoted to the study of $L(p,q)$-labellings. We refer the interested reader to the surveys of Calamoneri [7] and Yeh [25].

Generalisations of $L(p,q)$-labellings in which for each $i \ge 1$, a minimum gap of p_i is required for channels assigned to vertices at distance i, have also been studied (see for example the recent survey of Griggs and Král' [12], and consult also [3, 16, 17, 19]).

In the context of the channel assignment problem, the main goal is to minimise the number of channels used. Hence, we are interested in the *span* of an $L(p,q)$-labelling f, which is the difference between the largest and the smallest labels of f plus 1. The $\lambda_{p,q}$*-number* of G is $\lambda_{p,q}(G)$, the minimum span over all $L(p,q)$-labellings of G. In general, determining the $\lambda_{p,q}$-number of a graph is NP-hard [10]. In their seminal paper, Griggs and Yeh [13] observed that a greedy algorithm yields that $\lambda_{2,1}(G) \le \Delta^2+2\Delta+1$, where Δ is the maximum degree of the graph G. Moreover, they conjectured that this upper bound can be decreased to Δ^2+1.

Conjecture 1 ([13])**.** For every $\Delta \ge 2$ and every graph G of maximum degree Δ,

$$\lambda_{2,1}(G) \le \Delta^2+1\,.$$

This upper bound would be tight: there are graphs with degree Δ, diameter 2 and $\Delta^2 + 1$ vertices, namely the 5-cycle, the Petersen graph and the Hoffman-Singleton graph. Thus, their square is a clique of order $\Delta^2 + 1$, so the span of every $L(2,1)$-labelling is at least $\Delta^2 + 1$.

However, such graphs exist only for Δ being 2, 3, 7, and possibly 57, as shown by Hoffman and Singleton [14]. So one can ask how large may be the $\lambda_{2,1}$-number of a graph with large maximum degree. As it should be at least as large as the largest clique in its square, one can ask what is the largest clique number $\gamma(\Delta)$ of the square of a graph with maximum degree Δ. If Δ is a prime power plus 1, then $\gamma(\Delta) \geq \Delta^2 - \Delta + 1$. Indeed, in the projective plane of order $\Delta - 1$, each point is in Δ lines, each line contains Δ points, each pair of distinct points is in a line and each pair of distinct lines has a common point. Consider the *incidence graph* of the projective plane: it is the bipartite graph with vertices the set of points and lines of the projective plane, and every line is linked to all the points it contains. The properties of the projective plane imply that the set of points and the set of lines form two cliques in the square of this graph, and there are $\Delta^2 - \Delta + 1$ vertices in each.

Jonas [15] improved slightly on Griggs and Yeh's upper bound by showing that every graph of maximum degree Δ admits an $L(2,1)$-labelling with span at most $\Delta^2 + 2\Delta - 3$. Subsequently, Chang and Kuo [8] provided the upper bound $\Delta^2 + \Delta + 1$ which remained the best general upper bound for about a decade. Král' and Škrekovski [18] brought this upper bound down by 1 as the corollary of a more general result. And, using the algorithm of Chang and Kuo [8], Gonçalves [11] decreased this bound by 1 again, thereby obtaining the upper bound $\Delta^2 + \Delta - 1$. Note that Conjecture 1 is true for planar graphs of maximum degree $\Delta \neq 3$. For $\Delta \geq 7$ it follows from a result of van den Heuvel and McGuiness [24], and Bella et al. [4] proved it for the remaining cases.

We prove the following approximate version of the generalisation of Conjecture 1 to $L(p,1)$-labelling.

Theorem 2. *For any fixed integer p, there exists a constant C_p such that for every integer Δ and every graph of maximum degree Δ,*

$$\lambda_{p,1}(G) \leq \Delta^2 + C_p .$$

This result is obtained by combining a greedy algorithm (or any of the previously mentioned upper bounds, or their generalisation for $L(p,1)$-labellings) with the next theorem, which settles Conjecture 1 for sufficiently large Δ.

Theorem 3. *For any fixed integer p, there is a Δ_p such that for every graph G of maximum degree $\Delta \geq \Delta_p$,*

$$\lambda_{p,1}(G) \leq \Delta^2 + 1 .$$

Actually, we consider a more general setup. We are given a graph G_1 with vertex-set V, along with a spanning subgraph G_2. We want to find a $(p,1)$-*colouring* of (G_1, G_2) that is an assignment of integers from $\{1, 2, \ldots, k\}$ to the elements of V so that vertices adjacent in G_1 receive different colours and vertices adjacent in G_2 receive colours which differ by at least p. This setup is a particular case of the *constraint matrix* or *weighted graph* model (with unit demands), formalised in the early nineties. Recently, Broersma et al. [6] called this particular case the *backbone colouring problem* and Babilon et al. [3] studied its generalisation to real weights via the notion of lambda-graphs.

Typically the maximum degree of G_1 is much larger than the maximum degree of G_2. In the case of $L(p,1)$-labelling, G_1 is the square of G_2. We impose the condition that for some integer Δ, the graph G_1 has maximum degree at most Δ^2 and G_2 has maximum degree Δ. We show that under these conditions there exists a $(p,1)$-colouring for $k = \Delta^2 + 1$ provided that Δ is large enough. The bound is best possible since G_1 may be a clique of size $\Delta^2 + 1$. Formally, we prove the following result.

Theorem 4. *Let p be an integer. There is a Δ_p such that for every $\Delta \geq \Delta_p$, and $G_2 \subseteq G_1$ with $\Delta(G_1) \leq \Delta^2$ and $\Delta(G_2) \leq \Delta$, there exists a $(p,1)$-colouring of (G_1, G_2).*

In the next section we give an outline of the proof. In the section following that, we present some probabilistic tools we need. We then turn to the gory details.

In what follows, we use G_1-*neighbour* to mean a neighbour in G_1 and G_2-*neighbour* to indicate a neighbour in G_2. For every vertex v and every subgraph H of G_1, we let $\deg^1_H(v)$ be the number of G_1-neighbours of v in H. We omit the subscript if $H = G_1$.

Moreover, lots of inequalities are correct only when Δ is large enough. In such inequalities, we use the symbols $\leq^*$, $\geq^*$ $<^*$ and $>^*$ instead of $\leq$, $\geq$ $<$ and $>$, respectively. We do not explicit the value of the constant Δ_p, and make no attempt to minimise it.

We finish this section by pointing out that Theorem 2 can be further generalised as follows:

For every integers $p \geq 2$ and q and every real $c \in [0,1]$, there exists an integer $C_{p,q,c}$ such that for every graph G of maximum degree Δ^c,

$$\lambda_{p,q}(G) \leq q \cdot \Delta^c + C_{p,q,c} \,.$$

2 A Sketch of the Proof

We consider a counter-example to Theorem 4 chosen so as to minimise V. Thus, for every proper subset X of the vertices of G_1, there is a $(p,1)$-colouring c of $(G_1[X], G_2[X])$ using at most Δ^2+1 colours. Such a colouring is a *good colouring* of X. In particular, as $G_2 \subseteq G_1$, this implies that every vertex v has more than $\Delta^2 - (2p-2)\Delta$ neighbours in G_1, as otherwise we could complete a good colouring of $V(G_1) - v$ greedily. Indeed, for each vertex, a coloured G_2-neighbour forbids $2p-1$ colours, which is $2p-2$ more than being only a G_1-neighbour.

The next lemma follows by setting $d = 1000p\Delta$ and applying to G_1 a decomposition result due to Reed [22, Lemma 15.2].

Lemma 5. *There is a partition of V into disjoint sets $D_1, \ldots, D_\ell, S$ such that*

(a) every D_i has between $\Delta^2 - 8000p\Delta$ and $\Delta^2 + 4000p\Delta$ vertices;

(b) there are at most $8000p\Delta^3$ edges of G_1 leaving any D_i;

(c) a vertex has at least $\frac{3}{4}\Delta^2$ G_1-neighbours in D_i if and only if it is in D_i; and

(d) for each vertex v of S, the neighbourhood of v in G_1 contains at most $\binom{\Delta^2}{2} - 1000p\Delta^3$ edges.

We let H_i be the subgraph of G_1 induced by D_i and $\overline{H_i}$ its complementary graph. An *internal neighbour* of a vertex of D_i is a neighbour in H_i. An *external neighbour* of a vertex of D_i is a neighbour that is not internal.

Lemma 6. *For every i, the graph $\overline{H_i}$ has no matching of size at least $10^3p\Delta$.*

Proof. Suppose on the contrary that M is a matching of size $10^3p\Delta$ in $\overline{H_i}$.

Let R be the unmatched vertices in H_i. Then, $\Delta^2 - 10^4 p\Delta < |R| < \Delta^2 + 10^4 p\Delta$ by Lemma 5(a). For each pair of vertices u and v that are matched in M, the number of internal neighbours of u plus the number of internal neighbours of v is at least $\frac{3}{2}\Delta^2$, by Lemma 5(c). Thus there are at least $\frac{1}{2}\Delta^2 - (|H_i| - \Delta^2) - 2|M| >^* \frac{1}{3}|R|$ vertices in R that are adjacent to both of u and v in G_1. So on average, a vertex of R is adjacent in G_1 to both members of at least $\frac{1}{3}|M|$ pairs. This implies that at least $\frac{1}{5}|R| >^* \frac{1}{10}\Delta^2$ members of R are adjacent in G_1 to both members of at least $\frac{1}{10}|M|$ pairs. Let X be $\frac{1}{10}\Delta^2$ such vertices in R.

Every vertex of $R \setminus X$ that is adjacent in G_1 to less than half of X must have at least $\Delta^2 - (2p-2)\Delta - (|H_i| - \frac{1}{2}|X|) >^* \frac{1}{25}\Delta^2$ G_1-neighbours outside D_i. Thus, Lemma 5(b) implies that there are at least $|R \setminus X| - 200\,000 p\Delta \geq \frac{9}{10}\Delta^2 - 10^4 p\Delta - 200\,000 p\Delta \geq^* \frac{1}{2}\Delta^2$ vertices in $R - X$ that are adjacent in G_1 to at least half of X. Let Y be a set of $\frac{1}{2}\Delta^2$ such vertices.

We consider a good colouring of $V(G_1) \setminus D_i$. We obtain a contradiction by extending this good colouring to our desired (Δ^2+1)-colouring of $V(G_1)$ greedily, as follows.

1. Colour the vertices of M, assigning the same colour to both members of each matched pair. This is possible because each pair has at most $\frac{1}{2}\Delta^2 + 2|M|$ previously coloured G_1-neighbours (by Lemma 5(c)) and 2Δ previously coloured G_2-neighbours, so there are at least $\frac{1}{2}\Delta^2 + 1 - 1004p\Delta \geq^* 1$ colours available.

2. Colour the vertices of $H_i - Y - X - M$. This is possible since each such vertex has at most $\frac{1}{4}\Delta^2$ G_1-neighbours outside of D_1 (by Lemma 5(c)), and at most $|H_i| - |X| - |Y| <^* \frac{1}{2}\Delta^2$ previously coloured internal neighbours.

3. Colour the vertices of Y. This is possible since each vertex of Y has at least $\frac{1}{2}|X| = \frac{1}{20}\Delta^2$ uncoloured G_1-neighbours and hence at least $\frac{1}{20}\Delta^2 + 1 - (2p-2)\Delta \geq^* 1$ colours available.

4. Colour the vertices of X. This is possible since each vertex of X has at least $\frac{1}{10}|M| = 100p\Delta$ colours that appear twice in its neighbourhood, and thus has at least $98p\Delta$ colours available.

□

For each $i \in \{1, 2, \ldots, \ell\}$, we let M_i be a maximum matching of $\overline{H_i}$, and K_i be the clique $D_i - V(M_i)$. So, $|K_i| \geq \Delta^2 - 10^4 p\Delta$ by Lemmas 5(a) and 6.

We let B_i be the set of vertices in K_i that have more than $\Delta^{5/4}$ G_1-neighbours outside D_i, and we set $A_i := K_i \setminus B_i$. Considering Lemma 5(b) we can make the following observation.

Observation 7. *For every index $i \in \{1, 2, \dots, \ell\}$,*

$$|B_i| \le 8000p\Delta^{7/4} \text{ and so } |A_i| \ge \Delta^2 - 9000p\Delta^{7/4}.$$

We are going to colour the vertices in three steps. We first colour $V_1 := V \setminus \cup_{i=1}^{\ell} A_i$ except some vertices of S. Then we colour the vertices of $V_2 := \cup_{i=1}^{\ell} A_i$. We finish by colouring the uncoloured vertices of S greedily.

In order to extend the (partial) colouring of V_1 to V_2, we need some properties. We prove the following.

Lemma 8. *There is a good colouring c of a subset Y of V_1 such that*

(i) *every uncoloured vertex of V_1 is in S;*

(ii) *for each edge xy of every M_i, $c(x) = c(y)$;*

(iii) *for every uncoloured vertex v of V_1, there are at least $(2p-2)\Delta$ colours that appear on two G_1-neighbours of v; and*

(iv) *for every colour j and clique A_i there are at most $\frac{4}{5}\Delta^2$ vertices of A_i that have either a G_1-neighbour outside D_i coloured j or a G_2-neighbour outside D_i with a colour in $[j-p+1, j+p-1]$.*

We then establish that a colouring that verifies the conditions of Lemma 8 can be extended to $Y \cup V_2$.

Lemma 9. *Every good colouring of a subset Y of V_1 satisfying conditions (i)–(iv) of Lemma 8 can be completed to a good colouring of $Y \cup V_2$.*

By Lemma 8(iii), we can then complete the colouring by colouring the vertices of $V_1 - Y$ greedily.

Thus to prove our theorem, we need only prove Lemmas 8 and 9. Forthwith the details.

3 Probabilistic Preliminaries

In this section, we present a few probabilistic tools that are used in this paper. Each of these tools is presented in the book of Molloy and Reed [22], and most are presented in many other places.

The Lovász Local Lemma [9] *Let $A_1, ..., A_n$ be a set of random events so that, for each $i \in \{1, 2, \ldots, n\}$,*

(i) $\mathbf{Pr}(A_i) \leq p$ *and*

(ii) A_i *is mutually independent of all but at most d other events.*

If $pd \leq \frac{1}{4}$ then $\mathbf{Pr}(\overline{A_1} \cup ... \cup \overline{A_n}) > 0$.

The *binomial random variable* $\mathrm{BIN}(n, p)$ is the sum of n independent zero-one random variables where each is equal to 1 with probability p.

The Chernoff Bound [20, 1] *For every $t \in [0, np]$,*

$$\mathbf{Pr}\left(|\mathrm{BIN}(n, p) - np| > t\right) < 2 \exp\left(-\frac{t^2}{3np}\right).$$

Only in the proof of Lemma 20 do we use the following version of the Chernoff Bound: for every $t > 0$,

$$\mathbf{Pr}\left(|\mathrm{BIN}(n, p) - np| > t\right) < 2 \exp\left(t - \ln\left(1 + \frac{t}{np}\right)(np + t)\right).$$

The following is a simple corollary of Azuma's Inequality [2, 22].

The Simple Concentration Bound *Let X be a non-negative random variable determined by the independent trials $T_1, ..., T_n$. Suppose that for every set of possible outcomes of the trials*

(i) changing the outcome of any one trial can affect X by at most c.

Then

$$\mathbf{Pr}(|X - \mathbf{E}(X)| > t) \leq 2 \exp\left(-\frac{t^2}{c^2 n}\right).$$

Talagrand's Inequality requires another condition, but often provides a stronger bound when $\mathbf{E}(X)$ is much smaller than n. Rather than providing

Talagrand's original statement [23], we present the following useful corollary [22].

Talagrand's Inequality [23] *Let X be a non-negative random variable determined by the independent trials $T_1, ..., T_n$. Suppose that for every set of possible outcomes of the trials*

(i) changing the outcome of any one trial can affect X by at most c; and

(ii) for each $s > 0$, if $X \geq s$ then there is a set of at most rs trials whose outcomes certify that $X \geq s$.

Then for every $t \in [0, \mathbf{E}(X)]$,

$$\mathbf{Pr}\left(|X - \mathbf{E}(X)| > t + 60c\sqrt{r\,\mathbf{E}(X)}\right) \leq 4\exp\left(-\frac{t^2}{8c^2 r\,\mathbf{E}(X)}\right).$$

McDiarmid extended Talagrand's Inequality to the setting where X depends on independent trials and permutations, a setting that arises in this paper. Again, we present a useful corollary [22] rather than the original inequality [21].

McDiarmid's Inequality [21] *Let X be a non-negative random variable determined by the independent trials $T_1, \ldots, T_n$ and m independent permutations $\Pi_1, \ldots, \Pi_m$. Suppose that for every set of possible outcomes of the trials*

(i) changing the outcome of any one trial can affect X by at most c;

(ii) interchanging two elements in any one permutation can affect x by at most c; and

(iii) for each $s > 0$, if $X \geq s$ then there is a set of at most rs trials whose outcomes certify that $X \geq s$.

Then for every $t \in [0, \mathbf{E}(X)]$,

$$\mathbf{Pr}\left(|X - \mathbf{E}(X)| > t + 60c\sqrt{r\,\mathbf{E}(X)}\right) \leq 4\exp\left(-\frac{t^2}{8c^2 r\,\mathbf{E}(X)}\right).$$

In both Talagrand's Inequality and McDiarmid's Inequality, if

$$60c\sqrt{r\,\mathbf{E}(X)} \leq t \leq \mathbf{E}(X)$$

then by substituting $t/2$ for t in the above bounds, we obtain the more concise

$$\mathbf{Pr}\left(|X - \mathbf{E}(X)| > t\right) \leq 4\exp\left(-\frac{t^2}{32c^2r\,\mathbf{E}(X)}\right).$$

That is the bound that we usually use.

4 The Proof of Lemma 8

In this section, we want to find a good colouring for an appropriate subset Y of $G[V_1]$, which satisfies conditions (i)–(iv) of Lemma 8. We actually construct new graphs G_1^* and G_2^* and consider good colourings of these graphs. This helps us to ensure that the conditions of Lemma 8 hold.

4.1 Forming G_1^* and G_2^*

For $j \in \{1,2\}$, we obtain G'_j from G_j by contracting each edge of each M_i into a vertex (that is, we consider these vertex pairs one by one, replacing the pair xy with a vertex adjacent to all of the neighbours of both x and y in the graph). We let C_i be the set of vertices obtained by contracting the pairs in M_i. We set $V^* := V_1 \setminus \cup_{i=1}^{\ell} V(M_i) \bigcup \cup_{i=1}^{\ell} C_i$. For each $i \in \{1,2,\dots,\ell\}$, let Big_i be the set of vertices of V^* not in $B_i \cup C_i$ that have more than $\Delta^{9/5}$ neighbours in A_i. We construct G_1^* from G'_1 by removing the vertices of $\cup_{i=1}^{\ell} A_i$ and adding for each i an edge between every pair of vertices in Big_i. And G_2^* is obtained from G'_2 by removing the vertices of $\cup_{i=1}^{\ell} A_i$.

Note that $G_2^* \subseteq G_1^*$. Our aim is to colour the vertices of V^* except some of S such that vertices adjacent in G_1^* are assigned different colours, and vertices adjacent in G_2^* are assigned colours at distance at least p. Such a colouring is said to be *nice*. To every partial nice colouring of V^* is associated the $(p,1)$-colouring of V_1 obtained as follows: each coloured vertex of $V \cap V^*$ keeps its colour, and for each index i, every pair of matched vertices of M_i is assigned the colour of the corresponding vertex of C_i. So this partial good colouring satisfies condition (ii) of Lemma 8.

Definition 10. For every vertex u and every subset F of V^*,

- the number of G_1^*-neighbours of u in F is $\delta_F^1(u)$;
- the number of G_2^*-neighbours of u in F is $\delta_F^2(u)$; and

- $\delta^*_F(u) := \delta^1_F(u) + 4p\delta^2_F(u)$.

For all these notations, we omit the subscript if $F = V^*$.

The next lemma bounds these parameters.

Lemma 11. *Let v be a vertex of V^*. The following hold.*

(i) $\delta^2(v) \le 2\Delta$, *and if* $v \notin \cup_{i=1}^{\ell} C_i$ *then* $\delta^2(v) \le \Delta$;

(ii) *if* $v \in S \cap \mathrm{Big}_i$ *for some* i, *then* $\delta^1(v) \le \Delta^2 - 8p\Delta$;

(iii) $\delta^1(v) \le \Delta^2$, *and if* $v \notin S$ *then* $\delta^1(v) \le \frac{3}{4}\Delta^2$.

Proof. *(i)* To obtain G_2^*, we only removed some vertices and contracted some pairwise disjoint pairs of non-adjacent vertices. Consequently, the degree of each new vertex is at most twice the maximum degree of G_2, i.e. 2Δ, and the degree of the other vertices is at most their degree is G_2, hence at most Δ.

(ii) By Lemma 5(*b*), we have $|\mathrm{Big}_i| \le 8000p\Delta^{6/5}$ for each index i. Moreover, a vertex v can be in Big_i for at most $\Delta^{1/5}$ values of i. Recall that for each index i such that $v \in S \cap \mathrm{Big}_i$, the vertex v has at least $\Delta^{9/5}$ G_1-neighbours in A_i. So, in the process of constructing G_1^*, it looses at least $\Delta^{9/5}$ edges and gain at most $8000p\Delta^{7/5}$ edges. Consequently, the assertion follows because $\Delta^{9/5} \ge^* 8000p\Delta^{7/5} + 8p\Delta$.

(iii) By *(ii)*, if $v \in S$ then $\delta^1(v) \le \deg^1(v) \le \Delta^2$. Assume now that $v \notin S$, hence $v \in B_i \cup C_i$ for some index i. By Lemma 6, each set C_i has at most $1000p\Delta$ vertices and by Observation 7, each set B_i has at most $8000p\Delta^{7/4}$ vertices. Moreover, by Lemma 5(*c*), each vertex of D_i has at most $\frac{1}{4}\Delta^2$ G_1-neighbours outside of D_i. It follows that each vertex of $B_i \cup C_i$ has at most $\frac{1}{2}\Delta^2 + 1000p\Delta + 8000p\Delta^{7/4} + 8000p\Delta^{7/5} \le^* \frac{3}{4}\Delta^2$ G_1^*-neighbours.

□

Our construction of G_1' and G_2' is designed to deal with condition *(ii)* of Lemma 8. The edges we add between vertices of Big_i are designed to help with condition *(iv)*. The bound of $\frac{3}{4}\Delta^2$ on the degree of the vertices of $V^* \setminus S$ in the last lemma helps us to ensure that condition *(i)* holds.

To ensure that condition *(iii)* holds, we would like to use condition *(i)* and the fact that sparse vertices have many non-adjacent pairs of G_1-neighbours.

However, in constructing G_1^*, we contracted some pairs of non-adjacent vertices and added edges between some other pairs of non-adjacent vertices. As a result, possibly some vertices in S are no longer sparse. We have to treat such vertices carefully.

We define $\hat{S}$ to be those vertices in S that have at least $90p\Delta$ neighbours outside S. Then $\hat{S}$ contains all the vertices which may no longer be sufficiently sparse, as we note next.

Lemma 12. *Each vertex of $S \setminus \hat{S}$ has at least $450p\Delta^3$ pairs of G_1-neighbours in S that are not adjacent in G_1^*.*

Proof. Let $s \in S \setminus \hat{S}$. We know that s has at least $\Delta^2 - (2p-2)\Delta$ G_1-neighbours. Hence it has more than $\binom{\Delta^2}{2} - 4p\Delta^3$ pairs of G_1-neighbours. Thus, by Lemma 5(d), the vertex s has more than $996p\Delta^3$ pairs of G_1-neighbours that are not adjacent in G_1. Since $s \notin \hat{S}$, all but at most $90p\Delta^3$ such pairs lie in $N(s) \cap S$. Let Ω be the collection of pairs of G_1-neighbours of s in S that are not adjacent in G_1. Then $|\Omega| \geq 906p\Delta^3$. For convenience, we say that a pair of Ω is *suitable* if its vertices are not adjacent in G_1^*.

Let s_1 be a member of a pair of Ω. If s_1 does not belong to $\cup_{i=1}^{\ell} \mathrm{Big}_i$, then every vertex of S that is not adjacent to s_1 in G_1 is also not adjacent to s_1 in G_1^*. Thus every pair of Ω containing s_1 is suitable.

If $s_1 \in \cup_{i=1}^{\ell} \mathrm{Big}_i$, then for each index i such that $s_1 \in \mathrm{Big}_i$, the vertex s_1 has at least $\Delta^{9/5}$ G_1-neighbours in A_i. Hence, there are more than $\Delta^2 - 92p\Delta - (\Delta^2 - \Delta^{9/5}) = \Delta^{9/5} - 92p\Delta$ pairs of Ω containing s_1. Recall from the proof of Lemma 11 that the number of edges added to s_1 by the construction of G_1^* is at most $8000p\Delta^{7/5} <^* \frac{1}{2}\Delta^{9/5} - 46p\Delta$. Consequently, the number of suitable pairs of Ω containing the vertex s_1 is at least half the number of pairs of Ω containing s_1.

Therefore, we conclude that at least $\frac{1}{2}|\Omega| > 450p\Delta^3$ pairs of Ω are suitable. $\square$

It turns out that we will colour all of $\hat{S}$, which makes it easier to ensure that condition (*iii*) holds.

4.2 High Level Overview

Our first step is to colour some of S, including all of $\hat{S}$. We do this in two phases. In the first one, we consider assigning each vertex of S a colour at random. We show by analysing this random procedure that there is a partial nice colouring of S such that every vertex of $S \setminus \hat{S}$ satisfies condition (iii) of Lemma 8. In the second phase, we finish colouring the vertices of $\hat{S}$. We use an iterative quasi-random procedure. In each iteration but the last, each vertex chooses a colour, from those which do not yield a conflict with any already coloured neighbour, uniformly at random. The last iteration has a similar flavour.

We then turn to colouring the vertices in the sets B_i and C_i. Our degree bounds imply that we could do this greedily. However, we will mimic the iterative approach just discussed. We use this complicated colouring process because it allows us to ensure that condition (iv) of Lemma 8 holds for the colouring we obtain. At any point during the colouring process, $\mathrm{Notbig}_{i,j}$ is the set of vertices $v \in A_i$ such that v has either a G_1'-neighbour $u \notin \mathrm{Big}_i \cup D_i$ that has colour j or a G_2'-neighbour $u \notin \mathrm{Big}_i \cup D_i$ that has a colour in $[j-p+1, j+p-1]$. The challenge is to construct a colouring such that $\mathrm{Notbig}_{i,j}$ remains small for every index i and every colour j.

4.3 Colouring sparse vertices

As mentioned earlier, we colour sparse vertices in two phases. The first one provides a partial nice colouring of S satisfying condition (iii) of Lemma 8. The second one extends this nice colouring to all the vertices of $\hat{S}$, using an iterative quasi-random procedure.

We need a lemma to bound the size of $\mathrm{Notbig}_{i,j}$. We consider the following setting. We have a collection of at most Δ^2 subsets of vertices. Each set contains at most Q vertices, and no vertex lies in more than $\Delta^{9/5}$ sets. A random experiment is conducted, where each vertex is marked with probability at most $\frac{1}{Q \cdot \Delta^{2/5}}$. We moreover assume that, for any set of $s \geq 1$ vertices, the probability that all are marked is at most $\left(\frac{1}{Q \cdot \Delta^{2/5}}\right)^s$. Note that this is in particular the case if the vertices are marked independently.

Lemma 13. *Under the preceding hypothesis, the probability that at least* $\Delta^{37/20}$ *sets contain a marked vertex is at most* $\exp\left(-\Delta^{1/20}\right)$.

Proof. For every $i \in \{1, 2, \ldots, 9\}$, let E_i be the event that at least $\frac{1}{9}\Delta^{37/20}$ sets contain a marked member of T_i, where T_i is the set of vertices lying in between $\Delta^{(i-1)/5}$ and $\Delta^{i/5}$ sets. Note that if at least $\Delta^{37/20}$ sets contain at least one marked vertex, then at least one the events E_i holds.

The total number of vertices in the sets being at most $\Delta^2 Q$, we deduce that $|T_i| \leq \frac{\Delta^2 Q}{\Delta^{(i-1)/5}}$. Furthermore, if E_i holds then at least $\frac{1}{9}\Delta^{37/20 - i/5}$ vertices of T_i must be marked. Therefore,

$$\begin{aligned}
\mathbf{Pr}(E_i) &\le \binom{\Delta^2 Q/\Delta^{(i-1)/5}}{\frac{1}{9}\Delta^{37/20-i/5}} \cdot \left(\frac{1}{Q\Delta^{2/5}}\right)^{\frac{1}{9}\Delta^{37/20-i/5}} \\
&\le \left(\frac{e\Delta^2 Q/\Delta^{(i-1)/5}}{\frac{1}{9}\Delta^{37/20-i/5} \times Q\Delta^{2/5}}\right)^{\frac{1}{9}\Delta^{37/20-i/5}} \qquad \text{(by Stirling formula)} \\
&\le \left(\frac{9e}{\Delta^{1/20}}\right)^{\frac{1}{9}\Delta^{37/20-i/5}}.
\end{aligned}$$

Since $\frac{1}{9}\Delta^{37/20-i/5} \ge \frac{1}{9}\Delta^{1/20}$, the probability that E_i holds is at most $\frac{1}{9}\exp\left(-\Delta^{1/20}\right)$, and therefore the sought result follows. □

4.3.1 First step

Lemma 14. *There exists a nice colouring of a subset H of S with colours in $\{1, 2, \ldots, \Delta^2 + 1\}$ such that*

(i) every uncoloured vertex v of $S \setminus \hat{S}$ has at least $(2p-2)\Delta$ colours appearing at least twice in $N_S(v) := N_{G_1}(v) \cap S$;

(ii) every vertex of S has at most $\frac{19}{20}\Delta^2$ coloured G_1^*-neighbours;

(iii) for every index i and every colour j, the size of $\mathrm{Notbig}_{i,j}$ *is at most $\Delta^{19/10}$.*

Proof. For convenience, let us set $C := \Delta^2+1$. We use the following colouring procedure.

1. Each vertex of S is activated with probability $\frac{9}{10}$.
2. Each activated vertex is assigned a colour of $\{1, 2, \ldots, C\}$, independently and uniformly at random.
3. A vertex which gets a colour creating a conflict — i.e. assigned to one of its G_1^*-neighbours, or at distance less than p of a colour assigned to one of its G_2^*-neighbours — is uncoloured.

We aim at applying the Lovász Local Lemma to prove that, with positive probability, the resulting colouring fulfils the three conditions of the lemma. Let v be a vertex of G. We let $E_1(v)$ be the event that v does not fulfil condition (i), and $E_2(v)$ be the event that v does not fulfil condition (ii). For each i, j, let $E_3(i,j)$ be the event that the size of $\text{Notbig}_{i,j}$ exceeds $\Delta^{19/10}$. It suffices to prove that each of those events occurs with probability less than Δ^{-17}. Indeed, each event is mutually independent of all events involving vertices or dense sets at distance more than 4 in G_1^* or G_1'. Moreover, each vertex of any set A_i has at most $\Delta^{5/4}$ external neighbours in G, and $|A_i| \leq \Delta^2+1$. Thus, each event is mutually independent of all but at most Δ^{16} other events. Consequently, the Lovász Local Lemma applies since $\Delta^{-17} \times \Delta^{16} <^* \frac{1}{4}$, and yields the sought result.

Hence, it only remains to prove that the probability of each event is at most Δ^{-17}. We use the results cited in Section 3. Let us start with $E_2(v)$. We define W to be the number of activated neighbours of v. Thus, $\mathbf{Pr}(E_2(v)) \leq \mathbf{Pr}\left(W > \frac{19}{20}\Delta^2\right)$. We set $m := |N(v) \cap S|$, and we may assume that $m > \frac{19}{20}\Delta^2$. The random variable W is a binomial on m variables with probability $\frac{9}{10}$. In particular, its expected value $\mathbf{E}(W)$ is $\frac{9m}{10}$. Applying the Chernoff Bound to W with $t = \frac{m}{20}$, we obtain

$$\begin{aligned} \mathbf{Pr}\left(W > \tfrac{19}{20}\Delta^2\right) &\leq \mathbf{Pr}\left(|W - \mathbf{E}(W)| > \tfrac{m}{20}\right) \\ &\leq 2\exp\left(-\tfrac{m^2 \cdot 10}{400 \cdot 27m}\right) \leq^* \Delta^{-17}, \end{aligned}$$

since $\frac{19}{20}\Delta^2 < m \leq \Delta^2$.

Let $v \in S \setminus \hat{S}$. We now bound $\mathbf{Pr}(E_1(v))$. By Lemma 12, let Ω be a collection of $450p\Delta^3$ pairs of G_1-neighbours of v in S that are not adjacent in G_1^*. We consider the random variable X defined as the number of pairs of Ω whose members (i) are both assigned the same colour j, (ii) both retain that colour, and (iii) are the only two vertices in $N(v)$ that are assigned j. Thus, X is at most the number of colours appearing at least twice in $N_S(v)$. The probability that some non-adjacent pair of vertices u, w in $N(v)$ satisfies (i) is $\frac{9}{10} \cdot \frac{9}{10} \cdot \frac{1}{C}$. In total, the number of G_1^*-neighbours of v, u, w in H is at most $3\Delta^2$, and the number of G_2^*-neighbours of u and w is at most 4Δ. Therefore, given that they satisfy (i), the vertices u and w also satisfy (ii) and (iii) with probability at least $\left(1 - \frac{1}{C}\right)^{3\Delta^2} \cdot \left(1 - \frac{2p-2}{C}\right)^{4\Delta}$. Consequently,

$$\mathbf{E}(X) \geq 450p\Delta^3 \cdot \frac{81}{100C} \cdot \exp\left(-\frac{3\Delta^2}{C}\right) \exp\left(-\frac{(8p-8)\cdot\Delta}{C}\right) >^* 3p\Delta\,.$$

Hence, if $E_1(v)$ holds then X must be smaller than its expected value by at least $p\Delta$. But we assert that

$$\mathbf{Pr}\left(\mathbf{E}(X) - X > p\Delta\right) \leq^* \Delta^{-17}, \tag{1}$$

which will yield the desired result.

To establish Equation (1), we apply Talagrand's Inequality, stated in Section 3. We set X_1 to be the number of colours assigned to at least two vertices in $N(v)$, including both members of at least one pair in Ω, and X_2 is the number of colours that (i) are assigned to both members of at least one pair in Ω, and (ii) create a conflict with one of their neighbours, or are also assigned to at least one other vertex in $N(v)$. Note that $X = X_1 - X_2$. Therefore, by what precedes, if $E_1(v)$ holds then either X_1 or X_2 must differ from its expected value by at least $\frac{1}{2}p\Delta$. Notice that

$$\mathbf{E}(X_2) \leq \mathbf{E}(X_1) \leq C \cdot 450p\Delta^3 \cdot \frac{1}{C^2} \leq 450p\Delta\,.$$

If $X_1 \geq t$, then there is a set of at most $4t$ trials whose outcomes certify this, namely the activation and colour assignment for t pairs of variables. Moreover, changing the outcome of any random trial can only affect X_1 by at most 1, since X_1 can decrease by 1 in case the old colour is not counted anymore and increase by 1 in case the new colour was not counted before and is counted now. Thus Talangrand's Inequality applies and, since $\mathbf{E}(X_1) \geq \mathbf{E}(X) >^* 3p\Delta$, we obtain

$$\mathbf{Pr}\left(|X_1 - \mathbf{E}(X_1)| > \frac{1}{2}p\Delta\right) \leq 4\exp\left(-\frac{p^2\Delta^2}{4\cdot 32\cdot 1\cdot 4\cdot 450p\Delta}\right) \leq^* \frac{1}{2}\Delta^{-17}\,.$$

Similarly, if $X_2 \geq t$ then there is a set of at most $6t$ trials whose outcomes certify this fact, namely the activation and colour assignment of t pairs of vertices and, for each of these pairs, the activation and colour assignment of a colour creating a conflict to a neighbour of a vertex of the pair. As previously, changing the outcome of any random trial can only affect X_2 by at most $2p$. Therefore by Talagrand's Inequality, if $\mathbf{E}(X_2) \geq \frac{1}{2}p\Delta$ then

$$\mathbf{Pr}\left(|X_2 - \mathbf{E}(X_2)| > \frac{1}{2}p\Delta\right) \leq 4\exp\left(-\frac{p^2\Delta^2}{4\cdot 32\cdot 4p^2\cdot 6\cdot 450p\Delta}\right) \leq^* \frac{1}{2}\Delta^{-17}\,.$$

If $\mathbf{E}(X_2) < \frac{1}{2}p\Delta$, then we consider a binomial random variable that counts each vertex of $N_S(v)$ independently with probability $\frac{1}{4|N_S(v)|}p\Delta$. We let X'_2 be the sum of this random variable and X_2. Note that $\frac{1}{4}p\Delta \leq \mathbf{E}(X'_2) \leq \frac{3}{4}p\Delta$ by Linearity of Expectation. Moreover, observe that if $|X_2 - \mathbf{E}(X_2)| > \frac{1}{2}p\Delta$ then $|X'_2 - \mathbf{E}(X'_2)| > \frac{1}{4}p\Delta$. Therefore, by applying Talagrand's Inequality to X'_2 with $c = 2p$, $r = 6$, and $t = \frac{1}{4}p\Delta \in [60c\sqrt{r\,\mathbf{E}(X'_2)}, \mathbf{E}(X'_2)]$, we also obtain in this case

$$\mathbf{Pr}\left(|X_2 - \mathbf{E}(X_2)| > \frac{1}{2}p\Delta\right) \leq \mathbf{Pr}\left(|X_2' - \mathbf{E}(X_2')| > \frac{1}{4}p\Delta\right)$$
$$\leq 4\exp\left(-\frac{2\cdot p^2\Delta^2}{16\cdot 32\cdot 4p^2\cdot 6\cdot p\Delta}\right) \leq^* \frac{1}{2}\Delta^{-17}.$$

Consequently, we infer that $\mathbf{Pr}\left(\mathbf{E}(X) - X > \Delta\right) \leq^* \Delta^{-17}$, as desired.

It remains now to deal with $E_3(i,j)$. We use Lemma 13. For each i, every vertex of A_i has at most $\Delta^{5/4}$ external neighbours. Moreover, for each colour j, each such neighbour is activated and assigned a colour in $[j-p+1, j+p-1]$ with probability at most $\frac{9}{10}\cdot\frac{(2p-1)}{C} <^* \frac{1}{\Delta^{5/4}\cdot\Delta^{2/5}}$. As these assignments are made independently, the conditions of Lemma 13 are fulfilled, so we deduce that the probability that $E_3(i,j)$ holds is at most $\exp\left(-\Delta^{1/20}\right) \leq^* \Delta^{-17}$. Thus, we obtained the desired upper bound on $\mathbf{Pr}(E_3(i,j))$, which concludes the proof. □

4.3.2 Second step

In the second step, we extend the partial colouring of S to all the vertices of $\hat{S}$. To do so, we need the following general lemma, that will also be used in the next subsection to colour the vertices of the sets $B_i \cup C_i$. Its proof is long and technical, so we postpone it to Section 6.

Lemma 15. *Let F be a subset of V^* with a partial nice colouring, and H be a set of uncoloured vertices of F. For each vertex u of H, let $L(u)$ be the colours available to colour u, that is that create no conflict with the already coloured vertices of $F \cup H$. We assume that for every vertex u, $|L(u)| \geq 11p\Delta^{33/20}$ and $|L(u)| \geq \delta^1_H(u) + 6p\Delta$.*

Then, the partial nice colouring of F can be extended to a nice colouring of H such that for every index $i \in \{1, 2, \ldots, \ell\}$ and every colour j, the size of $\mathrm{Notbig}_{i,j}$ increases by at most $\Delta^{19/10}$.

Consider a partial nice colouring of S obtained in the first step. In particular, $|\mathrm{Notbig}_{i,j}| \leq \Delta^{19/10}$. We wish to ensure that every vertex of $\hat{S}$ is coloured. This can be done greedily, but to be able to continue the proof we need to have more control on the colouring. We apply Lemma 15 to the set H of uncoloured vertices in $\hat{S}$. For each vertex $u \in H$, the list $L(u)$ is initialised as the list of colours that can be assigned to u without creating any conflict. By Lemmas 11 and 14(ii), $|L(u)| \geq \frac{1}{20}\Delta^2 - 4p\Delta \geq^* 11p\Delta^{33/20}$.

Suppose that u is in no set Big_i. Then $\delta^1_S(u) \le \deg^1_S(u) \le \Delta^2 - 90p\Delta$, and u has at most Δ G_2^*-neighbours. Hence, we infer that $|L(u)| \ge \delta^1_H(u) + 88p\Delta$. Assume now that u belongs to some set Big_i. By Lemma 11(i) and (ii), we have $\delta^1(u) \le \Delta^2 - 8p\Delta$ and $\delta^2(u) \le \Delta$. So, $|L(u)| \ge \delta^1_H(u) + 8p\Delta - 2p\Delta = \delta^1_H(u) + 6p\Delta$.

Therefore, by Lemma 15 we can extend the partial nice colouring of S to $\hat{S}$ such that $|\mathrm{Notbig}_{i,j}| \le 2\Delta^{19/10}$ for every index i and every colour j.

4.4 Colouring the sets B_i and C_i

Let $H := \bigcup_{i=1}^{\ell}(B_i \cup C_i)$. At this stage, the vertices of H are uncoloured. We first apply Lemma 15 to extend the partial nice colouring of S to the vertices of H in such a way that $\mathrm{Notbig}_{i,j}$ does not grow too much, for every index i and colour j. Next, we show that the good colouring derived from this nice colouring satisfies the conditions of Lemma 8.

For each vertex u of H, let $L(u)$ be the list of colours that would not create any conflict with the already coloured vertices. By Lemma 11(iii), $\delta^1(u) \le \frac{3}{4}\Delta^2$. Hence, $|L(u)| \ge \frac{1}{4}\Delta^2 + \delta^1_H(u) - 4p\Delta \ge^* \max\left(11p\Delta^{33/20}, \delta^1_H(u) + 6p\Delta\right)$.

Therefore, by Lemma 15, we extend the partial nice colouring of the vertices of S to the vertices of $\bigcup_{i=1}^{\ell}(B_i \cup C_i)$. Moreover, for each index i and each colour j, the size of each $\mathrm{Notbig}_{i,j}$ is at most $3\Delta^{19/10}$.

Consider now the partial good colouring of V_1 associated to this nice colouring. Let us show that it satisfies the conditions of Lemma 8. By the definition, it satisfies conditions (i) and (ii). Condition (iii) follows from Lemma 14. Hence, it only remains to show that condition (iv) holds.

Fix an index i and a colour j. Recall that Big_i is a clique, so there is at most one vertex of Big_i of each colour. Consequently, the number of vertices of A_i with a G_1-neighbour in Big_i coloured j is at most $\max\left(2 \cdot \frac{1}{4}\Delta^2, \frac{3}{4}\Delta^2\right) = \frac{3}{4}\Delta^2$, by Lemma 5($c$). Besides, the number of vertices of A_i with a G_2-neighbour in Big_i with a colour in $[j-p+1, j+p-1]$ is at most $4p\Delta$. Finally, the number of vertices of A_i with either a G_1-neighbour not in $\mathrm{Big}_i \cup D_i$ coloured j, or a G_2-neighbour not in $\mathrm{Big}_i \cup D_i$ with a colour in $[j-p+1, j+p-1]$ is at most $|\mathrm{Notbig}_{i,j}| \le 3\Delta^{19/10}$. Thus, all together, the number of vertices of A_i with a G_1-neighbour not in $B_i \cup C_i$ coloured j, or a G_2-neighbour not in $B_i \cup C_i$ with a colour in $[j-p+1, j+p-1]$ is at most

$$\frac{3}{4}\Delta^2 + 3\Delta^{19/10} + 4p\Delta \le^* \frac{4}{5}\Delta^2\,,$$

as desired.

This concludes the proof of Lemma 8.

5 The Proof of Lemma 9

We consider a good colouring of V satisfying the conditions of Lemma 8. The procedure we apply is composed of two phases. In the first phase, a random permutation of a subset of the colours is assigned to the vertices of A_i. In doing so, we might create two kinds of conflicts: a vertex of A_i coloured j might have an external G_1-neighbour coloured j, or a G_2-neighbour with a colour in $[j-p+1, j+p-1]$. We shall deal with these conflicts in a second phase. To be able to do so, we first ensure that the colouring obtained in the first phase fulfils some properties.

Proposition 16.

$$|A_i| + |B_i| + \frac{1}{2}|V(M_i)| \leq \Delta^2 + 1 \,.$$

Proof. By the maximality of M_i, for every edge $e = xy$ of M_i there is at most one vertex v_e of K_i that is adjacent to both x and y in $\overline{H_i}$. Hence, every edge e of M_i has an endvertex $n(e)$ that is adjacent in H_i to every vertex of K_i except possibly one, called $x(e)$. By Lemmas 5 and 6,

$$|K_i| = |A_i| + |B_i| \geq \Delta^2 - 8000p\Delta - 2.10^3 p\Delta \geq^* 10^3 p\Delta > |M_i| \,.$$

So there exists a vertex $v \in A_i \cup B_i \setminus \cup_{e \in M_i} x(e)$. The vertex v is adjacent in G_1 to all the vertices of K_i (except itself) and all the vertices $n(e)$ for $e \in M_i$. So

$$|K_i| - 1 + \frac{1}{2}|V(M_i)| \leq \deg^1(v) \leq \Delta^2 \,.$$

□

Phase 1. For each set A_i, we choose a subset of $a_i := |A_i|$ colours as follows. First, we exclude all the colours that appear on the vertices of $B_i \cup C_i$. Moreover, if a colour j is assigned to at least $2p-1$ pairs of vertices matched by M_i, not only do we exclude the colour j but also the colours in $[j-p+1, j+p-1]$. By Proposition 16 and because every edge of M_i is monochromatic by Lemma 8(ii), we infer that at least a_i colours have not been excluded. Then we assign a random permutation of those colours to the vertices of A_i. We let Temp_i be the subset of vertices of A_i with an external G_1-neighbour of the same colour, or a G_2-neighbour with a colour at distance less than p.

Lemma 17. *With positive probability, the following hold.*

(i) *For each* i*,* $|\operatorname{Temp}_i| \le 3\Delta^{5/4}$*;*

(ii) *for each index* i *and each colour* j*, at most* $\Delta^{19/10}$ *vertices of* A_i *have a* G_1*-neighbour in* $\cup_{k\neq i} A_k$ *coloured* j *or a* G_2*-neighbour in* $\cup_k A_k$ *with a colour in* $[j-p+1, j+p-1]$*.*

Proof. We use the Lovász Local Lemma. For every index i, we let $E_1(i)$ be the event that $|\operatorname{Temp}_i|$ is greater than $3\Delta^{5/4}$. For each index i and each colour j, we define $E_2(i,j)$ to be the event that condition (ii) is not fulfilled. Each event is mutually independent of all events involving dense sets at distance greater than 2, so each event is mutually independent of all but at most Δ^9 other events. According to the Lovász Local Lemma, it is enough to show that each event has probability at most Δ^{-10}, since $\Delta^9 \times \Delta^{-10} <^* \frac{1}{4}$.

Our first goal is to upper bound $\mathbf{Pr}(E_1(i))$. We may assume that both the colour assignments for all cliques other than A_i, and the choice of the a_i colours to be used on A_i have already been made. Thus it only remains to choose a random permutation of those a_i colours onto the vertices of A_i. Since every vertex $v \in A_i$ has at most $\Delta^{5/4}$ external neighbours and Δ G_2-neighbours, the probability that $v \in \operatorname{Temp}_i$ is at most $(\Delta^{5/4}+2p\Delta)/a_i$. So we deduce that $\mathbf{E}(|\operatorname{Temp}_i|) \le \Delta^{5/4}+2p\Delta$. We define a binomial random variable B that counts each vertex of A_i independently with probability $\Delta^{5/4}/(2a_i)$. We set $X := |\operatorname{Temp}_i| + B$. By Linearity of Expectation,

$$\frac{1}{2}\Delta^{5/4} \le \mathbf{E}(X) = \mathbf{E}(|\operatorname{Temp}_i|) + \frac{1}{2}\Delta^{5/4} \le^* 2\Delta^{5/4}\,.$$

Moreover, if $|\operatorname{Temp}_i| > 3\Delta^{5/4}$ then $|\operatorname{Temp}_i| - \mathbf{E}(|\operatorname{Temp}_i|) > \Delta^{5/4}$, and hence $X - \mathbf{E}(X) > \frac{1}{2}\Delta^{5/4}$. We now apply McDiarmid's Inequality to show that X is concentrated. Note that if $|\operatorname{Temp}_i| \ge s$, then the colours to $2s$ vertices (that is, s members of Temp_i and one neighbour for each) certify that fact. Moreover, switching the colours of two vertices in A_i may only affect whether those two vertices are in Temp_i, and whether at most four vertices with a colour at distance less than 2 are in Temp_i. So we may apply McDiarmid's Inequality to X with $c = 6, r = 2$ and $t = \frac{1}{2}\Delta^{5/4} \in \left[60c\sqrt{r\,\mathbf{E}(X)}, \mathbf{E}(X)\right]$.

We deduce that the probability that the event $E_1(i)$ holds is at most

$$\begin{aligned}\mathbf{Pr}\left(|\operatorname{Temp}_i| - \mathbf{E}(|\operatorname{Temp}_i|) > \Delta^{5/4}\right) &\le \mathbf{Pr}\left(|X - \mathbf{E}(X)| > \frac{1}{2}\Delta^{5/4}\right) \\ &< 4\exp\left(-\frac{\Delta^{5/2}}{4 \times 32 \times 36 \times 2\Delta^{5/4}}\right) \\ &<^* \Delta^{-10}.\end{aligned}$$

We now upper bound $\mathbf{Pr}(E_2(i,j))$. To this end, we use Lemma 13. Recall that the vertices of A_i get different colours. Every vertex $v \in A_i$ has at most $\Delta^{5/4}$ external neighbours, and Δ G_2-neighbours. We set $Q := \Delta^{5/4} + \Delta$. We let $S(v)$ be the set of all vertices that are either external G_1-neighbours of v, or G_2-neighbours of v. Hence, $|S(v)| \le Q$. Note that each vertex is in at most $\Delta^{5/4}$ sets $S(v)$ for $v \in A_i$. Each vertex of a set $S(v)$ is assigned a colour in $[j-p+1, j+p-1]$ with probability at most

$$\max_k \frac{2p-1}{a_k} <^* \frac{1}{(2p-1)Q \times \Delta^{2/5}},$$

because $\min a_k \ge \Delta^2 - 9000p\Delta^{7/4}$ by Observation 7. Moreover, at most $2p-1$ vertices in each set A_k are assigned a colour in $[j-p+1, j+p-1]$. As the random permutations for different cliques are independent, Lemma 13 implies that the probability that more than $\Delta^{37/20}$ vertices of A_i have an external G_1-neighbour in some A_k coloured j, or a G_2-neighbour in some A_k coloured in $[j-p+1, j+p-1]$ is at most $\exp\left(-\Delta^{1/20}\right) <^* \Delta^{-10}$. This concludes the proof. ☐

Phase 2. We consider a colouring γ satisfying the conditions of Lemma 17. For each set A_i and each vertex $v \in \operatorname{Temp}_i$ we let $\operatorname{Swappable}_v$ be the set of vertices u such that

(a) $u \in A_i \setminus \operatorname{Temp}_i$;

(b) $\gamma(u)$ does not appear on an external G_1-neighbour of v;

(c) $\gamma(v)$ does not appear on an external G_1-neighbour of u;

(d) no colour of $[\gamma(u)-p+1, \gamma(u)+p-1]$ appears on a G_2-neighbour of v;

(e) no colour of $[\gamma(v)-p+1, \gamma(v)+p-1]$ appears on a G_2-neighbour of u.

Lemma 18. *For every* $v \in \mathrm{Temp}_i$*, the set* $\mathrm{Swappable}_v$ *contains at least* $\frac{1}{10}\Delta^2$ *vertices.*

Proof. Let us upper bound the number of vertices that are not in $\mathrm{Swappable}_v$. By Lemma 17(i), at most $3\Delta^{5/4}$ vertices of A_i violate condition (a) and at most $\Delta^{5/4}$ vertices violate condition (b) by the definition of A_i. As v has at most Δ neighbours in G_2, the number of vertices violating condition (d) is at most $2p\Delta$. According to Lemma 8(iv), the number of vertices of A_i violating conditions (c) or (e) because of a neighbour not in $\left(\cup_{k=1}^{\ell} A_k\right) \cup (B_i \cup C_i)$ is at most $\frac{4}{5}\Delta^2$. Moreover, by the way we chose the a_i colours for A_i, for any colour $\alpha \in [\gamma(v) - p + 1, \gamma(v) + p - 1] \setminus \{\gamma(v)\}$, at most $2 \cdot (2p - 2)$ vertices of M_i and one vertex of B_i are coloured α. Each of these vertices have at most Δ neighbours in G_2. Hence, as there are $2p - 2$ choices for the colour α, the number of vertices violating condition (e) because of a neighbour in $B_i \cup C_i$ is at most

$$(2p - 2) \cdot (2 \cdot (2p - 2) + 1) \cdot \Delta = (8p^2 - 14p + 6) \cdot \Delta\,.$$

Finally, the number of vertices violating conditions (c) or (e) because of a colour assigned during Phase 1 is at most $\Delta^{19/10}$ thanks to Lemma 17(ii). Therefore, we deduce that

$$|\,\mathrm{Swappable}_v\,| \geq |A_i| - \frac{4}{5}\Delta^2 - \Delta^{19/10} - 4\Delta^{5/4} - (8p^2 - 14p + 2p + 6)\cdot\Delta - 1 \geq^* \frac{1}{10}\Delta^2\,,$$

as $|A_i| \geq \Delta^2 - 9000p\Delta^{\frac{7}{4}}$ by Observation 7. □

For each index i and each vertex $v \in \mathrm{Temp}_i$, we choose 100 uniformly random members of $\mathrm{Swappable}_v$. These vertices are called *candidates* of v.

Definition 19. A candidate u of v is *unkind* if either

(a) u is a candidate for some other vertex;

(b) v has an external neighbour w that has a candidate w' with the same colour as u;

(c) v has a G_2-neighbour w that has a candidate w' with a colour in $[\gamma(u) - p + 1, \gamma(u) + p - 1]$;

(d) v has an external neighbour w that is a candidate for exactly one vertex w', with $\gamma(w') = \gamma(u)$;

(e) v has a G_2-neighbour w that is a candidate for exactly one vertex w', which has a colour in $[\gamma(u) - p + 1, \gamma(u) + p - 1]$;

(f) u has an external neighbour w that has a candidate w' with the same colour as v;

(g) u has a G_2-neighbour w that has a candidate w' with a colour in $[\gamma(v) - p + 1, \gamma(v) + p - 1]$;

(h) u has an external neighbour w that is a candidate for a vertex w' with the same colour as v; or

(i) u has a G_2-neighbour w that is a candidate for a vertex w' with a colour in $[\gamma(v) - p + 1, \gamma(v) + p - 1]$.

A candidate of v is *kind* if it is not unkind.

Lemma 20. *With positive probability, for each index i, every vertex of* Temp_i *has a kind candidate.*

We choose candidates satisfying the preceding lemma. For each vertex $v \in \mathrm{Temp}_i$ we swap the colour of v and one of its kind candidates. The obtained colouring is the desired one. So to conclude the proof of Lemma 9, it only remains to prove Lemma 20.

Proof of Lemma 20. For every vertex v in some Temp_i, let $E_1(v)$ be the event that v does not have a kind candidate. Each event is mutually independent of all events involving dense sets at distance greater than 2. So each event is mutually independent of all but at most Δ^9 other events. Hence, if we prove that the probability of each event is at most Δ^{-10}, then the conclusion would follow from the Lovász Local Lemma since $\Delta^{-10} \cdot \Delta^9 <^* \frac{1}{4}$.

Observe that the probability that a particular vertex of $\mathrm{Swappable}_v$ is chosen is $100/|\mathrm{Swappable}_v|$, which is at most $1000\Delta^{-2}$.

We wish to upper bound $\mathbf{Pr}(E_1(v))$ for an arbitrary vertex $v \in \mathrm{Temp}_i$, so we can assume that all vertices but v have already chosen candidates. By Lemma 17(i), the number of vertices that satisfy condition (a) of Definition 19 is at most $300\Delta^{5/4}$. Note that the vertex v has at most $\Delta^{5/4}$ external neighbours, each having at most 100 candidates. Since each colour appears on at most one member of $\mathrm{Swappable}_v$, we deduce that the number of vertices satisfying one of the conditions (b) and (d) is at most $101\Delta^{5/4}$. Similarly, the number of vertices satisfying one of the conditions (c) and (e) is at most $202p\Delta$.

We now deal with the remaining four conditions, starting with condition (f). The number of vertices of A_i that satisfy condition (f) is at most the number of edges with an endvertex in A_i and an endvertex in A_k with $k \neq i$, and such that the endvertex not in A_i has chosen a candidate with the colour of v. For each vertex $w \in \cup_{k\neq i} A_k$, we let N_w be the number of G_1-neighbours of w in A_i. So, $N_w \leq \Delta^{5/4}$. Note that $\sum N_w \leq 8000p\Delta^3$ by Lemma 5(b). We define the random variable F_w to be N_w if w has a candidate with the colour of v, and 0 otherwise. Thus, the number of vertices of A_i that satisfy condition (f) is at most the sum σ of the variables F_w for $w \in \cup_{k\neq i} A_k$. We aim at showing that

$$\mathbf{Pr}\left(\sigma > 2\Delta^{3/2}\right) < \frac{1}{8}\Delta^{-10}. \tag{2}$$

Since each vertex in some set Temp_k chooses its candidates independently, the variables F_w are independent. For each $r \in \{0, 1, \dots, \lceil \log_2\left(\Delta^{5/4}\right)\rceil\}$, let S_r be the set of vertices w of $\cup_{k\neq i} A_k$ such that $2^{r-1} < N_w \leq 2^r$. So,

$$\sigma \leq \sum_{r=0}^{\lceil \log_2\left(\Delta^{5/4}\right)\rceil} \sum_{w\in S_r} F_w \leq \sum_{r=0}^{\lceil \log_2\left(\Delta^{5/4}\right)\rceil} 2^r \sigma_r$$

where $\sigma_r := |\{w \in S_r : F_w \neq 0\}|$. Consequently, to prove (2) it suffices to show that for every index r,

$$\mathbf{Pr}\left(\sigma_r > t_r\right) < \frac{\Delta^{-10}}{8\left(\lceil \log_2(\Delta^{5/4})\rceil + 1\right)}$$

where $t_r := \frac{2\Delta^{3/2}}{2^r\left(\lceil \log_2\left(\Delta^{5/4}\right)\rceil+1\right)}$.

Fix an index r. As the variables F_w are independent, the probability that σ_r is more than t_r is no more than the probability that the binomial random variable $\mathrm{BIN}(n_r, p_r)$ with $n_r := \frac{8000p}{2^{r-1}}\Delta^3$ and $p_r := 1000\Delta^{-2}$ is more than t_r. Therefore, we deduce from Chernoff's Bound that

$$\begin{aligned}
\mathbf{Pr}\left(\sigma_r > t_r\right) &\leq^* \mathbf{Pr}\left(\mathrm{BIN}(n_r, p_r) - n_r p_r > \frac{t_r}{2}\right) \\
&< 2\exp\left(\frac{t_r}{2} - \left(n_r p_r + \frac{t_r}{2}\right)\ln\left(1 + \frac{t_r}{2n_r p_r}\right)\right) \\
&<^* \frac{\Delta^{-10}}{8\left(\lceil \log_2(\Delta^{5/4})\rceil + 1\right)},
\end{aligned}$$

as wanted.

A similar argument shows that, with probability at least $1 - \frac{1}{8}\Delta^{-10}$, at most $2\Delta^{3/2}$ vertices of A_i satisfy condition (g).

We now consider condition (h). A vertex u of A_i satisfies condition (h) if it has an external G_1-neighbour that was chosen as a candidate for a vertex with the same colour as v. We actually consider the number of edges with an endvertex in A_i and the other in some A_k with $k \neq i$, and such that the endvertex not in A_i is a candidate for a vertex with the same colour as v. We express this as the sum of several random variables.

Recall that N_w is the number of G_1-neighbours of w in A_i, for every $w \in \cup_{k\neq i} A_k$. So, $N_w \leq \Delta^{5/4}$. We define X_w to be N_w if w is a candidate for a vertex with the colour of v, and 0 otherwise. Thus, the probability that $X_w = N_w$ is at most $1000\Delta^{-2}$. The number of vertices of A_i satisfying condition (h) is at most the sum τ of the variables X_w for $w \in \cup_{k\neq i} A_k$. Our aim is to show that

$$\mathbf{Pr}\left(\tau > 2\Delta^{3/2}\right) < \frac{1}{8}\Delta^{-10}. \tag{3}$$

Recall that

$$S_r = \{w \in \cup_{k\neq i} A_k : 2^{r-1} < N_w \leq 2^r\}$$

for every $r \in \{0, 1, \ldots, \lceil \log_2 \left(\Delta^{5/4}\right)\rceil\}$. Hence,

$$\tau \leq \sum_{r=0}^{\lceil \log_2\left(\Delta^{5/4}\right)\rceil} \sum_{w\in S_r} X_w \leq \sum_{r=0}^{\lceil \log_2\left(\Delta^{5/4}\right)\rceil} 2^r \tau_r$$

where $\tau_r := |\{w \in S_r : X_w \neq 0\}|$. Consequently, to prove (3) it suffices to show that for every index r,

$$\mathbf{Pr}\left(\tau_r > t_r\right) < \frac{\Delta^{-10}}{8\left(\lceil \log_2\left(\Delta^{5/4}\right)\rceil + 1\right)} \tag{4}$$

where $t_r := \frac{2\Delta^{3/2}}{2^r\left(\lceil \log_2\left(\Delta^{5/4}\right)\rceil+1\right)}$.

Let us fix an index r. Observe that τ_r is at most $100\sum_{k\neq i} Z_r^k$ where each Z_r^k is a zero-one random variable, which is 1 if there is a vertex of $S_r \cap A_k$ that is a candidate for a vertex with the same colour as v, and 0 otherwise. In particular, $Z_r^k = 1$ with probability at most $1000|S_r \cap A_k|\Delta^{-2}$. Moreover, if $\tau_r > t_r$ then $\sum_{k\neq i} Z_r^k > \frac{t}{100}$. Let $R_r := 2^{1-r} \cdot 8000p\Delta^3$. By Lemma 5$(b)$, for every $k \neq i$ the size of $S_r \cap A_k$ is at most $M_r := \min\left(\Delta^2, R_r\right)$. We set

$$T_m := \{k \neq i : 2^{m-1} \leq |S_r \cap A_k| \leq 2^m\}$$

for every integer $m \in \{0, 1, \ldots, \lceil \log_2(M_r)\rceil\}$. Hence, $|T_m| \leq 2^{2-m-r} \cdot 8000p\Delta^3$, and

$$\tau_r \leq 100 \sum_{m=0}^{\lceil \log_2(M_r)\rceil} \sum_{k\in T_m} Z_r^k\,.$$

To prove (4), it suffices to show that $\forall m \in \{0, 1, \ldots, \lceil \log_2(M_r) \rceil\}$,

$$\mathbf{Pr}\left(\sum_{k \in T_m} Z_r^k > t'\right) < \frac{\Delta^{-10}}{8(\log_2(\lceil \Delta^{5/4})\rceil + 1)(\lceil \log M_r \rceil + 1)} \tag{5}$$

where

$$t'_r := \frac{t_r}{100 \cdot (\lceil \log_2(M_r) \rceil + 1)}.$$

Let us fix an index m. The variables Z_r^k for $k \in T_m$ are independent zero-one random variables, each being 1 with probability at most $2^m \cdot 1000\Delta^{-2}$. Observe that if $2^m \cdot 1000\Delta^{-2} > 1$, then $|T_m| \leq 32 \cdot 10^6 p\Delta \cdot 2^{-r}$ and hence (5) holds. Thus we assume in the sequel that $2^m \cdot 1000\Delta^{-2} \leq 1$. We define Y_m to be the sum of $2^{2-m-r} \cdot 8000p\Delta^3$ independent zero-one random variables, each being 1 with probability $2^m \cdot 1000\Delta^{-2}$. Thus, $\sum_{k \in T_m} Z_r^k \leq Y_m$. The expected value of Y_m is

$$\mathbf{E}(Y_m) = 32 \cdot 10^6 p \cdot 2^{-r}\Delta <^* \Delta^{3/2}.$$

We deduce from Chernoff's Bound that

$$\begin{aligned}\mathbf{Pr}\left(Y_m - \mathbf{E}(Y_m) > \frac{t'_r}{2}\right) &< 2\exp\left(\frac{t'_r}{2} - \ln\left(1 + \frac{t'_r}{2\,\mathbf{E}(Y_m)}\right)\left(\mathbf{E}(Y_m) + \frac{t'_r}{2}\right)\right) \\ &<^* \frac{\Delta^{-10}}{8\left(\lceil \log_2\left(\Delta^{5/4}\right)\rceil + 1\right)\left(\lceil \log_2\left(M_r\right)\rceil + 1\right)}.\end{aligned}$$

This yields (5), and thus (4), which in turn implies (3), as desired.

A similar argument shows that the probability that more than $2\Delta^{3/2} - 200p\Delta$ vertices of A_i satisfy condition (i) because of an external G_2-neighbour is at most $\frac{1}{8}\Delta^{-10}$. Moreover, at most $200p\Delta$ vertices satisfy condition (i) because of an internal G_2-neighbour.

Therefore, with probability at least $1 - \frac{1}{2}\Delta^{-10}$ the number of unkind members of Swappable_v is at most

$$8\Delta^{3/2} + 300\Delta^{5/4} + 101\Delta^{5/4} + 303\Delta <^* \Delta^{7/4}.$$

In this case, the probability that no candidate is kind is at most

$$\left(\frac{\Delta^{7/4}}{\Delta^2/10}\right)^{100} <^* \frac{1}{2}\Delta^{-10}.$$

Consequently, the probability that $E_1(v)$ holds is at most $\frac{1}{2}\Delta^{-10} + \frac{1}{2}\Delta^{-10} = \Delta^{-10}$, as desired. □

6 The Proof of Lemma 15

In this subsection we prove Lemma 15. We colour H using a two phase quasi-random procedure.

Phase 1. We fix a small real number $\varepsilon \in \left]0, \frac{1}{10000}\right]$, and carry out $K := 2\Delta^{\varepsilon}\log\Delta$ iterations. In each iteration, we analyse the following random procedure, which produces a partial colouring λ. Note that at every time of the procedure, $|L(v)| \geq \delta^1_U(v) + 2p\Delta$ for every vertex v of H, where U is the subgraph of H induced by the uncoloured vertices.

1. Each uncoloured vertex of H is activated with probability $\alpha := \Delta^{-\varepsilon}$;
2. each activated vertex v choose a uniformly random colour $\lambda(v) \in L(v)$;
3. if two activated neighbours create a conflict, both are uncoloured;
4. each activated vertex u that is still coloured is uncoloured with probability $q(v)$, where $q(v)$ is defined so that v has probability exactly $\frac{1}{2}\alpha$ of being activated and retaining its colour;
5. for each vertex v that retains a colour, we remove from the lists of each yet uncoloured vertex every colour whose assignment to this vertex would create a conflict.

First, we have to show that the parameter $q(v)$ is well-defined. Let $N_1(v)$ be the set of all uncoloured G^*_1-neighbours of v. Given that v is activated, the probability that it is uncoloured in the third step of the procedure is at most

$$\begin{aligned}
&\sum_{j\in L(v)} \mathbf{Pr}(\lambda(v)=j) \times \sum_{u\in N_1(v)} \alpha\,\mathbf{Pr}(\lambda(u) \in [j-p+1, j+p-1]) \\
=&\frac{\alpha}{|L(v)|} \sum_{u\in N_1(v)} \sum_{j\in L(v)} \mathbf{Pr}(\lambda(u) \in [j-p+1, j+p-1]) \\
\leq&\frac{\alpha}{|L(v)|} \sum_{u\in N_1(v)} \sum_{k\in L(u)} (2p-1)\cdot \mathbf{Pr}(\lambda(v)=k) \\
\leq&\frac{\alpha}{|L(v)|} \sum_{u\in N_1(v)} (2p-1) \\
=&(2p-1)\alpha\frac{|N_1(v)|}{|L(v)|} \leq (2p-1)\alpha <^* \frac{1}{2},
\end{aligned}$$

since $|L(v)| > |N_1(v)|$. Thus, the probability of being activated and not being uncoloured after the third step of the procedure is more than $\frac{1}{2}\alpha$. So $q(v)$ is well-defined.

Lemma 21. *After K iterations, with positive probability*

- (i) *each vertex of $\cup_{i=1}^{\ell} A_i$ has at most $\Delta^{200\varepsilon}$ uncoloured external neighbours in H;*
- (ii) *each vertex of H has at most $\Delta^{200\varepsilon}$ uncoloured neighbours in H; and*
- (iii) *for every i and every colour j, the size of $\mathrm{Notbig}_{i,j}$ grows by at most $\frac{1}{2}\Delta^{19/10}$.*

We postpone the proof of this lemma to the end of this section. We choose a partial colouring of H that verifies the conditions of the preceding lemma, and proceed with Phase 2.

Phase 2. For every uncoloured vertex of H, let $L_1(v)$ be the list of available colours after Phase 1. At most $\delta^*_H(v) \le \delta^1_H(v) + 4p\Delta$ colours have been removed from $L(v)$. Hence, $|L_1(v)| \ge 2p\Delta$. We apply the following procedure.

1. For each uncoloured vertex v of H, we choose a uniformly random subset $L'(v) \subset L_1(v)$ of size $2p\Delta^{200\varepsilon}$;
2. we colour all such vertices v from their sublist $L'(v)$, greedily one-at-a-time.

Observe that the second step is possible thanks to Lemma 21(ii). Thus, we obtain a good colouring of H. It only remains to prove that it fulfils the condition of Lemma 15. To this end, we first establish the following result about the colouring constructed in Phase 2.

Lemma 22. *With positive probability, for every i and every colour j, the size of $\mathrm{Notbig}_{i,j}$ grows by at most $\frac{1}{2}\Delta^{19/10}$ during Phase 2.*

Proof. We want to apply the Lovász Local Lemma. For each set A_i and each colour j, let $E(i,j)$ be the event that more than $\frac{1}{2}\Delta^{19/10}$ vertices of A_i have neighbours outside of $\mathrm{Big}_i \cup D_i$ with a colour in $[j-p+1, j+p-1]$ in their sublist. We bound $\mathbf{Pr}\left(E(i,j)\right)$ using Lemma 13. By lemma 21(i), every vertex of A_i has at most $Q := \Delta^{200\varepsilon}$ uncoloured external neighbours in H. Each such neighbour u chooses a colour in $[j-p+1, j+p-1]$ in its sublist with probability at most $4p^2\Delta^{200\varepsilon}/|L_1(u)| <^* \frac{1}{Q\Delta^{2/5}}$, because $|L_1(u)| \geq 2p\Delta$. Besides, these assignments are made independently. So, as $\frac{1}{2}\Delta^{19/10} >^* \Delta^{37/20}$, Lemma 13 yields that $\mathbf{Pr}\left(E(i,j)\right) < \exp(-\Delta^{1/20}) <^* \Delta^{-10}$.

Observe that each event is mutually independent of all events involving dense sets at distance more than 2, and each dense set is adjacent to at most $8000p\Delta^3$ other dense sets. As a result, each event is mutually independent of all but at most Δ^9 other events. Consequently, the Lovász Local Lemma applies and yields the conclusion. $\square$

Using the last two lemmas, we can prove Lemma 15.

Proof of Lemma 15. We consider a colouring obtained after Phases 1 and 2. By Lemmas 21(*iii*) and 22, $\mathrm{Notbig}_{i,j}$ grows by at most $\frac{1}{2}\Delta^{19/10}$ during each phase for every index i and every colour j. $\square$

Thus, to complete the proof, it only remains to prove Lemma 21. To this end, we inductively obtain an upper bound R_k on the number of uncoloured external G'_1-neighbours of a vertex of $\cup_{i=1}^{l} A_i$ after the k^{th} iteration, and lower and upper bounds $m_k^-(v)$ and $m_k^+(v)$ on the number of neighbours in U of a vertex v after the k^{th} iteration. Let $\theta := \left(1 - \frac{1}{2}\Delta^{-\varepsilon}\right)$. Note that $\theta > \frac{1}{2}$ since $\Delta^{\varepsilon} >^* 1$. We set

$$R_0 := \Delta^{5/4} \quad \text{and} \quad \forall k > 0,\ R_k := \theta R_{k-1} + R_{k-1}^{49/50}, \tag{6}$$

and for every vertex v,

$$m_0^+(v) := \delta_H^1(v) \quad \text{and} \quad \forall k > 0,\ m_k^+(v) := \theta m_{k-1}^+(v) + m_{k-1}^+(v)^{49/50}, \tag{7}$$

$$m_0^-(v) := \delta_H^1(v) \quad \text{and} \quad \forall k > 0,\ m_k^-(v) := \theta m_{k-1}^-(v) - m_{k-1}^-(v)^{49/50}. \tag{8}$$

These parameters fulfil some useful properties, as we note next.

Lemma 23. *The following hold.*

(i) If $R_k \geq \Delta^{150\varepsilon}$ then $R_k \leq^* 2\theta^k R_0$.

(ii) If $m_k^-(v) \geq \Delta^{150\varepsilon}$ then

$$\frac{1}{2}\theta^k \delta_H^1(v) \leq^* m_k^-(v) \leq m_k^+(v) \leq^* 2\theta^k \delta_H^1(v)\,.$$

Proof. (i) We prove the following statement by induction on the integer k, which yields the sought conclusion.

$$\forall k \geq 0, \quad R_k \leq \theta^k R_0 + \left(\theta^k R_0\right)^{99/100}$$

as long as $R_k \geq \Delta^{150\varepsilon}$. To see that this statement is indeed stronger, note that if $R_k \geq \Delta^{150\varepsilon}$, then as $\Delta^{150\varepsilon} \geq^* 2$ we infer that $\theta^k R_0 \geq^* 1$ which yields the conclusion.

The statement trivially holds when $k = 0$. Now, assume that it holds for some integer $k-1$, and let us prove that it holds for k. As in the previous remark, we obtain using the induction hypothesis that $\theta^{k-1} R_0 \geq 1$ because $R_{k-1} > R_k \geq \Delta^{150\varepsilon}$. Therefore,

$$\begin{aligned}
R_k &= \theta R_{k-1} + R_{k-1}^{49/50} \\
&\leq \theta \left(\theta^{k-1} R_0 + \left(\theta^{k-1} R_0\right)^{99/100}\right) + \left(2\theta^{k-1} R_0\right)^{49/50} \\
&< \theta^k R_0 + \theta^{99k/100} \theta^{1/100} R_0^{99/100} + \left(4\theta^k R_0\right)^{49/50} \\
&= \theta^k R_0 + \theta^{99k/100} R_0^{99/100} \left(\theta^{1/100} + 4^{49/50} \left(\theta^k R_0\right)^{-1/100}\right) \\
&\leq \theta^k R_0 + \left(\theta^k R_0\right)^{99/100},
\end{aligned}$$

where the third line follows from the fact that $\frac{1}{2} < \theta < 1$, and the last line follows from the fact that $\theta^{1/100} + 4^{49/50} \left(\theta^k R_0\right)^{-1/100} \leq^* 1$.

(ii) The proof of the rightmost inequality is identical to the preceding one, so we omit it. Moreover, $m_k^-(v) \leq m_k^+(v)$ by the definition. We now prove by induction on the integer k that

$$\forall k \geq 0, \quad m_k^-(v) \geq \theta^k \delta_H^1(v) - \left(\theta^k \delta_H^1(v)\right)^{99/100}.$$

The inequality is trivial if $k = 0$. Suppose that the inequality holds for an integer $k-1$, and let us prove it for k.

By the prior remarks, we know that

$$m_k^-(v) \leq m_k^+(v) \leq \theta^k \delta_H^1(v) + \left(\theta^{k-1} \delta_H^1(v)\right)^{99/100} \leq 2\theta^k \delta_H^1(v).$$

Thus,

$$\begin{aligned}
m_k^-(v) &= \theta m_{k-1}^-(v) - m_{k-1}^-(v)^{49/50} \\
&\geq \theta \left(\theta^{k-1} \delta_H^1(v) - \left(\theta^{k-1} \delta_H^1(v)\right)^{99/100}\right) - \left(2\theta^k \delta_H^1(v)\right)^{49/50} \\
&> \theta^k \delta_H^1(v) - \theta^{99k/100} \delta_H^1(v)^{99/100} \left(\theta^{1/100} + 2\left(\theta^k \delta_H^1(v)\right)^{-1/100}\right) \\
&> \theta^k \delta_H^1(v) - \left(\theta^k \delta_H^1(v)\right)^{99/100},
\end{aligned}$$

since $\theta^{1/100} <^* 1 - 2\left(\theta^k \delta^1_H(v)\right)^{-1/100}$.

□

Proof of Lemma 21. We apply the Lovász Local Lemma to each iteration of the procedure to prove inductively that with positive probability, after $k \leq K$ iterations the following hold.

(a) If $R_k \geq \frac{1}{2}\Delta^{200\varepsilon}$ then every vertex in $\cup_{i=1}^{l} A_i$ has at most R_k uncoloured external G'_1-neighbours in H;

(b) for every vertex v of H, if $m^-_k(v) \geq \frac{1}{8}\Delta^{200\varepsilon}$ then $m^-_{k-1}(v) \leq \delta^1_U(v) \leq m^+_{k-1}(v)$;

(c) for every index i and every colour j, the size of $\text{Notbig}_{i,j}$ increases by at most $\frac{1}{4\log\Delta}\Delta^{19/10-\varepsilon}$ during iteration k.

Assuming this, we can finish the proof as follows. Note that

$$2\theta^K\Delta^2 <^* 1 <^* \Delta^{150\varepsilon}.$$

Since $R_0 = \Delta^{5/4}$ and $\delta^1_H(v) \leq \Delta^2$ for every vertex v, the contrapositive of Lemma 23 implies that both R_K and $m^-_K(v)$ are less than $\Delta^{150\varepsilon} \leq^* \frac{1}{8}\Delta^{200\varepsilon}$. Furthermore, all these parameters decrease with k. Note that R_k and $m^+_k(v)$ decrease by less than half at each iteration, and this is also true for $m^-_k(v)$ provided it is large enough, e.g. if $m^-_k(v) \geq \Delta^{150\varepsilon}$. Therefore, as $\Delta^{200\varepsilon} < \Delta^{5/4}$, there exist two integers k_1 and $k_2(v)$, both at most K, such that

$$\frac{1}{2}\Delta^{200\varepsilon} \leq R_{k_1} < \Delta^{200\varepsilon}$$

and, if $\delta^1_H(v) > \Delta^{200\varepsilon}$ then

$$\frac{1}{8}\Delta^{200\varepsilon} \leq m^-_{k_2(v)}(v) < \frac{1}{4}\Delta^{200\varepsilon}.$$

Note that the number of uncoloured vertices cannot increase, therefore applying (a) at iteration k_1 yields (i). Similarly, applying (b) at iteration $k_2(v)$ yields (ii), since $m^+_{k_2(v)}(v) \leq 4m^-_{k_2(v)}(v) < \Delta^{200\varepsilon}$, using Lemma 23($ii$). Finally, ($iii$) follows from ($c$) because the number of iterations is $K = 2\Delta^\varepsilon \log\Delta$.

It only remains to prove (a), (b) and (c). We proceed by induction on k, the three assertions holding trivially when $k = 0$. Let k be a positive integer such that the assertions hold for all smaller integers.

For every uncoloured vertex v of $\cup_{i=1}^{l} A_i$, we define $E_1(v)$ to be the event that v violates (a). For every vertex u of H, we define $E_2(u)$ to be the event that u violates (b). For every index i and each colour j, we define $E_3(i,j)$ to be the event that $\text{Notbig}_{i,j}$ violates (c). Each event is mutually independent of all other events involving vertices or dense sets at distance more than 4 in G_1^*, and hence is mutually independent of all but at most Δ^{16} other events. We prove that each event $E_1(v)$, $E_2(v)$ and $E_3(i,j)$ occurs with probability at most Δ^{-17}. Consequently, the Lovász Local Lemma applies since $3\Delta^{-17} \cdot \Delta^{16} <^* \frac{1}{4}$, and therefore with positive probability none of these events occurs.

Bounding $\mathbf{Pr}(E_3(i,j))$.

Fix an index i and a colour j. We apply Lemma 13 with

$$Q := \max(R_{k-1}, \Delta^{200\varepsilon}).$$

By induction, we know that every vertex in A_i has at most Q uncoloured external G_1'-neighbours at the beginning of iteration k. Moreover, the probability that a vertex v of H is assigned a colour in $[j-p+1, j+p-1]$ is at most $\frac{2p}{|L(v)|}$. Note that these colour assignments are independent. Consequently, provided that $|L(v)| \geq 2pQ\Delta^{2/5}$, Lemma 13 implies that $\mathbf{Pr}(E_3(i,j)) < \exp\left(-\Delta^{1/20}\right) \leq^* \Delta^{-17}$, since $\frac{\Delta^{19/10-\varepsilon}}{4\log\Delta} \geq^* \Delta^{37/20}$.

So now, let us show that $|L(v)| \geq 2pQ\Delta^{2/5}$. Note that at most $\delta_H^*(v)$ colours can be removed from $L(v)$, so by hypothesis $|L(v)| \geq 2p\Delta$. This remark establishes the result if $Q \leq \Delta^{3/5}$. Notice that $\Delta^{200\varepsilon} < \Delta^{3/5}$, since $\varepsilon < \frac{3}{1000}$. So we may assume now that $R_{k-1} > \Delta^{3/5}$, and hence $Q = R_{k-1}$. Recall that at the beginning $|L(v)| \geq 11p\Delta^{33/20}$ by hypothesis. Thus, if $\delta_H^1(v) \leq 8p\Delta^{33/20}$, then $|L(v)| \geq 3p\Delta^{33/20} - 4p\Delta \geq^* 2p\Delta^{33/20} \geq 2pQ\Delta^{2/5}$ since $Q = R_{k-1} \leq R_0 = \Delta^{5/4}$. If $\delta_H^1(v) > 8p\Delta^{33/20}$ then as $R_{k-1} > \Delta^{3/5}$ observe that $m_k^-(v) \geq \Delta^{150\varepsilon}$. Indeed, $m_{k-1}^+(v) > \Delta^{3/5}$ since $m_0^+(v) = \delta_H^1(v) > R_0$. Hence $m_k^+(v) >^* \frac{1}{2} \cdot \Delta^{3/5}$. Consequently, $m_k^-(v) > \frac{1}{8} \cdot \Delta^{3/5} \geq \Delta^{150\varepsilon}$ by Lemma 23(ii). So by Lemma 23(i) and (ii), we deduce that

$$\begin{aligned} |L(v)| \geq \delta_U^1(v) \geq m_{k-1}^-(v) &\geq \frac{1}{2}\theta^{k-1}\delta_H^1(v) \\ &> 4p\theta^{k-1}R_0\Delta^{2/5} \\ &\geq^* 2pR_{k-1}\Delta^{2/5} = 2pQ\Delta^{2/5}. \end{aligned}$$

Bounding $\mathbf{Pr}(E_1(v))$.

Fix a vertex v of $\cup_{i=1}^{l} A_i$. We assume that $R_k \geq \frac{1}{2}\Delta^{200\varepsilon}$. Let m be the number of uncoloured external neighbours of v in H at the beginning of iteration k. By induction, $m \le R_{k-1}$. We define Y to be the number of those vertices that are coloured during iteration k. The probability of an uncoloured vertex becoming coloured during iteration k is exactly $\frac{1}{2}\Delta^{-\varepsilon}$. Hence, $\mathbf{E}(Y) = \frac{1}{2}\Delta^{-\varepsilon}m$. Consequently, if $E_1(v)$ holds then Y must differ from its expected value by more than $R_{k-1}^{49/50}$.

As in the proof of Lemma 14, we express Y as the difference of two random variables. Let Y_1 be the number of uncoloured external G_1'-neighbours of v that are activated during iteration k. Let Y_2 be the number of uncoloured external G_1'-neighbours of v that are activated and uncoloured during iteration k. Thus, $Y = Y_1 - Y_2$ and hence if $E_1(v)$ holds then either Y_1 or Y_2 differs from its expected value by more than $\frac{1}{2}R_{k-1}^{49/50}$.

Note that $Y_1 \le R_{k-1}$, hence $\mathbf{E}(Y_1) \le R_{k-1}$. Moreover, Y_1 is a binomial random variable, so Chernoff's Bound implies that

$$\mathbf{Pr}\left(|Y_1 - \mathbf{E}(Y_1)| > \frac{1}{2}R_{k-1}^{49/50}\right) \le 2\exp\left(-\frac{R_{k-1}^{49/25}}{12R_{k-1}}\right) \le^* \frac{1}{2}\Delta^{-10},$$

since $R_{k-1} \ge R_k > \frac{1}{2}\Delta^{200\varepsilon}$.

The random variable Y_2 is upper-bounded by the random variable Y_2', defined as the number of uncoloured external G_1'-neighbours of v that are activated and (i) uncoloured, or (ii) assigned a colour that is assigned to at least $\log\Delta$ G_1'-neighbours of v. Furthermore, we assert that $Y_2 = Y_2'$ with high probability. Indeed, if $Y_2 \ne Y_2'$ then there exists a colour assigned to at least $\log\Delta$ G_1'-neighbours of v. By Lemma 23(i), the number of uncoloured G_1'-neighbours of v in H is at most $d := 2\theta^{k-1}R_0$. Moreover, by the induction hypothesis, $\delta_U^1(u) \ge m_{k-1}^-(u) \ge^* \frac{1}{2}\theta^{k-1}\delta_H^1(u)$ for every G_1'-neighbour u of v in H. Therefore, the number of colours available for u is at least

$$\begin{aligned}
&\max\left(\delta_H^1(u) + 6p\Delta, 11p\Delta^{33/20}\right) - \delta_H^*(u) + m_{k-1}^-(u)\\
\ge&\max\left(\delta_H^1(u) + 6p\Delta, 11p\Delta^{33/20}\right) - 4p\Delta - \left(1 - \frac{1}{2}\theta^{k-1}\right)\delta_H^1(u)\\
\ge&\max\left(\delta_H^1(u), 8p\Delta^{33/20}\right)\cdot\left(1 - \left(1 - \frac{1}{2}\theta^{k-1}\right)\right)\\
\ge&\frac{1}{2}\theta^{k-1} \times 8p\Delta^{33/20}\\
\ge&2p\Delta^{2/5}d.
\end{aligned}$$

Consequently,

$$\mathbf{Pr}(Y_2 \ne Y_2') \le \Delta^2 \times \binom{d}{\log\Delta}\left(2p\Delta^{2/5}d\right)^{-\log\Delta} \le \Delta^2 \times \left(\frac{e}{2p\Delta^{2/5}}\right)^{\log\Delta} <^* \frac{1}{4}\Delta^{-17},$$

which proves the assertion.

Since $|Y_2 - Y_2'| \le \Delta^2$, this implies that $|\mathbf{E}(Y_2) - \mathbf{E}(Y_2')| = o(1)$. As a result, it is enough to establish that $\mathbf{Pr}\left(|Y_2' - \mathbf{E}(Y_2')| > \frac{1}{4}R_{k-1}^{49/50}\right) <^* \frac{1}{4}\Delta^{-17}$ to deduce that $\mathbf{Pr}\left(|Y_2 - \mathbf{E}(Y_2)| > \frac{1}{2}R_{k-1}^{49/50}\right) \le^* \frac{1}{2}\Delta^{-17}$.

We apply Talagrand's Inequality. For convenience, we consider that each vertex v of H is involved in two random trials. The first one, which combines steps 1 and 2 of our procedure, is to be assigned the label "unactivated" or "activated with colour j" for some colour j in $L(v)$. The former label is assigned with probability $1 - \Delta^{-\varepsilon}$, and the latter with probability $\frac{\Delta^{-\varepsilon}}{|L(v)|}$. The second random trial assigns to v the label "uncoloured" with probability $q(v)$, whatever the result of the first trial is. The technical benefit of this approach is to obtain independent random trials.

If $Y_2' \ge s$ then there is a set of at most $s \log \Delta$ random trials that certify this fact, i.e for each of the s vertices counted by Y_2', the activation and colour assignment of the vertex and either the choice to uncolour it in step 4, or the activation and assignment of a conflicting colour to a neighbour of that vertex, or the activation and assignment of the same colour to $\log \Delta - 1$ other G_1'-neighbours of v. Furthermore, changing the outcome of one of the random trials can affect Y_2' by at most $\log \Delta$. Recalling that $\mathbf{E}(Y_2) \le R_{k-1}$ and $R_{k-1} \ge \Delta^{200\varepsilon}$, Talagrand's Inequality yields that

$$\mathbf{Pr}\left(|Y_2' - \mathbf{E}(Y_2')| > \frac{1}{4}R_{k-1}^{49/50}\right) < 4\exp\left(-\frac{R_{k-1}^{49/25}}{16 \times 32 \log^3 \Delta R_{k-1}}\right) <^* \frac{1}{4}\Delta^{-17},$$

provided that $\mathbf{E}(Y_2) \ge \frac{1}{4}R_{k-1}^{49/50}$. If $\mathbf{E}(Y_2) < \frac{1}{4}R_{k-1}^{49/50}$ then we consider the random variable Y_2' defined to be the sum of Y_2 and a binomial random variable that counts each of the m uncoloured external neighbours of v in H independently at random with probability $\max\left(1, \frac{1}{8m}R_{k-1}^{49/50}\right)$. Notice that $\frac{1}{8}R_{k-1}^{49/50} \le \mathbf{E}(Y_2') \le \frac{3}{8}R_{k-1}^{49/50}$. Moreover, if $|Y_2 - \mathbf{E}(Y_2)| > \frac{1}{4}R_{k-1}^{49/50}$ then $|Y_2' - \mathbf{E}(Y_2')| > \frac{1}{8}R_{k-1}^{49/50}$. Applying Talagrand's Inequality to Y_2', we infer that the last inequality occurs with probability less than $\frac{1}{4}\Delta^{-17}$, as wanted. Therefore, we obtain $\mathbf{Pr}(E_1(v)) \le \Delta^{-17}$, as desired.

Bounding $\mathbf{Pr}(E_2(v))$.

We fix a vertex v of H, and we assume that $m_k^-(v) \ge \frac{1}{8}\Delta^{200\varepsilon}$. Our aim is to prove that $\mathbf{Pr}(E_2(v)) \le^* \Delta^{-17}$. To this end, we wish to use a similar approach to that for $E_1(v)$. However, for every G_1^*-neighbour u of v in H, the degree of u in H may be a lot bigger than the degree of v in H, which makes it more difficult to bound the analogue of $\mathbf{Pr}(Y_2 \ne Y_2')$.

For every vertex u, let $\tilde{\delta}(u) := \delta_U^1(u)$. Let L_u be the set of colours available to colour u. Recall that $|L_u| \geq \delta_H^1(u) + 2p\Delta > \tilde{\delta}(u)$.

We define Z_1 and Z_2 analogously to Y_1 and Y_2, that is we let Z_1 be the number of uncoloured G_1^*-neighbours of v that get activated, and we let Z_2 be the number of those activated neighbours of v that get uncoloured. Similarly as before, it suffices to prove that, with high probability, neither Z_1 nor Z_2 differs from its expected value by more than $\frac{1}{2}\left(m_{k-1}^-(v)\right)^{49/50}$. Observe that $Z_1 \leq m_{k-1}^+(v) < 4m_{k-1}^-(v)$, and so $\mathbf{E}(Z_1) \leq 4m_{k-1}^-(v)$. Therefore, Chernoff's Bound implies that

$$\mathbf{Pr}\left(|Z_1 - \mathbf{E}(Z_1)| > \frac{1}{2}\left(m_{k-1}^-(v)\right)^{49/50}\right) \leq 2\exp\left(-\frac{m_{k-1}^-(v)^{49/25}}{48 \cdot m_{k-1}^-(v)}\right) <^* \frac{1}{2}\Delta^{-17},$$

since $m_{k-1}^-(v) \geq \frac{1}{8}\Delta^{200\varepsilon}$.

We partition the neighbours of v in H into two parts N_A and N_B, where N_A contains those vertices u with $\tilde{\delta}(u) \geq \tilde{\delta}(v)^{3/4}$, and N_B those with $\tilde{\delta}(u) < \tilde{\delta}(v)^{3/4}$. We define Z_A and Z_B to be the number of vertices that get activated and uncoloured during this iteration in N_A and N_B, respectively. Thus, $Z_2 = Z_A + Z_B$.

We use a similar argument as the one for Y_2 to show that Z_A is concentrated. Let $Z_{A'}$ be the number of vertices in N_A that get activated and are (i) uncoloured, or (ii) assigned a colour that is assigned to at least $\tilde{\delta}(v)^{3/10}$ members of N_A. As $|N_A| \leq \tilde{\delta}(v)$, and $|L_u| \geq \tilde{\delta}(u) \geq \tilde{\delta}(v)^{3/4}$ for every vertex of $u \in N_A$, the probability that Z_A and Z'_A are different is at most

$$(\Delta^2+1)\binom{\tilde{\delta}(v)}{\tilde{\delta}(v)^{3/10}}\left(\tilde{\delta}(v)^{3/4}\right)^{-\tilde{\delta}(v)^{3/10}} < (\Delta^2+1)\left(\frac{e\tilde{\delta}(v)}{\tilde{\delta}(v)^{3/10}\tilde{\delta}(v)^{3/4}}\right)^{\tilde{\delta}(v)^{3/10}}$$

$$<^* \frac{1}{8}\Delta^{-17},$$

since $\tilde{\delta}(v) \geq m_{k-1}^-(v) > \frac{1}{8}\Delta^{200\varepsilon}$. As $|Z_A - Z'_A| \leq \Delta^2$, we infer that $|\,\mathbf{E}(Z_A) - \mathbf{E}(Z'_A)| = o(1)$.

By the same argument as for Y'_2, we deduce that if $Z'_A \geq s$ then there are at most $\tilde{\delta}(v)^{3/10} \cdot s$ trials whose outcomes certify this fact. Furthermore, each trial can affect Z'_A by at most $\tilde{\delta}(v)^{3/10}$. Therefore, if $\mathbf{E}(Z'_A) \geq \frac{1}{4}m_{k-1}^-(v)^{49/50}$

then Talagrand's Inequality yields that

$$\begin{aligned}
&\mathbf{Pr}\left(|Z'_A - \mathbf{E}(Z'_A)| > \frac{1}{4}\left(m^-_{k-1}(v)\right)^{49/50}\right) \\
\leq& 4\exp\left(-\frac{m^-_{k-1}(v)^{49/25}}{32 \cdot m^+_{k-1}(v)^{6/10} \cdot m^+_{k-1}(v)^{3/10} \cdot 4 \cdot m^-_{k-1}(v)}\right) \\
\leq& 4\exp\left(-\frac{m^-_{k-1}(v)^{49/25}}{128 \cdot 4^{9/10} \cdot m^-_{k-1}(v)^{6/10} \cdot m^-_{k-1}(v)^{3/10} \cdot m^-_{k-1}(v)}\right) \\
<^*& \frac{1}{8}\Delta^{-17}.
\end{aligned}$$

If $\mathbf{E}(Z'_A) < \frac{1}{4}m^-_{k-1}(v)^{49/50}$, then we define Z''_A to be the sum of Z'_A and a binomial random variable that counts each vertex of N_A independently with probability $\max\left(1, \frac{1}{8|N_A|}m^-_{k-1}(v)^{49/50}\right)$. By Linearity of Expectation, $\frac{1}{8}m^-_{k-1}(v)^{49/50} \leq \mathbf{E}(Z''_A) \leq \frac{3}{8}m^-_{k-1}(v)^{49/50}$. Furthermore, if $|Z'_A - \mathbf{E}(Z'_A)| > \frac{1}{4}m^-_{k-1}(v)^{49/50}$ then $|Z''_A - \mathbf{E}(Z''_A)| > \frac{1}{8}m^-_{k-1}(v)^{49/50}$. Applying Talagrand's Inequality to Z''_A yields that this latter inequality occurs with probability less than $\frac{1}{8}\Delta^{-17}$, as desired. Consequently,

$$\mathbf{Pr}\left(|Z_A - \mathbf{E}(Z_A)| > \frac{1}{2}m^-_{k-1}(v)^{49/50}\right) \leq^* \frac{1}{4}\Delta^{-17}. \tag{9}$$

We finish with considering Z_B. We first expose the assignments to all vertices other than N_B. Let $\mathcal{H}$ be this assignment. We now condition on $\mathcal{H}$. First, we consider the conditional expected value of Z_B regarding $\mathcal{H}$. We assert that

$$\mathbf{Pr}\left(|\,\mathbf{E}(Z_B|\mathcal{H}) - \mathbf{E}(Z_B)| > \frac{1}{2}m^-_{k-1}(v)^{49/50}\right) <^* \frac{1}{8}\Delta^{-17}. \tag{10}$$

To see this, let $\mu_\mathcal{H}$ be the conditional expectation $\mathbf{E}(Z_B|\mathcal{H})$. Note that the expected value of $\mu_\mathcal{H}$ over the space of random colourings of $H - N_B$ is equal to the expected value of Z_B over the space of random colourings of H. So our assertion is that $\mu_\mathcal{H}$ is indeed concentrated.

For each vertex u of N_B, let $F_u = F_u(\mathcal{H}) \subseteq L_u$ be the set of colours of L_u that conflict with the assignments made by $\mathcal{H}$ to the neighbours of u in H that are not in N_B. First, we use Talagrand's Inequality to prove that $|F_u|$ is concentrated.

The random variable $|F_u|$ is determined by the independent colour assignments to the vertices of $H - N_B$. If $|F_u| \geq s$ then there is a set of at most s assignments that certify this fact, namely the assignments of colours to s vertices. Observe that the assignments to one vertex can affect $|F_u|$ by at most $2p$. Moreover, $|F_u| \le |L_u|$ so $\mathbf{E}(F_u) \le |L_u|$. Therefore, Talagrand's Inequality implies that

$$\mathbf{Pr}\left(||F_u| - \mathbf{E}(|F_u|)| > \Delta^{-2/5}|L_u|\right) < 4\exp\left(-\frac{\Delta^{-4/5}|L_u|^2}{128p^2|L_u|}\right) <^* \frac{1}{8}\Delta^{-19},$$

since $|L_u| \ge 2\Delta$, in the case where $\mathbf{E}(|F_u|) \ge \Delta^{-2/5}|L_u|$. In the opposite case, we define F to be the sum of the random variable F_u and a binomial random variable that counts each colour of L_u independently at random with probability $\frac{1}{2}\Delta^{-2/5}$. Note that $\frac{1}{2}\Delta^{-2/5}|L_u| \le \mathbf{E}(F) \le \frac{3}{2}\Delta^{-2/5}|L_u|$. Moreover, if $||F_u| - \mathbf{E}(|F_u|)| > \Delta^{-2/5}|L_u|$ then $|F - \mathbf{E}(F)| > \frac{1}{2}\Delta^{-2/5}|L_u|$. Applying Talagrand's Inequality to F shows that this latter inequality occurs with probability less than $\frac{1}{8}\Delta^{-19}$, as desired.

Consequently, the probability that there is at least one vertex u of N_B for which $|F_u|$ differs from its expected value by more than $\Delta^{-2/5}|L_u|$ is at most $|N_B|\frac{1}{8}\Delta^{-19} \le \frac{1}{8}\Delta^{-17}$. Hence, we assume that there is no such vertex u, and we prove that this implies that $|\mu_{\mathcal{H}} - \mathbf{E}(\mu_{\mathcal{H}})| < \frac{1}{2}m^-_{k-1}(v)^{49/50}$.

Given a particular assignment $\mathcal{H}$ to $H \setminus \cup_{i=1}^{\ell} A_i$ and a colour $j \in L_u$, the probability that u keeps the colour j is 0 if $j \in F_u$, and at most

$$(1 - q(u)) \prod_{\substack{w \in N(u)\cap N_B \\ j \in L_w}} \left(1 - \frac{2p-1}{|L_w|}\right)$$

otherwise. Note that the product is at most 1, and does not depend on F_u. Hence, changing whether j belongs to F_u affects the probability that u retains its colour by at most $\frac{2p-1}{|L(u)|}$. So, as $|F_u|$ differs from its expected value by at most $\Delta^{-2/5}|L_u|$, the conditional probability that u is uncoloured differs from its expected value by at most $(2p-1)\Delta^{-2/5}$. Since $\mu_{\mathcal{H}}$ is the sum of these probabilities over all the vertices u of N_B, we deduce that

$$|\mu_{\mathcal{H}} - \mathbf{E}(\mu_{\mathcal{H}})| \le (2p-1)\Delta^{-2/5}|N_B| < \frac{1}{2}m^-_{k-1}(v)^{49/50},$$

because $|N_B| < 2m^-_{k-1}(v)$ and $m^-_{k-1}(v) \le \Delta^2$. This concludes the proof of our assertion.

We define Z'_B to be the number of vertices of N_B that are activated and uncoloured because (i) they are assigned a colour conflicting with a neighbour outside of N_B, or (ii) they are assigned a colour conflicting with a neighbour $w \in N_B$ and this colour is assigned to at least $\tilde{\delta}(v)^{3/10}$ vertices of $N_{G_1^*}(w) \cap N_B$.

If $Z_B \neq Z'_B$ then some vertex u of N_B receives the same colour as at least $\tilde{\delta}(v)^{3/10}$ of its neighbours. Since each vertex u of N_B has at most $\tilde{\delta}(v)^{3/4}$ neighbours, and $|L_w| \geq 2\Delta$ for every vertex w, we deduce that

$$\begin{aligned}
\mathbf{Pr}\left(Z_B \neq Z'_B\right) \leq & |N_B| \times \binom{\tilde{\delta}(v)^{3/4}}{\tilde{\delta}(v)^{3/10}} \cdot (2\Delta)^{-\tilde{\delta}(v)^{3/10}} \\
\leq & \Delta^2 \left(\frac{e\tilde{\delta}(v)^{3/4}}{2\tilde{\delta}(v)^{3/10}\Delta}\right)^{\tilde{\delta}(v)^{3/10}} = \Delta^2 \left(\frac{\tilde{\delta}(v)^{9/20}}{2\Delta}\right)^{\tilde{\delta}(v)^{3/10}} \\
\leq & \Delta^2 \left(\frac{\Delta^{18/20}}{2\Delta}\right)^{\tilde{\delta}(v)^{3/10}} <^* \frac{1}{8}\Delta^{-17},
\end{aligned} \tag{11}$$

as $\frac{1}{8}\Delta^{200\varepsilon} \leq \tilde{\delta}(v) \leq \Delta^2$. Since $|Z_B - Z'_B| \leq \Delta^2$ for every choice of $\mathcal{H}$, we infer that $|\,\mathbf{E}(Z_B|\mathcal{H}) - \mathbf{E}(Z'_B|\mathcal{H})| = o(1)$.

After conditioning on $\mathcal{H}$, the random variable Z'_B is determined by at most $\delta^1_H(v)$ assignments and each assignment can affect Z'_B by at most $\delta^1_H(v)^{1/3}$. Note that $\delta^1_H(v) \leq m^+_{k-1}(v) \leq 2m^-_{k-1}(v)$. So, for every choice of $\mathcal{H}$, the Simple Concentration Bound yields that

$$\begin{aligned}
&\mathbf{Pr}\left(|Z'_B - \mathbf{E}(Z'_B|\mathcal{H})| > \frac{1}{4}m^-_{k-1}(v)^{49/50}|\mathcal{H}\right) \\
&< 2\exp\left(-\frac{m^-_{k-1}(v)^{49/25}}{32 \times 2m^-_{k-1}(v)^{2/3} \times 2m^-_{k-1}(v)}\right) <^* \frac{1}{8}\Delta^{-17}.
\end{aligned} \tag{12}$$

Therefore, by (10)–(12), we infer that $\mathbf{Pr}\left(|Z_B - \mathbf{E}(Z_B)| > \frac{1}{2}m^-_{k-1}(v)^{49/50}\right) <^* \frac{1}{2}\Delta^{-17}$. Thus, along with (9), we deduce that

$$\mathbf{Pr}(E_2(v)) <^* \Delta^{-17},$$

which concludes the proof. □

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