Cubical coloring — fractional covering by cuts

Robert Šámal

Abstract

We introduce a new graph invariant that measures fractional covering of a graph by cuts. Besides being interesting in its own, it is useful for study of homomorphisms and tension-continuous mappings. We study the relations with chromatic number, bipartite density, and other graph parameters.

As a main result, we compute the parameter for infinitely many graphs based on hypercubes. These graphs play for our parameter the role that circular cliques play for the circular chromatic number. The fact, that the defined parameter attains on these graphs the 'correct' value suggests that the definition is a natural one.

In the proof we use the eigenvalue bound for maximum cut and a recent result of Engström, Färnqvist, Jonsson, and Thapper.

This paper is an extension of extended abstract, that appeared as [20]. Another previous treatment of this topics appears in the author's thesis [21].

1 Introduction / definitions

All graphs we consider are undirected loopless; to avoid trivialities we do not consider edgeless graphs. For a set $W \subseteq V(G)$ we let $\delta(W)$ denote the set of edges leaving W and we call any set of form $\delta(W)$ a *cut*. Other terminology we shall be using is standard, and can be found in, e.g., [8].

Let us call (cut) n/k*-cover of* G an n-tuple $(X_1, \dots, X_n)$ of cuts in G such that every edge of G is covered by at least k of them. We define

$$x(G) = \inf\left\{\frac{n}{k} \mid \text{exists } n/k\text{-cover of } G\right\}$$

and call $x(G)$ the *fractional cut-covering number* of G. Its 'rescaling'

$$\chi_q(G) = 2/(2 - x(G))$$

will be called the *cubical chromatic number* of G. (This terminology is motivated by analogy with the circular chromatic number, see the discussion following Equation (1).)

If $k = 1$, i.e., if we want to cover every edge at least once, then we need at least $\lceil \log_2 \chi(G) \rceil$ of them (see, e.g., [7]). Here we consider a fractional version. In this context we may find it surprising that $x(G) < 2$ for every G (Corollary 2.3).

From another perspective, $x(G)$ is the fractional chromatic number of a certain hypergraph: it has $E(G)$ as points and odd cycles of G as hyperedges. This suggests that $x(G)$ is a solution of a linear program, see Equations (2) and (3).

The parameter $x(G)$ has found surprising use in computer science. Färnqvist, Jonsson, and Thapper [12] study the approximability of MAXCUT and its generalizations (so-called H-COLORING) using a suitably defined metric space. The function used to define the metric is in [9] recognized as a natural generalization of fractional covering by cuts. (They need to cover by subgraphs that are homomorphic to H for other H than just $H = K_2$.) See the concluding remark for further discussion.

As the last of the introductory remarks, we note that $x(G)$ is a certain type of chromatic number, but instead of complete graphs (or Kneser graphs or circulants) which are used to define chromatic number (or fractional or circular chromatic number) it uses another graph scale. Let $Q_{n/k}$ denote a graph with $\{0, 1\}^n$ as the set of vertices, where xy forms an edge iff $d(x, y) \geq k$ (here $d(x, y)$ is the Hamming distance of x and y). It is easy to see that G has n/k-cover if and only if it is homomorphic to $Q_{n/k}$. That means that an alternative definition is

$$x(G) = \inf\left\{\frac{n}{k} \mid G \xrightarrow{hom} Q_{n/k}\right\}. \tag{1}$$

An immediate corollary is that $x(G)$ is a homomorphism invariant, that is if $G \xrightarrow{hom} H$ then $x(G) \leq x(H)$. This will be strengthened in Lemma 1.1.

Let $H^{\geq k}$ denote the graph with vertices $V(H)$ and edges uv for any $u, v \in V(H)$ with distance in H at least k. Further let Q_n denote the n-dimensional cube. Then $Q_{n/k} = Q_n^{\geq k}$. This corresponds to the definition of circular chromatic number, where the target graph is $C_n^{\geq k}$. This observation inspires the term cubical chromatic number. However, as we will see later (in Corollary 2.3), a rescaling of $x(G)$ is in order to make it behave like a version of chromatic number, thus the definition of χ_q.

The original motivation for defining $x(G)$ was the study [21] of cut-continuous mappings (defined in [7]). Given graphs G, H we call a mapping $f : E(G) \to E(H)$ *cut-continuous*, if for every cut $U \subseteq E(H)$, the preimage $f^{-1}(U)$ is a cut in G. The following lemma is straightforward, but useful.

Lemma 1.1 *Let G, H be graphs. Then if there is a cut-continuous mapping from G to H (in particular, if there is a homomorphism $G \xrightarrow{hom} H$), then $x(G) \leq x(H)$ and (equivalently) $\chi_q(G) \leq \chi_q(H)$.*

Proof: It suffices to show that whenever H has an n/k-cover, G has it as well. So let f be some cut-continuous mapping from G to H, let $X_1, \ldots, X_n$ be an n/k-cover and consider X_i'—a preimage of the cut X_i under f. By definition, X_i' is also a cut. If e is an edge of G, $f(e)$ is an edge of H, hence it is covered by at least k of the cuts X_i. Thus e is covered by at least k of the cuts X_i'. For the homomorphism part, one may observe that the mapping induced on edges by a homomorphism is cut-continuous [7], or just use the alternative definition in Equation (1). □

Note that each graph $Q_{n/k}$ is a Cayley graph on some power of $\mathbb{Z}_2$. It follows [21] that for every graph G the existence of a homomorphism from G to $Q_{n/k}$ is equivalent to the existence of a cut-continuous mapping from G to $Q_{n/k}$. Consequently, we may as well use cut-continuous mapping to $Q_{n/k}$ in Equation (1). This also provides an indirect proof of Lemma 1.1.

It is a standard exercise to show, that $x(G)$ is is the solution of the following linear program ($\mathcal{C}$ denotes the family of all cuts in G)

$$\text{minimize} \sum_{X \in \mathcal{C}} w(X) \text{ subject to: for every edge } e, \sum_{X, e \in X \in \mathcal{C}} w(X) \geq 1. \qquad (2)$$

We conclude that we can replace $\inf$ by $\min$ in the definition of $x(G)$—the infimum is always attained. We can also consider the dual program

$$\text{maximize} \sum_{e \in E(G)} y(e) \text{ subject to: for every cut } X, \sum_{e, e \in X} y(e) \leq 1. \qquad (3)$$

This program is useful for computation of $x(G)$ for some G. (Färnqvist, Jonsson, and Thapper [12] used a modification of this program. There is an optimal solution y^* of the above program, that respects symmetries of G: if there is an automorphism of G that maps edge e to edge f, then $y^*(e) = y^*(f)$. This decreases the size of the linear program.) Moreover, in the final section we use this dual program to discuss yet another definition of $x(G)$ in terms of bipartite subgraph polytope.

There is another possibility to dualize the notion of fractional cut covering, namely *fractional cycle covering*. Bermond, Jackson and Jaeger [3] proved that every bridgeless graph has a cycle $7/4$-cover (i.e., a collection of 7 cycles, that cover every edge at least 4 times), and Fan [11] proved that it has a $10/6$-cover. An equivalent formulation of Berge-Fulkerson conjecture claims that every cubic bridgeless graph has a $6/4$-cover. On the other hand, Edmonds characterization of the matching polytope implies that every cubic bridgeless graph has a $3k/2k$-cover (for some k).

2 Basic properties

We let $\mathrm{MAXCUT}(G)$ denote the number of edges in the largest cut in G and write $b(G) = \mathrm{MAXCUT}(G)/|E(G)|$ for the *bipartite density* of G.

Lemma 2.1 *For any graph G we have $x(G) \geq 1/b(G)$. If G is edge-transitive, then equality holds.*

Proof: Suppose $x(G) = n/k$ and let $X_1, \ldots, X_n$ be an n/k-cover. Then $\sum_{i=1}^{n} |X_i| \leq n \cdot b(G)|E(G)|$, on the other hand this sum is at least $k \cdot |E(G)|$, as every edge is counted at least k times. This proves the first part of the lemma. To prove the second part, let $\mathcal{X} = \{X_1, \ldots, X_n\}$ be all cuts of the maximal size (i.e., $|X_i| = b(G)|E(G)|$). From the edge-transitivity follows that every edge is covered by the same number (say k) of elements of $\mathcal{X}$. Now $k \cdot |E(G)| = \sum_{i=1}^{n} |X_i| = n \cdot b(G)|E(G)|$, which finishes the proof. $\square$

Corollary 2.2 *We have the following values of x (and so of χ_q).*

$$x(K_{2n}) = x(K_{2n-1}) = 2 - 1/n \qquad \chi_q(K_{2n}) = \chi_q(K_{2n-1}) = 2n$$
$$x(C_{2k+1}) = 1 + 1/(2k) \qquad \chi_q(C_{2k+1}) = 2 + 2/(2k-1)$$
$$x(\mathrm{Pt}) = 5/4 \qquad \chi_q(\mathrm{Pt}) = 8/3$$

Corollary 2.3 *For any graph G,*

$$2 + 2/(g_o(G) - 2) \le \chi_q(G) \le 2\lceil\chi(G)/2\rceil .$$

Equivalently, $1 + \frac{1}{g_o(G)-1} \le x(G) \le 2 - 1/\lceil\chi(G)/2\rceil$.

In particular, $\chi_q(G) \ge 2$ *and* $x(G) \in [1, 2)$.

Proof: Let $l = g_o(G)$, i.e., C_l is the shortest odd cycle that is a subgraph of G. Put $n = \chi(G)$. Then there are homomorphisms $C_l \to G \to K_n$, so it remains to use Lemma 1.1 and Corollary 2.2. □

By combining Lemma 1.1 and Corollary 2.2 we get that there is no cut-continuous mapping from K_{n+2} to K_n. As there is obviously a cut-continuous mapping (indeed, even a homomorphism) in the other direction, we may say that the even cliques K_{2n} form a strictly ascending chain in the poset defined by cut-continuous mappings. This application was the original point in defining $x(G)$, the result is not as innocuous as it appears (for example, there *is* a cut-continuous mapping $K_4 \to K_3$).

Next, we will study how good are the bounds of Corollary 2.3. While they obviously are tight for G equal to a complete graph, resp. odd cycle, they can be arbitrarily far off, as documented by Corollary 2.5 and Theorem 2.7.

We begin by looking at $\chi_f(G)$—the fractional chromatic number of G. This may be defined by $\chi_f(G) = \inf\{n/k \mid G \xrightarrow{hom} K(n,k)\}$, where $K(n,k)$ is the *Kneser graph*. Its vertex set consists of all k-element subsets of $[n] = \{1, 2, \dots, n\}$, two vertices are connected iff they are disjoint subsets of $[n]$.

Lemma 2.4 *Let k, n be integers such that $0 < 2k \le n$. Then*

1. $b(K(n,k)) \ge 2k/n$.
2. $x(K(n,k)) \le n/(2k)$.

Consequently, for any graph G we have $x(G) \le \frac{1}{2}\chi_f(G)$.

Proof: For the first part we let $U = \{S \subseteq [n] \mid 1 \in S\}$ and observe that $\delta(U)$ contains $\binom{n-1}{k-1}\binom{n-k}{k}$ edges. As Kneser graphs are edge-transitive, the second part follows by Lemma 2.1. The rest follows by Lemma 1.1 and the definition of fractional chromatic number. Note that the bound is only useful if $k > n/4$. □

Corollary 2.5 *For every $\varepsilon > 0$ and every integer b there is a graph G such that*

$$\chi_q(G) < 2 + \varepsilon \quad \textit{and} \quad \chi(G) > b\,.$$

Proof: Let $G = K(n, k)$, for $n = 2k + t$, $k = 2^t$ and t large enough. Then by Corollary 2.4 we have $x(G) \le n/2k = 1 + t/2^{t+1}$, thus (for t large enough) $\chi_q(G) \le 2 + \varepsilon$. By [17] we have $\chi(G) = n - 2k + 2 = t + 2$. □

By Corollary 2.3, we can view Corollary 2.5 as a strengthening of the well-known fact that there are graphs with no short odd cycle and with a large chromatic number. It also shows that the converse of Lemma 1.1 is far from being true: just take G from the Corollary 2.5 and let $H = K_{b/2}$. Then $x(G)$ is close to 1 and $x(H)$ close to 2 (that is as far apart as these values can be), still by an application of results of [7] there is no cut-continuous mapping from G to H.

It is interesting to find how various graph properties affect $x(G)$. We saw already, that small $\chi(G)$ makes $x(G)$ small, while large $\chi(G)$ does not force it to be large. Also small $g_o(G)$ makes $x(G)$ large. In this context we ask:

Question 2.6 *Let G be a cubic graph with no cycle of length $\le c$. How large can $x(G)$ (resp. $\chi_q(G)$) be?*

For $c = 3$, it follows from Brook's theorem that $x(G) \le x(K_3) = 3/2$ ($\chi_q(G) \le 4$). For $c = 17$, it is known [6] that G has a cut-continuous mapping to C_5, hence $x(G) \le x(C_5) = 5/4$ ($\chi_q(G) \le 8/3$). On the other hand, there is $\varepsilon > 0$ such that cubic graphs of arbitrary high girth exist such that $b(G) < 1 - \varepsilon$ (by a result of McKay), hence with $x(G) > 1 + \varepsilon$ ($\chi_q(G) > 2 + 2\varepsilon$).

We finish by a result explaining why we in Question 2.6 restrict to cubic graphs. Note that much sharper results on MAXCUT of graphs without short cycles were conjectured in [10] and (some of them) proved in [1, 2].

Theorem 2.7 *For any integers k, l there is a graph G such that $\chi_q(G) > k$ and G contains no circuit of length at most l.*

Proof: We modify the famous Erdős' proof of existence of high-girth graphs of high chromatic number, we only need to ensure high cubical chromatic number. Let $p = n^{\alpha-1}$ (where $\alpha \in (0, 1/l)$) and consider the random graph $G(n, p)$.

The expected number of circuits of length at most l is $O((pn)^l) = o(n)$, therefore by Markov inequality with probability at least 1/2 the graph $G(n, p)$ contains at most n circuits of length at most l. We delete one edge from each of them and let G' be the resulting graph. We use modification of Lemma 3.1 for $\delta = n^{-\alpha/3}$. By Claim 1 from the proof of Lemma 3.1, the number of edges of $G(n, p)$ is a.a.s. $\Omega(n^{1+\alpha})$, hence the deletion of n edges creates only a $(1-o(1))$ factor in the estimate for the number of edges, and thus (as the size of MAXCUT cannot increase by deleting edges) we only get a $(1+o(1))$ factor in the estimate for $b(G(n,p))$. An application of Lemma 2.1 and a choice of sufficiently large n finishes the proof. $\square$

The next lemma shows that χ_q and x enjoy some of the properties of other chromatic numbers. ($G_1 \square G_2$ denotes the cartesian product of graphs, $G_1 \times G_2$ the categorical one.)

Lemma 2.8 *1.* $x(G) = \max\{x(G') \mid G'$ *is a component of* $G\}$

2. $x(G) = \max\{x(G') \mid G'$ *is a 2-connected block of* $G\}$ *for a connected graph* G*.*

3. $x(G_1 \square G_2) = \max\{x(G_1), x(G_2)\}$

4. $x(G_1 \times G_2) \le \min\{x(G_1), x(G_2)\}$
The same formulas are true for χ_q *in place of* x*.*

Proof: We will prove that if G', G'' are graphs that share at most one vertex, then $x(G' \cup G'') = \max\{x(G'), x(G'')\}$. Clearly, this proves 1 and 2. Let $X'_1, \ldots, X'_n$ be an optimal cover of G', $X''_1, \ldots, X''_m$ an optimal cover of G'', thus $x(G') = n/k$, and $x(G'') = m/l$. Consider the collection of mn cuts $\{X'_i \cup X''_j\}$ (these are cuts, indeed, as G' and G'' share at most one vertex). An edge of G' is covered at least mk times, an edge of G'' at least nl times. Hence $x(G) \le \frac{mn}{\min\{mk,nl\}} = \max\{\frac{n}{k}, \frac{m}{l}\} = \max\{x(G'), x(G'')\}$. On the other hand, both G' and G'' are subgraphs of G, hence by Lemma 1.1 the other inequality follows.

Part 3 follows from Lemma 1.1, as between $G_1 \square G_2$ and $G_1 \cup G_2$ exists a cut-continuous mapping in both directions.

Part 4 follows from Lemma 1.1 as there are homomorphisms (and therefore TT mappings) $G_1, G_2 \to G_1 \times G_2$

As $\chi_q = 2/(2-x)$ (which is an increasing function for the values that x can attain), the results for χ_q follow immediately. $\square$

3 Cubical chromatic number of random graphs

In this section we briefly consider the value of cubical chromatic number of random graphs.

Lemma 3.1 *Let* p, δ *be functions of* n *such that* $p, \delta \in [0,1]$ *and* $\delta^2 p \geq 7 \log n/n$. *Then* $b(G(n,p)) \leq \frac{1}{2}(1 + O(1/n) + O(\delta))$ *a.a.s. In particular we have*

$$b(G(n,p)) \leq \frac{1}{2} + O\left(\sqrt{\frac{\log n}{pn}}\right) \qquad \textit{a.a.s.}$$

Proof: We will prove that almost all graphs have "many edges but no huge cut".

Claim 1. $|E(G(n,p))| > (1-\delta)p\binom{n}{2}$ a.a.s.

To prove this we use Chernoff inequality (as stated in Corollary 2.3 of [15]) for random variable $X = |E(G(n,p))|$. It claims

$$\Pr[X \leq \mathbb{E}X - \delta\mathbb{E}X] \leq 2e^{-\frac{\delta^2}{3}\mathbb{E}X}$$

And as $\mathbb{E}X = p\binom{n}{2}$, Claim 1 follows.

Claim 2. $\mathrm{MAXCUT}(G(n,p)) < (1+\delta)p\frac{n^2}{4}$ a.a.s.

For a set $A \subseteq V(G(n,p))$ we let X_A be the random variable that counts the edges leaving A, and put $a = |A| \leq n/2$. By another variant of Chernoff inequality

$$\Pr[X_A \geq \mathbb{E}X_A + \delta\mathbb{E}X_A] \leq 2e^{-\frac{\delta^2}{3}\mathbb{E}X_A}$$

and substituting $\mathbb{E}X = pa(n-a)$ we get

$$\Pr[X_A \geq (1+\delta)pn^2/4] \leq 2e^{-\frac{\delta^2}{3}pa(n-a)} \leq 2e^{-\frac{\delta^2 pan}{6}}\,.$$

It remains to estimate the total probability of a large cut:

$$\begin{aligned}\Pr[(\exists A)X_A \geq (1+\delta)pn^2/4] &\leq \sum_{a=1}^{n/2}\binom{n}{a}2e^{-\frac{\delta^2 pan}{6}} \\ &\leq 2[(1+e^{-\frac{\delta^2 pn}{6}})^n - 1]\,.\end{aligned}$$

For $\delta^2 p \geq 7\log n/n$ the last expression tends to zero, which finishes the proof of Claim 2. The rest of the proof of the lemma is a simple calculation. □

Theorem 3.2

$$\Omega\left(\sqrt{n/\log n}\right) \leq \chi_q(G(n,1/2)) \leq O\left(n/\log n\right) \qquad \textit{a.a.s.}$$

Proof: The lower bound follows by Lemma 3.1, the upper one by an application of Corollary 2.3 and the well-known fact that $\chi(G(n, 1/2)) = O(n/\log n)$. $\square$

Note that we could have used the known result on clique number of a random graph for a direct proof of the lower bound in Theorem 3.2, but this way we would have obtained only $\chi_q(G(n, 1/2)) \geq \Omega(\log n)$. By using known results on $b(G_{n,1/2})$ [5] we could have improved the lower bound to $\Omega(\sqrt{n})$. We decided to present an easy self-contained argument instead. In a subsequent paper, the asymptotics of $\chi_q(G_{n,1/2})$ will be shown to be $\Theta(\sqrt{n})$ by using semidefinite approximation.

4 Measuring the scale

In this section we will discuss the 'invariance property' of cubical chromatic number. In analogy with $\chi(K_n) = n$, $\chi_c(C_n^{\geq k}) = n/k$, $\chi_f(K(n, k)) = n/k$, and 'dimension of product of n complete graphs is n' we would like to prove that $x(Q_{n/k}) = n/k$. The following lemma shows, that the situation is not that simple for x.

Lemma 4.1 *Let $1 \leq k \leq n$ be integers. Then we have $x(Q_{n/k}) \leq \frac{n}{k}$. If k is odd, then $x(Q_{n/k}) \leq \frac{n+1}{k+1}$.*

Proof: For the first part, it suffices to consider the identical homomorphism $Q_{n/k} \xrightarrow{hom} Q_{n/k}$. For the second part, mapping $V(Q_{n/k}) \to V(Q_{(n+1)/(k+1)})$ given by $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, x_1 + \cdots + x_n \bmod 2)$ is a homomorphism whenever k is odd. $\square$

Another complication is that by Corollary 2.3 we have $x(G) < 2$ for any graph G. However, with this exception, the bounds in Lemma 4.1 are optimal:

Theorem 4.2 *Let k, n be integers such that $k \leq n < 2k$. Then $x(Q_{n/k}) = \frac{n}{k}$ if k is even and $x(Q_{n/k}) = \frac{n+1}{k+1}$ if k is odd.*

This theorem was announced as a conjecture in the author's thesis [21], together with a beginning of a possible proof. The proof was finished by Engström, Färnqvist, Jonsson, and Thapper [9, Lemma 4.4], who did prove the inequality in Lemma 4.6.

We'll use the following results (Lemma 13.7.4 and 13.1.2 of [13]).

Lemma 4.3 *Let G be a graph with n vertices and m edges, let λ_n be the largest eigenvalue of the Laplacian of G. Then $b(G) \leq \frac{n}{m} \frac{\lambda_n}{4}$.*

Lemma 4.4 *Let G be an r-regular graph with n vertices, let eigenvalues of G be $\Theta_1 \geq \Theta_2 \geq \cdots \geq \Theta_n$. Then the eigenvalues of the Laplacian of G are given by $\lambda_i = r - \Theta_i$.*

We will also use an expression for spectra of graphs with transitive automorphism group ([16], see also Problem 11.8 in [18]).

Lemma 4.5 *Let G be a graph whose automorphism group contains a commutative subgroup Γ. Suppose Γ is regular, that is for each pair $x, y \in V(G)$ there is exactly one element $\gamma_{x,y} \in \Gamma$ that moves x to y. Let χ be a character of Γ and u any vertex of V. Then*

$$\sum_{v;uv \in E(G)} \chi(\gamma_{u,v})$$

is an eigenvalue of G; moreover all eigenvalues are of this form.

The following lemma was proved (using a clever induction) by Engström, Färnqvist, Jonsson, and Thapper [9, Lemma 4.4], resolving thus a question from [21].

Lemma 4.6 *Let k, n be integers such that $k \leq n < 2k$ and k is even, let x be an integer such that $1 \leq x \leq n$. Then*

$$\sum_{\text{odd } t} \binom{x}{t}\binom{n-x}{k-t} \leq \binom{n-1}{k-1}.$$

Proof: (of Theorem 4.2) Lemma 4.1 provides the upper bound, we will establish the lower bound now. Suppose first that k is even. We shall use a spanning subgraph of $Q_{n/k} = Q_n^{\geq k}$, that contains only edges of length precisely k; we shall use $Q_n^{=k}$ to denote this subgraph.

By Lemma 1.1 and 2.1 we have that $x(Q_{n/k}) \geq x(Q_n^{=k}) = 1/b(Q_n^{=k})$. By Lemma 4.3 and Lemma 4.4 it is enough to determine the smallest eigenvalue Θ of $Q_n^{=k}$. As $Q_n^{=k}$ is $\binom{n}{k}$-regular, we have

$$\frac{1}{b(Q_n^{=k})} \geq \frac{|E(Q_n^{=k})|}{|V(Q_n^{=k})|} \frac{4}{\binom{n}{k} - \Theta} = \frac{2\binom{n}{k}}{\binom{n}{k} - \Theta}.$$

Now we use Lemma 4.5 to find that eigenvalues of $Q_n^{=k}$. We suppose $V(Q_n^k) = \mathbb{Z}_2^n$ and take $\Gamma \simeq \mathbb{Z}_2^n$; therefore $\gamma_{u,v}$ corresponds to $u + v$ (operation modulo 2 in each coordinate) and the characters are $\chi_y : v \mapsto (-1)^{\sum_{i=1}^n v_i y_i}$ for each $y \in \mathbb{Z}_2^n$.

Now put $u = \vec{0}$ and suppose that the weight of y is x (that is $y_i = 1$ for exactly x values of i). By Lemma 4.5 the eigenvalue corresponding to y equals

$$\sum_{v \text{ of weight } k} \chi_y(v) = \sum_{t=0}^{k} (-1)^t \binom{x}{t} \binom{n-x}{k-t},$$

here t is the number of bits that y and v have in common. (This is the definition of Krawtchouk polynomial $K_k^n(x)$ – which doesn't seem to be helpful, though.) By using Vandermonde's identity and Lemma 4.6, we get that the above sum is $\geq \binom{n}{k}(1 - 2k/n)$, which is equal to the sum for $x = 1$. Thus the smallest eigenvalue Θ equals $\binom{n}{k}(1 - 2k/n)$, and we obtain $x(Q_{n/k}) \geq n/k$ as desired.

For odd values of k we cannot use the same method, as then $Q_n^{=k}$ is bipartite, hence $b(Q_n^{=k}) = 1$. However, observe that $Q_{(n+1)/(k+1)} \xrightarrow{hom} Q_{n/k}$, hence by Lemma 1.1 and the result for (even) $k+1$ we have

$$x(Q_{n/k}) \geq x(Q_{(n+1)/(k+1)}) \geq (n+1)/(k+1)\,.$$

□

5 Concluding Remarks

Bipartite subgraph polytope For a bipartite subgraph $B \subseteq G$, let c_B be the characteristic vector of $E(B)$. Bipartite subgraph polytope $P_B(G)$ is the convex hull of points c_B, for all bipartite graphs $B \subseteq G$. The study of this polytope was motivated by the max-cut problem: to look for a weighted maximum cut of G simply means to solve a linear program over $P_B(G)$. Thus, for graphs where $P_B(G)$ has simple description, we can have polynomial-time algorithm for max-cut; this in particular happens for weakly bipartite graphs (which include planar graphs), see [14]. We apply P_B to yield yet another definition of x.

Theorem 5.1 $x(G) = \max\{\sum_{e \in E(G)} y_e \mid y \cdot c \leq 1$ *defines a facet of* $P_B(G)\}$

Proof: By LP duality $x(G)$ is a solution to the program (3). This means, that we are maximizing over such y, that for each cut X satisfy $y \cdot c_X \leq 1$. As the convex hull of vectors c_X is P_B, we are maximizing the sum of coordinates of an element of the dual polytope P_B^*. This maximum is attained for some vertex of P_B^*, that is for y such that $y \cdot c \leq 1$ defines a facet of P_B. □

'Natural' facets of $P_B(G)$ are defined by $\sum_{e \in E(H)} y_e \leq \mathrm{MAXCUT}(H)$ for some $H \subseteq G$. (This inequality is satisfied for every graph H, but it doesn't always define a face of maximal dimension.) This proves the following observation (we add a direct proof, too).

Lemma 5.2 $x(G) \geq 1/(\min_{H \subseteq G} b(H))$

Proof: Let $H \subseteq G$. Then $H \xrightarrow{TT_2} G$, which by Lemma 1.1 and 2.1 implies $1/b(H) \leq x(G)$. □

Let us return to Lemma 2.1 for a while. In general $x(G)$ and $1/b(G)$ can be as distant as possible: Let G be a disjoint union of a K_n and $K_{N,N}$. Now $x(G)$ is close to 2 (because G is homomorphically equivalent to K_n, hence $x(G) = x(K_n)$) and $b(G)$ is close to 1 (provided N is sufficiently large). This motivates Lemma 5.2, which improves the original bound. A natural question is, whether this improvement gives the correct size of x. It turns out it does not (contrary to a conjecture in the author's thesis). In [9] it is shown, that the circular clique $K_{11/4}$ is a counterexample.

A failed attempt The proof of Theorem 4.2 could be attempted by another way: First, observe that the Kneser graph $K(n,r)$ is a subgraph of $Q_{n/2r}$. By Lemma 1.1 and 2.1 we have $x(Q_{n/2r}) \geq x(K(n,r)) \geq \frac{1}{b(K(n,r))}$. Thus, if we knew the value of $b(K(n,r))$ (and it turned out to be $2r/n$ for the range of r we are interested in), we would be done.

In [19] it is claimed that if $2r \leq n \leq 3r$ then, indeed, $b(K(n,r)) = 2r/n$. This would imply the conjecture for even k less than $3/2 \cdot n$; unfortunately the proof in [19] is incomplete (as already observed by [4]).

Generalizations and future work As already mentioned in the introduction, the metric that is used in [12, 9] to study approximability of MAX-H-COLORING can be computed from a generalization of fractional covering by cuts. One only needs to consider by more general edge sets, namely those of graphs that are homomorphic to H. Then the cube $Q_{n/k}$ in Equation (1) is replaced by appropriately defined power of H. One may also use this motivation to define H-continuous mappings as follows. We call a subset $X \subseteq E(G)$ an H-cut in G whenever there is a mapping $g : V(G) \to V(H)$ for which $g^{-1}(E(H)) = X$. We say a mapping $f : E(G_1) \to E(G_2)$ is *H-continuous* whenever a preimage of each H-cut is an H-cut. This definition deserves more attention.

The topic of this paper is being studied further by the author (and coauthors). A paper about approximating cubical chromatic number will follow shortly. It turns out, that $\chi_q(G)$ can be approximated within fraction 0.878567 using semidefinite programming.

This, possibly with computing more values of χ_q explicitly, can make Lemma 1.1 a useful 'no-homomorphism lemma' – a tool to prove there is no homomorphism between a given pair of graphs.

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