Long pairs of paths in faulty hypercubes

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Abstract

Let F be a set of faulty vertices of the hypercube Q_n and let A and B be two different sets of two vertices of $Q_n - F$. We prove that if $|F| \leq n-3$, then $Q_n - F$ contains two vertex disjoint paths of total length at least $2^n - 2|F| - 3$ such that each them has one end-vertex in A and the other in B.

1 Introduction

The n-dimensional hypercube Q_n is the (bipartite) graph with all binary vectors of length n as vertices and edges joining every two vertices that differ in exactly one coordinate. The bipartite classes of Q_n consist of vertices with even, respectively odd, weight, where the *weight* $|u|$ of a vertex $u \in V(Q_n) = \{0,1\}^n$ is defined as the number of 1's in u. A set $F \subseteq V(Q_n)$ in which all vertices are from the same bipartite class, is called a *monopartite* set.

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Applications of the hypercube in the theory of interconnection networks inspired many questions related to its robustness. In particular, if some faulty (or busy) vertices $F \subseteq V(Q_n)$ and all incident edges are removed from Q_n, is there a path in the remaining graph, denoted by $Q_n - F$, between given pair of vertices, which covers 'almost' all vertices? And how many vertices in the worst-case can be removed?

It is well known that Q_n contains a Hamiltonian path between every two vertices of the opposite parity. Lewinter and Widulski [4] studied the hypercube with one faulty vertex.

Proposition 1.1 (Lewinter and Widulski [4])**.** *Let $n \geq 2$ and u, v, w be distinct vertices in Q_n such that u and v have the same parity opposite to the parity of w. Then, $Q_n - \{w\}$ has a Hamiltonian uv-path.*

Generally, if $F \cup \{u, v\}$ is monopartite, the length of any path between u and v in $Q_n - F$ cannot exceed $2^n - 2|F| - 2$. This leads to the following definition. A path of length at least $2^n - 2|F| - 2$ in $Q_n - F$ is called a *long F-free path in Q_n*. Let $N(u)$ be the set of neighbors of a vertex u in Q_n.

Theorem 1.2 (Fink and Gregor [1])**.** *Let F be a set of at most $2n - 4$ faulty vertices of Q_n and $n \geq 5$. For every two vertices u and v of $Q_n - F$, there exists a long F-free uv-path in Q_n if and only if $N(u) \not\subseteq F \cup \{v\}$ and $N(v) \not\subseteq F \cup \{u\}$.*

Note that for $|F| \leq n-2$, the right side of the equivalence in Theorem 1.2 is always satisfied. Hence, we obtain the following direct corollary.

Corollary 1.3 (Fu [2])**.** *For every set F of at most $n-2$ vertices of Q_n and $n \geq 2$, there is a long F-free uv-path in Q_n between every two vertices u and v of $Q_n - F$.*

In this paper, we consider a modification of this problem for two paths with prescribed endvertices. Assume that we have two different (but not necessarily disjoint) sets $A = \{u, v\}$ and $B = \{x, y\}$ of vertices of $Q_n - F$. A path P between a vertex of A and a vertex of B is called an *AB-path*. Its length $|P|$ is the number of edges in P. A pair P_1, P_2 of vertex-disjoint AB-paths in $Q_n - F$ is called an *F-free AB-routing in Q_n*. Moreover, it is said to be *long* if $|P_1| + |P_2| \geq 2^n - 2|F| - 3$. Note that if A and B are not disjoint, say $A \cap B = \{u = x\}$, then any long F-free AB-routing consists of the uu-path of length 0 and an vy-path of length at least $2^n - 2|F| - 3$.

Theorem 1.4. *For every set F of at most $n - 3$ vertices in Q_n and $n \geq 4$, there exists a long F-free AB-routing in Q_n between every two different sets $A, B \subseteq V(Q_n) \setminus F$ such that $|A| = |B| = 2$ and $A \cup B$ is not monopartite.*

Note that by a simple parity argument it follows that the condition on $A \cup B$ not being monopartite is necessary in Theorem 1.4. Furthermore, the bound $|F| \leq n-3$ is tight. Indeed, let $F \cup \{b, c\}$ be the set of neighbors of some vertex $a \in V(Q_n)$ and $|F| = n - 2$. Clearly, for $A = \{a, b\}$ and $B = \{b, c\}$, the only possible two vertex-disjoint AB-paths in $Q_n - F$ are $P_1 = (a, c)$ and $P_2 = (b)$ of length 1 and 0, respectively, but $2^n - 2|F| - 3 \geq 11$ for $n \geq 4$.

As a consequence, if $F \cup \{u, v\}$ is not monopartite, we obtain an uv-path in $Q_n - F$ of length at least $2^n - 2|F| - 1$, which is more than is guaranteed by long paths.

Corollary 1.5. *For every set F of at most $n - 2$ vertices of Q_n and $n \geq 4$, the graph $Q_n - F$ has an uv-path of length at least $2^n - 2|F| - 1$ for every two vertices $u, v \in V(Q_n) \setminus F$ such that $F \cup \{u, v\}$ is not monopartite.*

Hung et al. [3] showed that even longer path exists if F is not monopartite.

Proposition 1.6 (Hung et al. [3])**.** *Let $n \geq 4$, $F \subseteq V(Q_n)$ such that $|F| \leq n - 2$ and F is not monopartite, and let $u, v \in V(Q_n) \setminus F$ be distinct vertices. Then, $Q_n - F$ has an uv-path of length at least $2^n - 2|F|$.*

2 Proofs

In this section we prove Theorem 1.4 and its Corollary 1.5.

In order to apply induction, we need to split the hypercube Q_n into two $(n-1)$-dimensional subcubes Q_L and Q_R. This is obtained by fixing some coordinate $i \in [n]$. Formally, we define the subcube Q_L as the subgraph of Q_n induced by vertices that have 0 on the i-th coordinate. Similarly, the subcube Q_R is the subgraph of Q_n induced by vertices that have 1 on the i-th coordinate. For a vertex x of Q_L, let x_R be the (only) neighbor of x in Q_R. Similarly for a vertex x of Q_R, let x_L be the (only) neighbor of x in Q_L.

Assume that F is a given set of faulty vertices of Q_n. The vertices of Q_n which are not in F are called F-*free*. We denote the sets of faulty vertices in Q_L and Q_R by F_L and F_R, respectively.

In the following two lemmas we start with dimensions $n = 3$ and $n = 4$. Note that Lemma 2.1 is needed for Lemma 2.2, whereas Lemma 2.2 serves us as a base of induction for Theorem 1.4.

Lemma 2.1. *For every set F of at most 1 vertex of Q_3, there exists a long F-free AB-routing in Q_3 between every two disjoint sets $A, B \subseteq V(Q_3) \setminus F$ such that $|A| = |B| = 2$ and $A \cup B$ is not monopartite.*

Proof. It is trivial to verify the statement by inspection of all cases. First, consider all possible sets A, B in case $F = \emptyset$ when we search for AB-routing P_1, P_2 in Q_3 such that $|P_1| + |P_2| \geq 5$. Then, consider the case $|F| = 1$ when we need $|P_1| + |P_2| \geq 3$. □

Note that the disjointness of the sets A and B is necessary in Lemma 2.1. Indeed, for $A = \{001, 110\}$, $B = \{111, 110\}$, and $F = \{000\}$, observe that there is no path between 001 and 111 in $Q_3 - \{000, 110\}$ of length at least 3, and consequently, no long F-free AB-routing in Q_3.

Lemma 2.2. *For every set F of at most 1 vertex of Q_4, there exists a long F-free AB-routing in Q_4 between every two different sets $A, B \subseteq V(Q_4) \setminus F$ such that $|A| = |B| = 2$ and $A \cup B$ is not monopartite.*

Proof. Case 1: First, we consider the case when $A = \{u, v\}$ and $B = \{x, v\}$ intersect at some vertex v. Then, we can treat v as a new faulty vertex in the set $F' = F \cup \{v\}$, so it suffices to find an ux-path in $Q_4 - F'$ of length at least $2^4 - 2|F'| - 1$. If u, x are of opposite parity, such path exists by Corollary 1.3. Now u and x are of the same parity.

If $F' = \{v\}$, then the requested ux-path exists by Proposition 1.1 since $A \cup B = \{u, x, v\}$ is not monopartite. Now we have $F' = \{f, v\}$. If f and v have opposite parity, then the requested path exists by Proposition 1.6.

Since $A \cup B$ is not monopartite, it remains to consider the case when f and v have the same parity opposite to the parity of u and x. We split Q_4 into Q_L and Q_R so that f and v are in separate subcubes, say $F'_L = \{f\}$ and $F'_R = \{v\}$, and we distinguish two subcases.

Subcase (i): If vertices u, x are in the same subcube, say $u, x \in V(Q_L)$, then from Proposition 1.1 we obtain ux-path P_L in $Q_L - F'_L$ of length 6. Let ab be an edge of P_L such that $a_R, b_R \neq v$. From Corollary 1.3 we obtain $a_R b_R$-path P_R in $Q_R - F'_R$ of length 5. After interconnecting P_R and $P_L - ab$ by edges aa_R, bb_R we get the desired ux-path in $Q_4 - F'$ of length $12 \geq 2^4 - 2|F'| - 1$.

Subcase (ii): Now vertices u, x are in different subcubes, say $x \in V(Q_L)$ and $u \in V(Q_R)$. We choose a vertex $a \in V(Q_L)$ with the opposite parity than u, $a \neq f$, and $a_R \neq u$. Note that $a \neq x$ and $a_R \neq v$. From Corollary 1.3 we obtain ax-path P_L in $Q_L - F'_L$ of length 5, and from Proposition 1.1 we obtain ua_R-path P_R in $Q_R - F'_R$ of length 6. By interconnecting these paths with the edge aa_R we obtain the desired ux-path in $Q_4 - F'$ of length $12 \geq 2^4 - 2|F'| - 1$.

Case 2: Second, we consider the case when $A = \{u, v\}$ and $B = \{x, y\}$ are disjoint. Then, we split Q_4 into Q_L and Q_R so that x, y are in different subcubes, say $x \in V(Q_L)$ and $y \in V(Q_R)$, and we distinguish two subcases depending on the vertices of A.

Subcase (i): If vertices u, v are in the same subcube, say $A \subseteq V(Q_L)$, we choose a vertex $a \in V(Q_L) \setminus F_L$ with the same parity as y, $a_R \notin F_R$, and $a \notin \{u, v, x\}$. Note that such vertex exists, since there are 4 candidate vertices in Q_L with the same parity as y, the set F blocks at most one of them, and the set $\{u, v, x\}$ blocks at most two of them, otherwise $A \cup B$ would be monopartite. For a set $B' = \{x, a\}$ it follows that A, B' are disjoint and $A \cup B'$ is not monopartite. Hence by Lemma 2.1, there is an AB'-routing P_1', P_2' in $Q_L - F_L$ such that $|P_1'| + |P_2'| \geq 2^3 - 2|F_L| - 3$. Assume that a is the endvertex of the path P_1'. By Corollary 1.3, there is an $a_R y$-path in $Q_R - F_R$ of length at least $2^3 - 2|F_R| - 1$ since a_R and y have opposite parity. By interconnecting P_1' and P_R with the edge aa_R, we obtain AB-routing P_1, P_2' in $Q_4 - F$ such that $|P_1| + |P_2'| = |P_1'| + |P_R| + 1 + |P_2'| \geq 2^4 - 2|F| - 3$.

Subcase (ii): Now vertices u, v are in different subcubes, say $u \in V(Q_L)$ and $v \in V(Q_R)$. If u and x, or v and y are of opposite parity, then from Corollary 1.3 we obtain a long F_L-free ux-path P_L in Q_L and a long F_R-free vy-path P_R in Q_R such that $|P_L| + |P_R| \geq 2^4 - 2|F| - 3$. Hence P_L, P_R is a long F-free AB-routing in Q_4.

Since $A \cup B$ is not monopartite, it remains to consider the case when u and x have the same parity opposite to the parity of v and y. We choose two vertices $a, b \in V(Q_L) \setminus F_L$ with the same parity opposite to the parity of u, and $a_R, b_R \notin F_R$. Note that such vertices exist since there are 4 candidate vertices in Q_L with the parity opposite to u and the set F blocks at most one of them. It follows that $A_L = \{u, x\}$, $B_L = \{a, b\}$ are disjoint and $A_L \cup B_L$ is not monopartite. Hence, by Lemma 2.1 there is a long F_L-free $A_L B_L$-routing P_1', P_2' in Q_L. Moreover, since both paths P_1', P_2' have odd length, we have $|P_1'| + |P_2'| \geq 2^3 - 2|F_L| - 2$. Assume that the $A_L B_L$-routing joins the vertex u with b, otherwise we switch the roles of a and b in what follows. By the definition of a, b, the sets $A_R = \{b_R, v\}$, $B_R = \{a_R, y\}$ are disjoint and $A_R \cup B_R$ is not monopartite. Hence, by Lemma 2.1 there is a long F_R-free $A_R B_R$-routing P_3', P_4' in Q_R. By interconnecting P_1', P_2' and P_3', P_4' with edges aa_R, bb_R we obtain AB-routing P_1, P_2 in $Q_4 - F$ such that $|P_1| + |P_2| = |P_1'| + |P_2'| + |P_3'| + |P_4'| + 2 \geq 2^4 - 2|F| - 2$. □

Now we are ready to prove Theorem 1.4, which says that for every set F of at most $n - 3$ vertices in Q_n and $n \geq 4$, there exists a long F-free AB-routing in Q_n between every two different sets $A, B \subseteq V(Q_n) \setminus F$ such that $|A| = |B| = 2$ and $A \cup B$ is not monopartite.

Proof of Theorem 1.4. We proceed by induction on the dimension n. For

$n = 4$ we apply Lemma 2.2. Now assume $n \geq 5$.

First, we split Q_n into Q_L and Q_R such that we separate two arbitrarily chosen faulty vertices from F if $|F| \geq 2$, otherwise we split Q_n arbitrarily. It follows that $|F_L|, |F_R| \leq n - 4$. Thus, we may apply induction both in Q_L and Q_R. We consider the following cases.

Case 1: If both A, B are in one subcube, say $A, B \subseteq V(Q_L)$, then by induction, there is a long F_L-free AB-routing P_1', P_2' in Q_L. Let ab be an edge of P_1' or P_2', such that $a_R, b_R \notin F_R$. Such edge exists, otherwise $2^{n-1} - 2|F_L| - 3 \leq |P_1'| + |P_2'| \leq 2|F_R|$, which yields a contradiction $2^{n-1} - 3 \leq 2|F| \leq 2n - 6$ for $n \geq 5$. From Corollary 1.3 we obtain an $a_R b_R$-path P_R in $Q_R - F_R$ of length $2^{n-1} - 2|F_R| - 1$ since a_R and b_R have different parity. After interconnecting P_R and P_1' or P_2' with the edges aa_R, bb_R we get the AB-routing P_1, P_2 in $Q_n - F$ such that $|P_1| + |P_2| = |P_1'| + |P_2'| + |P_R| + 1 \geq 2^n - 2|F| - 3$.

Case 2: If A is in one subcube and B in the other subcube, say $A = \{u, v\} \subseteq V(Q_L)$ and $B = \{x, y\} \subseteq V(Q_R)$, we distinguish two subcases.

Subcase (i): If u and v have different parity, then from Corollary 1.3 we obtain an uv-path P_L in $Q_L - F_L$ of length at least $2^{n-1} - 2|F_L| - 1$. Let ab be an edge of P_L such that $A' = \{a_R, b_R\}$ is disjoint with F_R and $A' \neq B$. Such edge exists, otherwise $|P_R| \leq 2|F_R| + 1$, which yields a contradiction $2^{n-1} - 2 \leq 2|F| \leq 2n - 6$ for $n \geq 5$. Since $A' \cup B$ is not monopartite, there is a long F_R-free $A'B$-routing P_1', P_2' in Q_R. By interconnecting $P_L - ab$ and P_1', P_2' with the edges aa_R, bb_R, we get an AB-routing P_1, P_2 in $Q_n - F$ such that $|P_1| + |P_2| = |P_L| + |P_1'| + |P_2'| + 1 \geq 2^n - 2|F| - 3$.

Subcase (ii): Now u and v are of the same parity. We choose vertices $B' = \{a, b\} \subseteq V(Q_L) \setminus F_L$ of the same parity opposite to the parity of u such that $A' = \{a_R, b_R\}$ is disjoint with F_R. Such vertices exists, since there are 2^{n-2} candidates in Q_L with parity opposite to the parity of u, and at most $n-3$ of them are blocked by F. Clearly, $A \neq B'$ and $A \cup B'$ is not monopartite. Thus, there is a long F_L-free AB'-routing P_1', P_2' in Q_L. Moreover, since both P_1', P_2' have odd length, we have $|P_1'| + |P_2'| \geq 2^{n-1} - 2|F_L| - 2$. In the other subcube Q_R, at least one vertex of $B = \{x, y\}$ has the opposite parity to the parity of a_R, b_R, u, and v. It follows that $A' \neq B$ and $A' \cup B$ is not monopartite, and hence, there is a long F_R-free $A'B$-routing P_3', P_4' in Q_R. By interconnecting P_1', P_2' and P_3', P_4' with edges aa_R, bb_R we get an AB-routing P_1, P_2 such that $|P_1| + |P_2| = |P_1'| + |P_2'| + |P_3'| + |P_4'| + 2 \geq 2^n - 2|F| - 3$.

Case 3: If A is one subcube, and B in both subcubes, say $A = \{u, v\} \subseteq V(Q_L)$, $x \in V(Q_L)$, $y \in V(Q_R)$, then we proceed similarly as in Case 2, Subcase (i) of Lemma 2.2. We choose a vertex $a \in V(Q_L) \setminus F_L$ with the same parity as y, $a_R \notin F_R$, and $a \notin \{u, v, x\}$. Note that such vertex exists, since there are 2^{n-2} candidate vertices in Q_L with the same parity as y, the faulty vertices block at most $n-3$ of them, the set $\{u, v, x\}$ blocks at most 3 of them, and $2^{n-2} - (n-3) - 3 \geq 1$ for $n \geq 5$. For a set $B' = \{x, a\}$ it follows that A, B' are disjoint and $A \cup B'$ is not monopartite. Hence by induction, there is an AB'-routing P_1', P_2' in $Q_L - F_L$ such that $|P_1'| + |P_2'| \geq 2^{n-1} - 2|F_L| - 3$. Assume that a is the endvertex of the path P_1'. By Corollary 1.3, there is an $a_R y$-path in $Q_R - F_R$ of length at least $2^{n-1} - 2|F_R| - 1$ since a_R and y have opposite parity. By interconnecting P_1' and P_R with the edge aa_R, we obtain AB-routing P_1, P_2' in $Q_n - F$ such that $|P_1| + |P_2'| = |P_1'| + |P_R| + 1 + |P_2'| \geq 2^n - 2|F| - 3$.

Case 4: If A, B are both subcubes, say $u, x \in V(Q_L)$ and $v, y \in V(Q_R)$, then we proceed similarly as in Case 2, Subcase (ii) of Lemma 2.2. If u and x, or v and y are of opposite parity, then from Corollary 1.3 we obtain a long F_L-free ux-path P_L in Q_L and a long F_R-free vy-path P_R in Q_R such that $|P_L| + |P_R| \geq 2^n - 2|F| - 3$. Hence P_L, P_R is a long F-free AB-routing in Q_n.

Since $A \cup B$ is not monopartite, it remains to consider the case when u and x have the same parity opposite to the parity of v and y. We choose two vertices $a, b \in V(Q_L) \setminus F_L$ with the same parity opposite to the parity of u, and $a_R, b_R \notin F_R$. Note that such vertices exist since there are 2^{n-2} candidate vertices in Q_L with the parity opposite to the parity of u, the faulty vertices block at most $n-3$ of them, and $2^{n-2} - (n-3) \geq 2$ for $n \geq 5$. It follows that $A_L = \{u, x\}$, $B_L = \{a, b\}$ are disjoint and $A_L \cup B_L$ is not monopartite. Hence, by induction there is a long F_L-free $A_L B_L$-routing P_1', P_2' in Q_L. Moreover, since both paths P_1', P_2' have odd length, we have $|P_1'| + |P_2'| \geq 2^{n-1} - 2|F_L| - 2$. Assume that the $A_L B_L$-routing joins the vertex u with b, otherwise we switch the roles of a and b in what follows. By the definition of a, b, the sets $A_R = \{b_R, v\}$, $B_R = \{a_R, y\}$ are disjoint and $A_R \cup B_R$ is not monopartite. Hence, by induction there is a long F_R-free $A_R B_R$-routing P_3', P_4' in Q_R. By interconnecting P_1', P_2' and P_3', P_4' with edges aa_R, bb_R we obtain AB-routing P_1, P_2 in $Q_n - F$ such that $|P_1| + |P_2| = |P_1'| + |P_2'| + |P_3'| + |P_4'| + 2 \geq 2^n - 2|F| - 2$. □

Finally, we prove Corollary 1.5 that says for every set F of at most $n-2$ vertices of Q_n and $n \geq 4$, the graph $Q_n - F$ has an uv-path of length at least $2^n - 2|F| - 1$ for every two vertices $u, v \in V(Q_n) \setminus F$ such that $F \cup \{u, v\}$ is not monopartite.

Proof of Corollary 1.5. If $F = \emptyset$, then u and v have opposite parity, and the statement follows from a well-known fact that Q_n contains a Hamiltonian path between every two vertices of opposite parity. Otherwise, there exists $f \in F$ such that $\{u, v, f\}$ is not monopartite. Applying Theorem 1.4 for

$A = \{u, f\}$, $B = \{v, f\}$, $F' = F \setminus \{f\}$ we obtain vertex-disjoint paths P_1, P_2 such that P_1 joins u and v, P_2 contains only f, and $|P_1|+|P_2| \geq 2^n - 2|F'| - 3$. Hence $|P_1| \geq 2^n - 2|F| - 1$, and P_1 is the desired path. □

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