THE EXTREMAL FUNCTION FOR PARTIAL BIPARTITE TILINGS

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Abstract. For a fixed bipartite graph H and given $\alpha \in (0,1)$ we determine the threshold $T_H(\alpha)$ which guarantees that any n-vertex graph with at least $T_H(\alpha)\binom{n}{2}$ edges contains $(1-o(1))\frac{\alpha}{v(H)}n$ vertex-disjoint copies of H.

1. Introduction

The Turán Theorem [12], the single most important result in Extremal Graph Theory, gives a sharp threshold, denoted $\mathrm{ex}(n, K_r)$, for the maximum number of edges of an n-vertex graph with no copy of K_r. Even though the Turán Theorem applies to any pair of values n and r, the interesting instances are rather those when n is large compared to r. Erdős and Stone [2] extended the result by determining the asymptotic behaviour of the function $\mathrm{ex}(n, H)$ for a fixed non-bipartite graph H. The same problem in the case that H is a fixed bipartite graph is — despite considerable effort — wide open for most graphs H. Let us recall that when H has colour classes of sizes s and t, $s \le t$, then the Kövari-Sós-Turán Theorem [8] asserts that

$$\mathrm{ex}(n, H) \le O(n^{2-1/s}) = o(n^2)\ . \tag{1}$$

On the other hand, a standard random graph argument gives that $\mathrm{ex}(n, K_{s,t}) \ge \Omega(n^{2-(s+t-2)/(st-1)})$.

It is natural to extend the above *existential questions* to *tiling questions*. In such a setting one asks for the maximum number of edges of an n-vertex graph which does not contain ℓ vertex-disjoint copies of a graph H. This quantity denotes $\mathrm{ex}(n, \ell \times H)$. Erdős and Gallai [3] gave a complete solution to the problem in the case when $H = K_2$.

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Theorem 1 (Erdős-Gallai, 1959). *Suppose that $\ell \leq n/2$. Then*

$$\mathrm{ex}(n, \ell \times K_2) = \max\left\{(\ell-1)(n-\ell+1) + \binom{\ell-1}{2}, \binom{2\ell-1}{2}\right\} .$$

Given $n, x \in \mathbb{N}$, $x \leq n$, we define the graph $M_{n,x}$ as an n-vertex graph whose vertex set is split into sets A and B, $|A| = x$, $|B| = n-x$, A induces a clique, B induces an independent set, and $M_{n,x}[A, B] \simeq K_{x,n-x}$. The graph $L_{n,x}$ is an n-vertex graph whose edges induce a clique of order x. Obviously, $e(M_{n,\ell-1}) = (\ell-1)(n-\ell+1) + \binom{\ell-1}{2}$, and $e(L_{n,2\ell-1}) = \binom{2\ell-1}{2}$. Moreover, it is easy to check that there are not ℓ disjoint edges in either of the graphs $M_{n,\ell-1}$, $L_{n,2\ell-1}$. Therefore, when $\ell < \frac{2}{5}n - O(1)$, the graph $M_{n,\ell-1}$ is (the unique) graph showing that $\mathrm{ex}(n, \ell \times K_2) \geq (\ell-1)(n-\ell+1) + \binom{\ell-1}{2}$. The graph $L_{n,2\ell-1}$ is the unique extremal graph for the problem otherwise.

Moon [10] started the investigation of $\mathrm{ex}(n, \ell \times K_r)$. Allen, Böttcher, Hladký, and Piguet [1] only recently determined the behaviour of $\mathrm{ex}(n, \ell \times K_r)$ for the whole range of ℓ in the case $r = 3$, and they made a substantial progress for larger values of r. Simonovits [11] determined the value $\mathrm{ex}(n, \ell \times H)$ for a non-bipartite graph H, fixed value of ℓ and large n.

Another density parameter which can be considered in the context of tiling questions is the minimum degree of the host graph. That is, we ask what is the largest possible minimum degree of an n-vertex graph which does not contain ℓ vertex-disjoint copies of H. In the case $H = K_r$ the precise answer is given by the Hajnal-Szemerédi Theorem [4]. An asymptotic threshold for a general fixed graph H was determined by Komlós [5].

In the present paper we determine asymptotic behaviour of the function $\mathrm{ex}(n, \ell \times H)$ for a fixed bipartite graph. Let H be an arbitrary bipartite graph. Suppose that $b : V(H) \to [2]$ is a proper coloring of H which minimizes $|b^{-1}(1)|$. We define quantities $s(H) := |b^{-1}(1)|$, $t(H) := |b^{-1}(2)|$. Obviously, $s(H) \leq t(H)$, and $s(H) + t(H) = v(H)$. Further, define $V_1(H) := b^{-1}(1)$ and $V_2(H) := b^{-1}(2)$. The sets $V_1(H)$ and $V_2(H)$ are uniquely defined provided that H does not contain a balanced bipartite graph as its component; this will always be the case in the rest of the paper.

Given $s, t \in \mathbb{N}$, we define a function $T_{s,t} : (0, 1) \to (0, 1)$ by setting

$$T_{s,t}(\alpha) := \max\left\{\frac{2s\alpha}{s+t}\left(1 - \frac{s\alpha}{2(s+t)}\right), \alpha^2\right\} , \tag{2}$$

for $\alpha \in (0,1)$. Note that $T_{s',t'} = T_{s,t}$ when $s' = ks$ and $t' = kt$. Also, note that

$$T_{s,s}(\alpha)\binom{n}{2} = \mathrm{ex}\left(n, \frac{\alpha n}{2} \times K_2\right) + o(n^2)\ . \tag{3}$$

Our main result is the following.

Theorem 2. *Suppose that H is a bipartite graph with no isolated vertices, $s := s(H), t := t(H)$. Let $\alpha \in (0,1)$ and $\varepsilon > 0$. Then there exists an $n_0 = n_0(s,t,\alpha,\varepsilon)$ such that for any $n \geq n_0$, any graph G with n vertices and at least $T_{s,t}(\alpha)\binom{n}{2}$ edges contains more than $(1-\varepsilon)\frac{\alpha}{s+t}n$ vertex-disjoint copies of the graph H.*

Thus Theorem 2 asserts that $\mathrm{ex}(n, \beta n \times H) \leq T_{s(H),t(H)}(\beta(s+t))\binom{n}{2} + \varepsilon n^2$ for any bipartite graph H with no isolated vertices, $\varepsilon > 0$ and large n. This asymptotically matches the lower bound which comes — as in Theorem 1 — from graphs $M_{n,\beta s(H)n-1}$ and $L_{n,\beta(s(H)+t(H))n-1}$. Note however that for most values of ℓ the graphs $M_{n,\beta s(H)n-1}$ and $L_{n,\beta(s(H)+t(H))n-1}$ are not extremal for the problem. For example, we can replace the independent set in the graph

$$L_{n,\beta(s(H)+t(H))n-1}$$

by any H-free graph. This suggests that a precise result is not within the reach of current techniques.

The assumption on H to contain no isolated vertices in Theorem 2 is made just for the sake of compactness of the statement. Indeed, let H' be obtained from H by removing all the isolated vertices. Then there is a very simple relation of the sizes of optimal coverings by vertex disjoint copies of H and H' in an n-vertex graph G. Let x and x' be the number of vertices covered by a maximum family of vertex-disjoint copies of H and of H' in G, respectively. We have that

$$x = \min\left\{v(H)\left\lfloor\frac{n}{v(H)}\right\rfloor, \frac{x'v(H)}{v(H')}\right\}\ .$$

Our proof of Theorem 2 borrows ideas from [5].

If $\mathcal{F}$ is a family of graphs, and G is a graph, an *$\mathcal{F}$-tiling* in G is a set of vertex-disjoint subgraphs of G, each of them isomorphic to a graph in $\mathcal{F}$. If $\mathcal{F} = \{H\}$ then we simply say H-tiling. $V(F)$ denotes the vertices of G covered by an $\mathcal{F}$-tiling F, and $|F| = |V(F)|$ is the *size* of the tiling F. For $n \in \mathbb{N}$, we write $[n]$ to denote the set $\{1, 2, \ldots, n\}$.

2. Tools for the proof of the main result

Our main tool is Szemerédi's regularity lemma (see [7, 9] for surveys). To state it we need some more notation.

Let $G = (V, E)$ be an n-vertex graph. If A, B are disjoint nonempty subsets of $V(G)$, the *density* of the pair (A, B) is $d(A, B) = e(A, B)/(|A||B|)$. We say that (A, B) is an *ε-regular pair* if $|d(X, Y) - d(A, B)| < \varepsilon$ for every $X \subset A, |X| > \varepsilon|A|$ and $Y \subset B, |Y| > \varepsilon|B|$.

The following statement asserts that large subgraphs of regular pairs are also regular.

Lemma 3. *Let (A, B) be an ε-regular pair with density d, and let $A' \subset A, |A'| \geq \alpha|A|, B' \subset B, |B'| \geq \alpha|B|$, $\alpha \geq \varepsilon$. Then (A', B') is an ε'-regular pair with $\varepsilon' = \max\{\varepsilon/\alpha, 2\varepsilon\}$, and for its density d' we have $|d' - d| < \varepsilon$.*

Let $\varepsilon > 0$ and $d \in [0, 1]$. An *(ε, d)-regular partition* of G with *reduced graph* $R = (V', E')$ is a partition $V_0 \dot\cup V_1 \dot\cup \ldots \dot\cup V_k$ of V with $|V_0| \leq \varepsilon n$, $|V_i| = |V_j|$ for any $1 \leq i < j \leq k$, $V(R) = \{V_1, V_2, \ldots, V_k\}$, such that (V_i, V_j) is an ε-regular pair in G of density greater than d whenever $V_iV_j \in E(R)$, and the subgraph $G' \subset G$ induced by the ε-regular pairs corresponding to the edges of R has more than $e(G) - (d + 3\varepsilon)n^2/2$ edges. In this case, we also say that G has an (ε, d)-reduced graph R, and call the sets $V_i, 1 \leq i \leq k$, the *clusters* of G.

The following lemma is a consequence of the so-called degree version of the Regularity Lemma [7, Theorem 1.10].

Lemma 4 (Regularity lemma). *For every $\varepsilon > 0$ and $m \in \mathbb{N}$ there is an $M = M(\varepsilon, m)$ such that, if G is any graph with more than M vertices and $d \in [0, 1]$ is any real number, then G has an (ε, d)-reduced graph R on k vertices, with $m \leq k \leq M$.*

Given four positive numbers a, b, x, y we say that the pair a, b *dominates* the pair x, y, if $\max\{x, y\}/\min\{x, y\} \geq \max\{a, b\}/\min\{a, b\}$. The following easy lemma states that $K_{a,b}$ has an almost perfect $K_{s,t}$-tiling provided that a, b dominates s, t.

Lemma 5. *For any $s, t \in \mathbb{N}$ there exists a constant C such that the following holds. Suppose that the pair $a, b \in \mathbb{N}$ dominates s, t. Then the graph $K_{a,b}$ contains a $K_{s,t}$-tiling containing all but at most C vertices of $K_{a,b}$.*

Proof. If $s = t$ then necessarily $a = b$. There obviously exists a $K_{s,t}$-tiling containing all but at most $C := 2(s - 1)$ vertices of $K_{a,b}$.

With no loss of generality, we may suppose that $a \le b$ and $s < t$. Then $as \le bt$ and $bs \le at$. A tiling with $\lfloor (bt-as)/(t^2-s^2) \rfloor$ copies of $K_{s,t}$ with the s-part of the $K_{s,t}$ placed in the a-part of the $K_{a,b}$ and $\lfloor (at-bs)/(t^2-s^2) \rfloor$ copies placed the other way misses at most $C := 2(s+t-1)$ vertices of $K_{a,b}$. $\square$

The next lemmas, versions of the Blow-up Lemma [6], assert that regular pairs have almost as good tiling properties as complete bipartite graphs.

Lemma 6. *For every $d > 0, \gamma \in (0,1)$ and any two graphs R and H, there is an $\varepsilon = \varepsilon(H,d,\gamma) > 0$ such that the following holds for all positive integers s. Let R_s be the graph obtained from R by replacing every vertex of R by s vertices, and every edge of R by a complete bipartite graph between the corresponding s-sets. Let G be the graph obtained similarly from R by replacing the edges with ε-regular pairs of density at least d. If R_s contains an H-tiling of size at least $\gamma v(R_s)$ then so does G.*

Lemma 7. *For every bipartite graph H and every $\gamma, d > 0$ there exists an $\varepsilon = \varepsilon(H,d,\gamma) > 0$ such that the following holds. Suppose that there is an H-tiling in $K_{a,b}$ of size x. Let (A,B) be an arbitrary ε-regular pair with density at least d, $|A| = a$, $|B| = b$. Then the pair (A,B) contains an H-tiling of size at least $x - \gamma(a+b)$.*

Finally, let us state a straightforward corollary of the König Matching Theorem.

Fact 8. *Let $G = (A \dot\cup B, E)$ be a bipartite graph with color classes A and B. If G has no matching with $l+1$ edges, then $e(G) \le l \max\{|A|,|B|\}$.*

3. The proof

In this section, we first state and prove the main technical result, Lemma 9. Then, we show how it implies Theorem 2.

For $s,t \in \mathbb{N}$, we set $\mathcal{F}_1 := \{K_{s,t}, K_{s,t-1}, K_2\}$ and $\mathcal{F}_2 := \{K_{st,t^2}, K_{st-1,(t-1)t}, K_{st,(t-1)t}, K_2\}$. Let us note that when $s < t$, the sizes of the two color classes of any graph from $\mathcal{F}^* := \mathcal{F}_1 \cup \mathcal{F}_2$ dominate s and t.

Suppose that F is an $K_{s,t}$-tiling in a graph G, $s < t$. We say that a pair of matchings $E_0, E_1 \subseteq E(G)$ is an *F-augmentation* if $E_0 \subseteq E_G[V(G) - V(F), V_1(F)]$, $E_1 \subseteq E_G[V_2(F)]$ and each copy of $K_{s,t}$ in F contains at most one vertex matched by E_0 and at most one vertex matched by E_1. Moreover, we require that if $K \in F$ contains a vertex matched by E_0, then it also contains a vertex matched by E_1.

The main step in our proof of Theorem 2 is the following lemma.

Lemma 9. *Let $t > s \geq 1, \alpha \in (0,1)$ and $\varepsilon > 0$. Suppose G is an n-vertex graph with $n \geq h(s,t,\alpha,\varepsilon)$ and $e(G) \geq T_{s,t}(\alpha)\binom{n}{2}$, and F is a maximum $K_{s,t}$-tiling in G with $|F| \leq (1-\varepsilon)\alpha n$. Then there exists an $\varepsilon' = \varepsilon'(s,t,\alpha,\varepsilon) > 0$ such that one of the following is true:*

(i) there exists an $\mathcal{F}_1$-tiling F' in G with $|F'| \geq |F| + \varepsilon' n$, or
(ii) there exists an F-augmentation E_0, E_1 such that E_0 contains at least $\varepsilon' n$ edges.

Proof. Set

$$\varepsilon' := \frac{1}{4}\min\left\{\frac{\varepsilon\alpha^2}{3t+1}, \frac{\varepsilon s\alpha}{(3t+1)(s+t)}\right\} ,$$

and let $h(s,t,\alpha,\varepsilon)$ be sufficiently large.

Suppose for a contradiction that the assertions of the lemma are not true.

Set $L := V(G) - V(F)$ and $m := |L|$. Let $\mathcal{C} := \{V_1(K) : K \in F\}, \mathcal{D} := \{V_2(K) : K \in F\}$ and $C := \bigcup \mathcal{C}, D := \bigcup \mathcal{D}$. We call members of $\mathcal{C}$ *lilliputs* while members of $\mathcal{D}$ are *giants*. We say that giant $V_2(K)$ ($K \in F$) is *coupled* with lilliput $V_1(K)$.

As F is a maximum $K_{s,t}$-tiling in G, by (1) we have that

$$e(G[L]) = o(n^2) . \tag{4}$$

Let r be the number of copies of $K_{s,t}$ in F. Then $r \leq (1-\varepsilon)\alpha n/(s+t)$. Moreover, we have

$$m = n - (s+t)r . \tag{5}$$

Let us define an auxiliary graph $H = (V', E')$ as follows. The vertex-set of H is $V' := \mathcal{C} \cup \mathcal{D} \cup L$. For any $x \in L$ and $K \in F$ the edge $xV_1(K)$ belongs to E' iff $\mathrm{N}_G(x) \cap V_1(K) \neq \emptyset$. Similarly, the edge $xV_2(K)$ belongs to E' iff $\mathrm{N}_G(x) \cap V_2(K) \neq \emptyset$. Finally, for any distinct $K, K' \in F$ the edge $V_2(K)V_2(K')$ belongs to E' iff $E_G(V_2(K), V_2(K')) \neq \emptyset$. The vertices L and the vertices $\mathcal{C}$ induce two independent sets in H.

As (i) does not hold, $H[L, \mathcal{D}]$ does not contain a matching with at least $\varepsilon' n$ edges. It follows from Fact 8 that

$$e_G(L, D) \leq \varepsilon' n t \max\{m, r\} . \tag{6}$$

Let M be a maximum matching in $H[L, \mathcal{C}]$ with l edges. Obviously, $l \leq r$. By Fact 8, we have that

$$e_G(L, C) \leq ls \max\{m, r\} \; . \tag{7}$$

Let $\mathcal{C}' \subseteq \mathcal{C}$ be the lilliputs matched by M. We write $\mathcal{D}' \subseteq \mathcal{D}$ for the giants coupled with $\mathcal{C}'$. Set $D' = \bigcup \mathcal{D}'$.

Suppose for a moment that $H[\mathcal{D}'] \cup H[\mathcal{D}', \mathcal{D} - \mathcal{D}']$ contains a matching T with at least $\varepsilon' n$ edges. Let $\mathcal{D}''$ be the giants matched by T and M' the set of edges in M matching the lilliputs coupled with $\mathcal{D}''$. Then M' and T give rise to an F-augmentation E_0, E_1 in G with $|E_0| = |M'| \geq |T| \geq \varepsilon' n$, contradicting our assumption that (ii) does not hold.

So $H[\mathcal{D}'] \cup H[\mathcal{D}', \mathcal{D} - \mathcal{D}']$ does not contain a matching with at least $\varepsilon' n$ edges. Applying Theorem 1 and passing to the graph G, we get

$$e(G[D'] \cup G[D', D - D']) \leq t^2 \mathrm{ex}(r, \varepsilon' n) + r\binom{t}{2} \leq 2t^2 \varepsilon' n r + r\binom{t}{2} \; .$$

Therefore,

$$\begin{aligned}
&e(G[C \cup D]) = \\
&= e(G[D'] \cup G[D', D - D']) + e(G[D - D']) + e(G[C]) + e_G(C, D) \\
&\leq 2t^2 \varepsilon' n r + r\binom{t}{2} + \binom{r-l}{2} t^2 + \binom{rs}{2} + r^2 st
\end{aligned} \tag{8}$$

Summing up the bounds (4), (6), (7), and (8) we get

$$\begin{aligned}
e(G) &= e(G[L]) + e_G(L, D) + e_G(L, C) + e(G[C \cup D]) \\
&\leq o(n^2) + t\varepsilon' n^2 + ls \max\{m, r\} + 2\varepsilon' n r t^2 \\
&\qquad\qquad + r\binom{t}{2} + \binom{r-l}{2} t^2 + r^2 st + \binom{rs}{2} \\
&\overset{(5)}{\leq} o(n^2) + 3t\varepsilon' n^2 + r\binom{t}{2} + r^2 st + \binom{rs}{2} \\
&\qquad\qquad + \max\left\{\binom{r}{2} t^2, rs(n - (s+t)r)\right\} \\
&< \max\left\{\binom{(s+t)r}{2}, \binom{rs}{2} + rs(n - rs)\right\} + (3t+1)\varepsilon' n^2 \\
&< T_{s,t}(\alpha)\binom{n}{2} \, ,
\end{aligned}$$

a contradiction. □

Suppose $G = (V, E)$ is a graph and $r \in \mathbb{N}$. The *r-expansion* of G is the graph $G' = (V', E')$ defined as follows. The vertex set of G' is $V \times [r]$. For $a, b \in [r]$, an edge $((u, a), (v, b))$ belongs to E' iff uv belongs to E. Note that there is a natural projection $\pi_{G'} : V' \to V$ that maps every vertex (u, a) from G' to the vertex u in G. We are interested in the following property of r-expansions. Suppose that K is a copy of any graph from $\mathcal{F}^*$ in G. Then $\pi_{G'}^{-1}(V(K))$ contains a complete bipartite graph B with color classes of sizes $s(K)r$ and $t(K)r$. By Lemma 5 we can tile B almost perfectly with copies of $K_{s,t}$. If F is an $\mathcal{F}^*$-tiling in G, we can apply the above operation on each member $K \in F$ and obtain a new tiling F' — which we call *retiling* — in the graph G'.

We are now ready to prove Theorem 2.

Proof of Theorem 2. Note that it suffices to prove the theorem for $H \simeq K_{s,t}$.

We first deal with the particular case $t = s$. Set $\alpha' := (1 - \varepsilon/4)\alpha$. Let $\varepsilon_1 := \frac{1}{15}(T_{s,t}(\alpha) - T_{s,t}(\alpha'))$, and ε_2 be given by Lemma 7 for input parameters H, $d := \varepsilon_1$ and $\gamma := \alpha\varepsilon/8$. Suppose that k_0 is sufficiently large. Let M be the bound from Lemma 4 for precision $\varepsilon_R := \min\{\varepsilon_1, \varepsilon_2\}$ and minimal number of clusters k_0. Let C be given by Lemma 5 for the input parameters s, t. Fix $n_0 \gg MC$. Suppose that G is an n-vertex graph, $n \geq n_0$, with at least $T_{s,t}(\alpha)\binom{n}{2}$ edges. We apply Lemma 4 on G to obtain an (ε_R, d)-reduced graph R with k clusters, $k_0 \leq k \leq M$. We have that

$$\begin{aligned} e(R) &\geq (T_{s,t}(\alpha) - d - 3\varepsilon_1)\binom{k}{2} \\ &= (T_{s,t}(\alpha') + \frac{1}{4}(T_{s,t}(\alpha) - (T_{s,t}(\alpha')))\binom{k}{2} \overset{(3)}{>} \mathrm{ex}\left(k, \frac{\alpha' k}{2} \times K_2\right) . \end{aligned}$$

Therefore, R contains at least $\frac{\alpha' k}{2}$ independent edges. These edges correspond to regular pairs in G which can be tiled almost perfectly with copies of $K_{s,t}$, by means of Lemma 5 and Lemma 7. Elementary calculations give that in this way we get a tiling of size at least $(1 - \varepsilon)\alpha n$.

Consequently we may suppose that $t > s$. We first define a handful of parameters. Set

$$\alpha' := \frac{6 - 4\varepsilon}{6 - 3\varepsilon}\alpha, \quad \gamma := (1 - \varepsilon/2)\alpha', \quad d := \frac{2}{5}(T_{s,t}(\alpha) - T_{s,t}(\alpha')) .$$

Note that $\gamma = (1 - 2\varepsilon/3)\alpha$.

Let ε_R be given by Lemma 6 for input graph $K_{s,t}$, density $d/2$ and approximation parameter γ. We may suppose that ε_R is sufficiently small such that $\gamma(1-\varepsilon_R) > (1-\varepsilon)\alpha$ and $\varepsilon_R < d/2$. Let C be given by Lemma 5 for input s, t. Further, let ε' and h be given by Lemma 9 for input parameters α' and $\varepsilon/4$. Set

$$p := t^2 \left\lceil \frac{4C}{\varepsilon'} \right\rceil, \quad q := \left\lceil \frac{2t}{\varepsilon'} \right\rceil$$

Let M be the upper bound on the number of clusters given by Lemma 4 for input parameters h (for the minimal number of clusters) and $\varepsilon_R p^{-q}/2$ (for the precision). Let $n_0 > Mp^q$ be sufficiently large.

Suppose now that G is a graph with $n > n_0$ vertices and at least $T_{s,t}(\alpha)\binom{n}{2}$ edges. We first apply Lemma 4 to G with parameters $\varepsilon_R p^{-q}/2$ and h. In this way we obtain an $(\varepsilon_R p^{-q}/2, d)$-reduced graph R with at least h vertices.

Let us now define a sequence of graphs $R^{(i)}$ by setting $R^{(0)} = R$ and letting $R^{(i)}$ be the p-expansion of $R^{(i-1)}$, $i = 1, 2, \ldots, q$. Note that $e(R^{(i)}) \geq T_{s,t}(\alpha')\binom{v(R^{(i)})}{2}$ for every $i \in \{0, 1, \ldots, q\}$.

Let $F^{(i)}$ be a maximum $K_{s,t}$-tiling in $R^{(i)}$ for $i = 0, 1, \ldots, q$. We claim that

$$|F^{(i)}| \geq \min\left\{ \frac{i\varepsilon' v(R^{(i)})}{2t}, \left(1 - \frac{\varepsilon}{2}\right)\alpha' v(R^{(i)}) \right\} . \tag{9}$$

To this end it suffices to show that for any $i \geq 1$,

(C1) if $|F^{i-1}| > (1-\varepsilon/4)\alpha' v(R^{(i-1)})$, then $\frac{|F^{(i)}|}{v(R^{(i)})} \geq \frac{|F^{(i-1)}|}{v(R^{(i-1)})} - \frac{\varepsilon\alpha'}{4}$, and

(C2) if $|F^{i-1}| \leq (1-\varepsilon/4)\alpha' v(R^{(i-1)})$, then $\frac{|F^{(i)}|}{v(R^{(i)})} \geq \frac{|F^{(i-1)}|}{v(R^{(i-1)})} + \frac{\varepsilon'}{2t}$.

In the case (C1), the retiling of $F^{(i-1)}$ in $R^{(i)}$ has size at least $(1-\varepsilon/2)\alpha' v(R^{(i)})$, thus proving the statement.

Consequently we may suppose that we are in case (C2). Apply Lemma 9 to the graph $R^{(i-1)}$ and the tiling $F^{(i-1)}$, with parameters α' and $\varepsilon/4$.

Suppose first that assertion (i) of the lemma holds. Then $R^{(i-1)}$ contains an $\mathcal{F}_1$-tiling F with $\frac{|F|}{v(R^{(i-1)})} \geq \frac{|F^{(i-1)}|}{v(R^{(i-1)})} + \varepsilon'$. By retiling F, we get a $K_{s,t}$-tiling in $R^{(i)}$ with size at least $|F|(p-C) > i\varepsilon' v(R^{(i)})/(2t)$, thus proving the statement.

Suppose now that assertion (ii) of Lemma 9 is true. Then $R^{(i-1)}$ contains an $F^{(i-1)}$-augmentation E_0, E_1 with $|E_0| \geq \varepsilon' v(R^{(i-1)})$. Let $r = p/t$. We shall denote by T the t-expansion of $R^{(i-1)}$ and by T' the r-expansion of T. Note that T' is isomorphic to $R^{(i)}$.

Let us build an $\mathcal{F}_2$-tiling in T in the following way.

For every edge $e = (u, v) \in E_0$ with $u \in V(F^{(i-1)})$ we choose an edge $e' = (u', v')$ in T with $\pi_T(u') = u$ and $\pi_T(v') = v$. We shall denote by w_e the vertex u' corresponding to u.

For every edge $e = (u, v) \in E_1$ we choose a set S_e of t independent edges in $\pi_T^{-1}(e)$.

For every $K \in F^{(i-1)}$ we shall also choose a subgraph K' of T. We distinguish the following cases. If K has no vertex matched by E_0 or E_1, then we let $K' := T[\pi_T^{-1}(K)]$. If K has a vertex u matched by E_1 but no vertex matched by E_0, we let $K' := T[\pi_T^{-1}(K - u)]$. Then $K' \simeq K_{st,(t-1)t}$. Finally, if K has a vertex u matched by an edge $e \in E_0$ and a vertex v matched by an edge in E_1, we let $K' := T[\pi_T^{-1}(K - v)] - w_e$. Note that in this last case $K' \simeq K_{st-1,(t-1)t}$.

It is easy to see that

$$F := \{e' : e \in E_0\} \cup \{K' : K \in F^{(i-1)}\} \cup \left(\bigcup_{e \in E_1} S_e \right)$$

is an $\mathcal{F}_2$-tiling in T. Moreover, we have that $\frac{|F|}{v(T)} \geq \frac{|F^{(i-1)}|}{v(R^{(i-1)})} + \frac{\varepsilon'}{t}$. So the retiling of F in T' has size at least $|F|(r - C) \geq i\varepsilon' v(R^{(i)})/(2t)$. This proves (C2) and also (9).

Using Lemma 3, we may subdivide every cluster corresponding to a vertex of R into p^q equal-sized parts, by discarding some vertices if necessary. This gives us an $(\varepsilon_R, d/2)$-reduced graph R'. By construction $R' \simeq R^{(q)}$. By (9), there is a $K_{s,t}$-tiling F in R' with size at least $(1 - \varepsilon/2)\alpha' v(R')$. Let G' be the subgraph of G induced by the clusters corresponding to the vertices of R'. By applying Lemma 6 to R', we see that G' has a $K_{s,t}$-tiling of size at least $\gamma v(G') \geq \gamma(1 - \varepsilon_R)v(G) > (1 - \varepsilon)\alpha v(G)$, and so does G.

This finishes the proof of Theorem 2. $\square$

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