# Forbidden Graphs for Tree-depth

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**Abstract**

For every $k \geq 0$, we define $\mathcal{G}_k$ as the class of graphs with tree-depth at most $k$, i.e. the class containing every graph $G$ admitting a valid colouring $\rho : V(G) \to \{1, \dots, k\}$ such that every $(x, y)$-path between two vertices where $\rho(x) = \rho(y)$ contains a vertex $z$ where $\rho(z) > \rho(x)$. In this paper we study the set of graphs not belonging in $\mathcal{G}_k$ that are minimal with respect to the minor/subgraph/induced subgraph relation (obstructions of $\mathcal{G}_k$). We determine these sets for $k \leq 3$ for each relation and prove a structural lemma for creating obstructions from simpler ones. As a consequence, we obtain a precise characterization of all acyclic obstructions of $\mathcal{G}_k$ and we prove that there are exactly $\frac{1}{2}2^{2^{k-1}-k}(1 + 2^{2^{k-1}-k})$. Finally, we prove that each obstruction of $\mathcal{G}_k$ has at most $2^{2^{k-1}}$ vertices.

**Keywords:** tree-depth, obstructions, graph enumeration, vertex ranking.

## 1 Introduction

The graph parameter of tree-depth (also known as the vertex ranking problem [1], or the ordered colouring problem [4]) has received much attention, mostly because of the theory of graph classes of bounded expansion, developed by Nešetřil and Ossona de Mendez in [7, 10, 8, 9, 6]. Furthermore, the tree-depth of a graph is equivalent to the minimum-height of an elimination tree of a graph [2, 3, 7] (this measure is of importance for the parallel Cholesky factorization of matrices [5]).

The *tree-depth* of a graph $G$ is defined as the minimum $k$ for which there is a valid colouring $\rho : V(G) \to \{1,\ldots,k\}$ such that every $(x,y)$-path between two vertices where $\rho(x)=\rho(y)$ contains a vertex $z$ where $\rho(z)>\rho(x)$. Given a non-negative integer $k$, we define $\mathcal{G}_k$ as the class of all graphs with tree-depth at most $k$.

We say that a graph $H$ is a *minor* of a graph $G$ if $H$ can be obtained from a subgraph of $G$ by applying edge contractions. We use the notation $\mathbf{obs}_{\leq}(\mathcal{G}_k)$ for the set of minor-minimal graphs not in $\mathcal{G}_k$. If instead of the minor relation, we consider the subgraph or the induced subgraph relation we define the sets $\mathbf{obs}_{\subseteq}(\mathcal{G}_k)$ and $\mathbf{obs}_{\sqsubseteq}(\mathcal{G}_k)$ respectively.

In this paper we examine the sets $\mathbf{obs}_{\leq}(\mathcal{G}_k)$, $\mathbf{obs}_{\subseteq}(\mathcal{G}_k)$, and $\mathbf{obs}_{\sqsubseteq}(G_k)$. From the Robertson and Seymour theorem [12] it follows that $\mathbf{obs}_{\leq}(\mathcal{G}_k)$ is finite for each $k \geq 0$. The finiteness of $\mathbf{obs}_{\subseteq}(\mathcal{G}_k)$ follows from [7]. Also it is easy to verify that $\mathbf{obs}_{\sqsubseteq}(\mathcal{G}_k)$ is finite (see Observation 4).

Our first result is an upper bound of $2^{2^{k-1}}$ to the order of the graphs in $\mathbf{obs}_{\sqsubseteq}(\mathcal{G}_k)$ for $k \geq 0$. This bound also holds for $\mathbf{obs}_{\subseteq}(\mathcal{G}_k)$ and $\mathbf{obs}_{\leq}(\mathcal{G}_k)$ as $\mathbf{obs}_{\leq}(\mathcal{G}_k) \subseteq \mathbf{obs}_{\subseteq}(\mathcal{G}_k) \subseteq \mathbf{obs}_{\sqsubseteq}(\mathcal{G}_k)$ (Observation 3). Our next result is a structural lemma that constructs new obstructions from simpler ones. This permits us to identify, for each $k \geq 0$, all acyclic obstructions and prove that are exactly $\frac{1}{2}2^{2^{k-1}-k}(1+2^{2^{k-1}-k})$ for all relations. So far, such a parameterized set of acyclic obstructions is known only for classes of bounded pathwidth [14] and variations of it such us search number [11], proper-pathwidth [14], linear-width [15] (see [13] for similar results on graphs with bounded feedback vertex set number). However, this is the first time where an exact enumeration of parameterized obstructions is derived. Our final result is the identification of the sets $\mathbf{obs}_{\leq}(\mathcal{G}_k)$, $\mathbf{obs}_{\subseteq}(\mathcal{G}_k)$, and $\mathbf{obs}_{\sqsubseteq}(G_k)$ for $k \leq 3$. For $k=3$ these sets have 12, 14, and 29 graphs respectively.

## 2 Preliminaries

In this paper we consider simple graphs without loops and parallel edges. We denote by $P_n$ the path that has $n$ vertices and length $n-1$ and by $\mathcal{C}(G)$ the connected components of a graph $G$. We say that two graphs $G_1, G_2$ are *hom-equivalent* if $G_1$ is homomorphic to $G_2$ and $G_2$ is homomorphic to $G_1$.

Moreover, an automorphism $f$ of a graph is called *involution* if and only if $f \circ f = \mathbf{id}$.

For a graph $H$ we say that it is

- an *induced subgraph* of a graph $G$, denoted $H \sqsubseteq G$, if it can be obtained from $G$ by applying vertex deletions.
- a *subgraph* of a graph $G$, denoted $H \subseteq G$, if it can be obtained from $G$ by applying edge and vertex deletions.
- a *minor* of a graph $G$, denoted $H \leq G$, if it can be obtained from $G$ by applying edge and vertex deletions and edge contractions, where to contract an edge $e = \{x, y\}$ of a graph $G$ is to remove it and then replace its ends by a single vertex incident to all the edges which were incident to either $x$ or $y$ without allowing parallel edges.

A graph $G$ admits a *$k$-vertex ranking* if there exists a valid colouring $\rho : V(G) \to \{1, \ldots, k\}$ such that every $(x, y)$-path between two vertices where $\rho(x) = \rho(y)$ contains a vertex $z$ where $\rho(z) > \rho(x)$. The *tree-depth* of a graph $G$, $\mathbf{td}(G)$, is defined as the minimum $k$ such that $G$ admits a $k$-vertex ranking. [7] Moreover, we give the following (equivalent) definition for the tree-depth of a connected graph $G$.

$$\mathbf{td}(G) = \begin{cases} 1 & \text{if } |V(G)| = 1 \\ 1 + \min_{v \in V(G)} \mathbf{td}(G \setminus v) & \text{if } |V(G)| > 1 \end{cases}$$

If follows from that for any non-negative integer $n$, $\mathbf{td}(P_n) = \lceil \log_2(n+1) \rceil$ (see [7]). For every non-negative integer $k$ we denote by $\mathcal{G}_k$ the class of graphs with tree-depth at most $k$, i.e. $\mathcal{G}_k = \{G \mid \mathbf{td}(G) \leq k\}$. It is known from [1, 7] that if $H$ is a minor of $G$, then $\mathbf{td}(H) \leq \mathbf{td}(G)$. A direct consequence is that for any non-negative integer $k$, $\mathcal{G}_k$ is minor-closed. For every $\mathsf{R} \in \{\sqsubseteq, \subseteq, \leq\}$, we denote by $\mathbf{obs}_{\mathsf{R}}(\mathcal{G}_k)$ the set of the graphs with tree-depth strictly bigger than $k$ that are minimal with respect to the relation $\mathsf{R}$.

**Lemma 1** ([7])**.** *Let $k \geq 1$ be an integer. Then, the class $\mathcal{G}_k$ includes a finite subset $\hat{\mathcal{G}}_k$ such that, for every graph $G \in \mathcal{G}_k$, there exists $\hat{G} \in \hat{\mathcal{G}}_k$ which is hom-equivalent to $G$ and isomorphic to an induced subgraph of $G$.*

By Lemma 1 a tower function bound can be derived for the order of the forbidden subgraphs. However, as we prove in the next section, a direct argument shows a much better bound.

## 3 Upper bound on the order of obstructions for $\mathcal{G}_k$

**Observation 1.** *For any graph $G$, $\mathbf{td}(G) = \max\{\mathbf{td}(C) \mid C \in \mathcal{C}(G)\}$.*

**Observation 2.** *For every $k \geq 0$, all graphs in $\mathbf{obs}_{\sqsubseteq}(\mathcal{G}_k), \mathbf{obs}_{\subseteq}(\mathcal{G}_k)$ and $\mathbf{obs}_{\leq}(\mathcal{G}_k)$ are connected.*

*Proof.* Follows directly from Observation 1. □

**Observation 3.** *For every non-negative integer $k$, $\mathbf{obs}_{\leq}(\mathcal{G}_k) \subseteq \mathbf{obs}_{\subseteq}(\mathcal{G}_k) \subseteq \mathbf{obs}_{\sqsubseteq}(\mathcal{G}_k)$.*

**Observation 4.** *Let $G$ be a graph such that $G \in \mathbf{obs}_{\sqsubseteq}(\mathcal{G}_k)$, for some integer $k$. Then there exists $G' \in \mathbf{obs}_{\subseteq}(\mathcal{G}_k)$ such that $V(G) = V(G')$ and $E(G') \subseteq E(G)$.*

*Proof.* Let $G$ be a counterexample of minimal size. Then there exists an edge $e$ such that $G' = G \setminus e$ also belongs to $\mathbf{obs}_{\sqsubseteq}(\mathcal{G}_k)$ and $V(G) = V(G')$ and $E(G') \subseteq E(G)$. □

**Theorem 1.** *For any integer $k > 0$, if $G$ is a graph with $\mathbf{td}(G) > k$, then $G$ contains a connected subgraph $H$ with $\mathbf{td}(H) > k$ and $|V(H)| \leq 2^{2^{k-1}}$.*

*Proof.* We may assume that $G$ is connected, otherwise from Observation 1 we focus on the component of $G$ that determines its tree-depth. Also, without loss of generality, $\mathbf{td}(G) = k + 1$. We prove the statement by induction:

If $\mathbf{td}(G) = 2$, then $G$ contains at least one edge, and we may set $H = K_2$. If $\mathbf{td}(G) = 3$, then $G$ is not a star forest, i.e., it contains $P_4$ or $K_3$ as a subgraph.

Suppose now that $\mathbf{td}(G) = k+1$ for $k \geq 3$, and assume that the statement holds for all smaller values of tree-depth. If $G$ contains $P_{2^k}$ as a subgraph, then we may set $H = P_{2^k}$. Otherwise, each two vertices in $G$ are connected by a path of length at most $2^k - 2$.

Since $\mathbf{td}(G) > k-1$, by induction hypothesis $G$ contains a subgraph $H_0$ with $\mathbf{td}(H_0) \geq k$ and $m \leq 2^{2^{k-2}}$ vertices $v_1, \ldots, v_m$. For each $i = 1, \ldots, m$, the graph $G \setminus v_i$ has tree-depth greater than $k-1$, hence $G \setminus v_i$ contains a subgraph $H_i$ with at most $2^{2^{k-2}}$ vertices and tree-depth at least $k$.

If there exists $i$ such that $V(H_0) \cap V(H_i) = \emptyset$, then we let $H$ consist of $H_0$, $H_i$ and the shortest path that connects them. For every vertex $v$ of $H$, the graph $H \setminus v$ contains $H_0$ or $H_i$ as a subgraph, hence the tree-depth of $H \setminus v$ is at least $k$ and $\mathbf{td}(H) > k$. Also, $|V(H)| \leq 2^{2^{k-2}+1} + 2^k - 3 \leq 2^{2^{k-1}}$ (for $k \geq 3$).

On the other hand, if all the graphs $H_i$ intersect $H_0$, then we set $H = H_0 \cup H_1 \cup \ldots \cup H_m$. Since all the graphs $H_i$ are connected, the graph $H$ is connected as well, and it has at most $m + m(2^{2^{k-2}} - 1) \leq 2^{2^{k-1}}$ vertices. Similarly to the previous case, the graphs $H \setminus v_i$ contain $H_i$ as a subgraph (for $i = 1, \ldots, m$), and the graph $H \setminus v$ for $v$ different from $v_1, \ldots, v_m$ contains $H_0$ as a subgraph, hence $\mathbf{td}(H) > k$. □

From Theorem 1 and Observations 3 and 4 we obtain the following corollary,

**Corollary 1.** *All graphs in* $\mathbf{obs}_{\sqsubseteq}(\mathcal{G}_k)$ *(and therefore, also in* $\mathbf{obs}_{\subseteq}(\mathcal{G}_k)$ *and* $\mathbf{obs}_{\leq}(\mathcal{G}_k)$*) have at most* $2^{2^{k-1}}$ *vertices.*

# 4 A structural Lemma for the obstructions of tree-depth

In this section we prove a lemma for tree-depth that permits us to build obstructions from simpler ones. We first consider the following observations.

**Observation 5.** *Let* $G$ *be a connected graph such that* $\mathbf{td}(G) = k$ *and* $\rho : V(G) \to [k]$ *a* $k$*-vertex ranking of* $G$*. Then* $|\rho^{-1}(k)| = 1$.

*Proof.* If $v_1$ and $v_2$ are two (non-adjacent) vertices in $\rho^{-1}(k)$, then there exists a path with end-vertices $v_1, v_2$. Observe that all internal vertices of this path have colour smaller than $k$, a contradiction. □

**Observation 6.** *If* $G \in \mathbf{obs}_{\sqsubseteq}(\mathcal{G}_k)$ *(or* $\mathbf{obs}_{\subseteq}(\mathcal{G}_k)$ *or* $\mathbf{obs}_{\leq}(\mathcal{G}_k)$*) then for every* $v \in V(G)$ *there exists a* $(k+1)$*-vertex ranking* $\rho$ *such that* $\rho(v) = k+1$*.*

*Proof.* As $G \in \mathbf{obs}_{\sqsubseteq}(\mathcal{G}_k)$ (or $\mathbf{obs}_{\subseteq}(\mathcal{G}_k)$ or $\mathbf{obs}_{\leq}(\mathcal{G}_k)$), $G \setminus v$ admits a $k$-vertex ranking $\rho$. Then $\rho \cup (v, k+1)$ is the required $(k+1)$-vertex ranking of $G$. □

Let $G_1$ and $G_2$ be two disjoint graphs and let $v_i \in V(G_i)$, for $i = 1, 2$. We define $\mathbf{j}(G_1, G_2, v_1, v_2) = (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2) \cup \{\{v_1, v_2\}\})$.

**Observation 7.** *Let* $G_1$ *and* $G_2$ *be disjoint graphs where* $\mathbf{td}(G_1) \leq k$ *and* $\mathbf{td}(G_2) \leq k$*. Let* $v_i \in V(G_i), i = 1, 2$*. Then the graph* $G = \mathbf{j}(G_1, G_2, v_1, v_2)$ *has tree-depth at most* $k+1$*.*

*Proof.* Let $\rho_i$ be a k-vertex ranking of $G_i, i = 1, 2$. Then $\rho = \rho_1 \cup \rho_2 \setminus \{(v_1, \rho_1(v_1))\} \cup \{(v_1, k+1)\}$ is a $(k+1)$-vertex ranking of $G$. □

**Observation 8.** *Let* $G_1$ *and* $G_2$ *be disjoint, connected graphs such that* $\mathbf{td}(G_1) \geq k$ *and* $\mathbf{td}(G_2) \geq k$*. Let* $v_i \in V(G_i), i = 1, 2$*. Then the graph* $G = \mathbf{j}(G_1, G_2, v_1, v_2)$ *has tree-depth at least* $k+1$*.*

*Proof.* Assume in contrary that there exists a $k$-vertex ranking $\rho : V(G) \to [k]$. Notice that $\rho^{-1}(k) \neq \emptyset$, otherwise $\mathbf{td}(G) < k$ contradicting the fact that $\mathbf{td}(G_1) \geq k$. Combining this fact with Observation 5, $G$ has a unique vertex $v$ where $\rho(v) = k$. W.l.o.g. we assume that $v \in V(G_1)$. Then the restriction of $\rho$ to $G_2$ gives a $(k-1)$-vertex ranking of it, a contradiction. □

**Lemma 2.** *Let* $k$ *be a positive integer and let* $\mathsf{R} \in \{\sqsubseteq, \subseteq, \leq\}$ *. Let* $G_1$ *and* $G_2$ *be disjoint graphs such that* $G_1, G_2 \in \mathbf{obs}_{\mathsf{R}}(\mathcal{G}_{k-1})$ *and let* $v_1 \in V(G_1), v_2 \in V(G_2)$*. Then* $\mathbf{j}(G_1, G_2, v_1, v_2) \in \mathbf{obs}_{\mathsf{R}}(\mathcal{G}_k)$*.*

*Proof.* Let $G_1$ and $G_2$ such that $G_1, G_2 \in \mathbf{obs}_{\mathsf{R}}(\mathcal{G}_{k-1})$ and let $v_i \in V(G_i), i = 1, 2$. We set $G = \mathbf{j}(G_1, G_2, v_1, v_2)$. We first prove that $\mathbf{td}(G) = k+1$. Indeed, Observation 7 yields $\mathbf{td}(G) \leq k+1$ and Observation 8 yields $\mathbf{td}(G) \geq k+1$.

We now have to prove that if $G'$ is the result of the removal or the contraction of some edge $e$ in $G$, then $\mathbf{td}(G') \leq k$ (this also covers the case of a vertex removal as, from Observation 1, $G$ is connected and thus the removal of a vertex implies the removal of at least one edge).

We examine first the case where $e = \{v_1, v_2\}$. If $G' = G \setminus e$, then from Observation 1, $\mathbf{td}(G) = \max\{\mathbf{td}(G_1), \mathbf{td}(G_2)\} \leq k$. If $G' = G \,/\, e$, then from Observation 6, there exists a $k$-vertex ranking $\rho_i$ of $G_i$ such that $\rho_i(v_i) = k, i = 1, 2$. Then if $v_{\text{new}}$ is the result of the contraction of $e$ we have that $\rho : V(G') \to [k]$ where

$$\rho(x) = \begin{cases} \rho_1(x) & \text{if } x \in V(G_1) \setminus \{v_1\} \\ \rho_2(x) & \text{if } x \in V(G_2) \setminus \{v_2\} \\ k & \text{if } x = v_{\text{new}} \end{cases}$$

is a $k$-vertex ranking of $G'$, therefore $\mathbf{td}(G') \leq k$.

Finally, we examine the case where $e$ is an edge of $G_1$ or $G_2$. Without loss of generality we assume that $e_1 \in E(G_1)$. Because $G_1 \in \mathbf{obs}_{\subseteq}(\mathcal{G}_{k-1})$, there exists a $(k-1)$-vertex ranking $\rho'_1$ of $G_1 \setminus e$ (and $G_1 \,/\, e$). By Observation 6, since $G_2 \in \mathbf{obs}_{\subseteq}(\mathcal{G}_{k-1})$, there exists a $k$-vertex ranking $\rho_2$ of $G_2$ such that $\rho_2(v_2) = k$. It is easy to see that $\rho'_1 \cup \rho_2$ is a $k$-vertex ranking of $G'$, thus $\mathbf{td}(G') \leq k$ and this completes the proof of the lemma. $\square$

# 5 Acyclic obstructions for tree-depth

For every integer $k \geq 0$, we recursively define the class $\mathcal{T}_k$ as follows. Let $\mathcal{T}_0 = \{K_1\}$ and for every $k \geq 1$ we set

$$\mathcal{T}_k = \{\mathbf{j}(G_1, G_2, v_1, v_2) \mid G_1, G_2 \in \mathcal{T}_{k-1}, v_i \in V(G_i), i = 1, 2\}$$

The above definition permits us to state Lemma 2 as follows.

**Observation 9.** *For every integer $k \geq 0$ and every $\mathsf{R} \in \{\sqsubseteq, \subseteq, \leq\}$, $\mathcal{T}_k \subseteq \mathbf{obs}_{\mathsf{R}}(\mathcal{G}_k)$.*

**Lemma 3.** *For any positive integer $k$, if $G \in \mathcal{T}_k$, then for any vertex $v \in V(G)$ there exists a leaf $u \neq v$ of $G$ such that the tree created from $G \setminus u$ by adding a leaf adjacent to $v$ also belongs to $\mathcal{T}_k$.*

*Proof.* Assume, that this holds for any tree in $\mathcal{T}_{k-1}$, $k \geq 2$. Let $G_1, G_2 \in \mathcal{T}_{k-1}$ and $v_i \in V(G_i), i = 1, 2$ such that $G = \mathbf{j}(G_1, G_2, v_1, v_2)$. Consider an arbitrary vertex $v \in V(G)$, and let us show that there exists a leaf $u$ of $G$ that we can move to $v$ while preserving membership in $\mathcal{T}_k$. Without loss of generality, we may assume that $v \in V(G_1)$. By the induction hypothesis, there exists a vertex $u' \in V(G_1)$ such that the tree created from $G_1 \setminus u'$ by adding a leaf adjacent to $v$ is also in $\mathcal{T}_{k-1}$. If $u' \neq v_1$ we may set $u = u'$. Otherwise, let $u''$ be the leaf of $G_2$ that can be moved to $v_2$. In this case, we can set $u = u''$: Moving the leaf $u''$ to $v$ has the same result as moving it to $v_2$, moving the leaf $u'$ to $v$, and replacing the edge $e$ by an edge between $u''$ and the vertex of $G_1$ that used to be adjacent to $u'$. □

In Lemma 2 we described a procedure that for any non-negative integer $k$ constructs graphs $G \in \mathbf{obs}_{\sqsubseteq}(\mathcal{G}_{k+1})$ from disjoint graphs $G_1, G_2 \in \mathbf{obs}_{\sqsubseteq}(\mathcal{G}_k)$ (adding an edge that connects a vertex $v_1$ of $G_1$ and a vertex $v_2$ of $G_2$). With the following Lemma we fully characterize and construct all the acyclic graphs in $\mathbf{obs}_{\sqsubseteq}(\mathcal{G}_{k+1})$ for every non-negative integer $k$.

**Lemma 4.** *Let $G$ be a tree in $\mathbf{obs}_{\sqsubseteq}(\mathcal{G}_k)$ for $k \geq 1$. Then there exists an edge $e \in E(G)$ such that if $\{G_1, G_2\} = \mathcal{C}(G \setminus \{e\})$ then $G_1, G_2 \in \mathbf{obs}_{\sqsubseteq}(\mathcal{G}_{k-1})$.*

*Proof.* We examine the non-trivial case where $k \geq 2$ assuming that the statement holds for all acyclic obstructions of smaller tree-depth. From Observation 7, we obtain that for each edge $e = \{v_1, v_2\} \in E(G)$, at least one of the connected components $G_1, G_2$ of $G \setminus e$ has tree-depth at least $k$. We claim that $G$ contains at least one edge $e = \{v_1, v_2\}$ such that both connected components of $G \setminus e$ have tree-depth $k$. Suppose that this is not correct. Then we can direct each edge $e = \{v_1, v_2\}$ of $E(G)$ such that its tail belongs to the connected component of $G \setminus e$ that has tree-depth $< k$. We denote this directed tree by $\tilde{T}$. As $k \geq 2$, $\tilde{T}$ contains internal vertices. Moreover, all edges of $\tilde{T}$ that are incident to a leaf are directed away from it. It follows that $\tilde{T}$ contains an internal vertex $v$ of out-degree 0. This means that each, say $G_i$, connected component of $G \setminus v$ has a $(k-1)$-vertex ranking $\rho_i$. Then $\rho = \{(v, k)\} \cup \bigcup_{i=1,\ldots,m} \rho_i$ is a $k$-vertex ranking of $G$, a contradiction and this completes the proof of the claim.

Let now $G_i$ be the connected component of $G \setminus e$ that contains $v_i$, $i = 1, 2$. If one, say $G_1$, is not in $\mathbf{obs}_{\sqsubseteq}(\mathcal{G}_{k-1})$ then it contains an induced subgraph $G_1'$ such that $G_1' \in \mathbf{obs}_{\sqsubseteq}(\mathcal{G}_{k-1})$. Additionally, there is a unique path $P$ in $G$

that connects $G_1'$ with $G_2$. Observe that since $G \in \mathbf{obs}_{\sqsubseteq}(\mathcal{G}_{k-1})$, $G$ is exactly the union of $G_1', G_2$ and $P$. We need to show that $P$ has no inner vertices. Suppose that this is not the case, and let $w$ be the inner vertex of $P$ adjacent to a vertex $v \in V(G_1)$. By the induction hypothesis, $G_1'$ and $G_2$ satisfy the conditions of Lemma 3, thus $G_1$ contains a leaf $u$ such that the graph obtained from $G_1$ by moving the leaf $u$ to $v$ belongs to $\mathbf{obs}_{\sqsubseteq}(\mathcal{G}_{k-1})$. This implies that we may remove the vertex $u$ from $G$ and consider $w$ to be its replacement. The created graph is a proper induced subgraph of $G$ and has tree-depth $k+1$, a contradiction. This completes the proof of the lemma. □

Observe now that the following is a direct consequence of Lemmata 2 and 4.

**Theorem 2.** *Let $k$ be a non-negative integer. Then $\mathcal{T}_k$ is the set of all acyclic graphs in $\mathbf{obs}_{\sqsubseteq}(\mathcal{G}_k)$.*

**Corollary 2.** *For every non-negative integer $k$, $\mathcal{T}_k$ is the set of all acyclic graphs in $\mathbf{obs}_{\sqsubseteq}(\mathcal{G}_k)$ (or in $\mathbf{obs}_{\leq}(\mathcal{G}_k)$).*

*Proof.* Follows directly from Observations 3 and 9. □

# 6 Lower bound on the number of obstructions for $\mathcal{G}_k$

In this section, we prove that $|\mathcal{T}_k| = \frac{1}{2}2^{2^{k-1}-k}(1+2^{2^{k-1}-k}), k \geq 1$. This gives a lower bound on $|\mathbf{obs}_{\leq}(\mathcal{G}_k)|, k \geq 2$. As we shall see later we can identify the elements of the sets $\mathbf{obs}_{\subseteq}(\mathcal{G}_i), \mathbf{obs}_{\leq}(\mathcal{G}_i), \mathbf{obs}_{\sqsubseteq}(\mathcal{G}_i)$ for $i = 0, 1, 2, 3$.

For a tree $G \in \mathcal{T}_k$ such that $G = \mathbf{j}(G_1, G_2, v_1, v_2)$, we call $v_1v_2$ the *middle edge* of $G$.

**Observation 10.** *If $k$ is a non-negative integer then every graph in $\mathcal{T}_k$ has exactly $2^k$ vertices. This implies that the middle edge of a graph $G \in \mathcal{T}_k$ is unique.*

Consider also the following.

**Observation 11.** *Let $T^1, T^2$ be two trees and $e^i = \{v_1^i, v_2^i\} \in E(T^i), i = 1, 2$. If $\phi$ is an isomorphism from $T^1$ to $T^2$ such that $\phi(v_i^1) = v_i^2, i = 1, 2$ and $T_i^j$ is the connected component of $T^j \setminus e^j$ that contains $v_i^j, i = 1, 2, j = 1, 2$, then $\phi_i = \{(x, y) \in \phi \mid x \in V(T_i^1)\}$ is an isomorphism from $T_i^1$ to $T_i^2, i = 1, 2$.*

We use notation $\mathbf{Aut}(G)$ for the automorphism group of a graph $G$. Observation 11 easily implies the following.

**Observation 12.** *Let $T$ be a tree and $e = \{v_1, v_2\} \in E(T)$. If $\phi \in \mathbf{Aut}(T)$ satisfies $\phi(v_i) = v_{3-i}, i = 1, 2$ and $T_i$ is the connected component of $T \setminus e$ that contains $v_i, i = 1, 2$, then $\phi' = \{(x, y) \in \phi \mid x \in V(T_1)\}$ is an isomorphism from $T_1$ to $T_2$.*

**Observation 13.** *Let $G_1, G_2$ be disjoint graphs such that $G_1, G_2 \in \mathcal{T}_k, k \geq 1$ and $v_i \in V(G_i), i = 1, 2$. If $\phi \in \mathbf{Aut}(G)$, where $G = \mathbf{j}(G_1, G_2, v_1, v_2)$, then $\phi(e) = e$.*

*Proof.* Follows directly from Observation 10. □

**Lemma 5.** *Let $G \in \mathcal{T}_k$ for $k \geq 1$, $e = \{v_1, v_2\} \in E(G)$ the middle edge and $\phi \in \mathbf{Aut}(G)$. If there exists $v \in V(G)$ such that $\phi(v) = v$, then $\phi(v_i) = v_i, i = 1, 2$.*

*Proof.* We examine the non-trivial case where $k \geq 2$. Suppose, in contrary, that $\phi(v_i) = v_{3-i}, i = 1, 2$. We denote by $G_1, G_2$ the connected components of $G \setminus e$ where, w.l.o.g, $v, v_1 \in V(G_1)$. By Observation 12, $\phi' = \{(v_1, v_2) \in \phi \mid v_1 \in V(G_1)\}$ is an isomorphism of $G_1$ to $G_2$, a contradiction since $\phi'(v) = \phi(v) = v$. □

We now proceed to the proof of the following.

**Lemma 6.** *Let $k$ be a non-negative integer. For any $G \in \mathcal{T}_k$ and $\phi \in \mathbf{Aut}(G)$, if there exists $v \in V(G)$ such that $\phi(v) = v$ then $\phi = \mathbf{id}$.*

*Proof.* We use induction on $k$. For $k = 0$ the claim is trivial. Assume now that the claim holds for $k = n \geq 0$. Let $k = n + 1$. We denote by $e = \{v_1, v_2\} \in E(G)$ the middle edge and by $G_1, G_2$ the connected components of $G \setminus e$, where $v_i \in V(G_i), i = 1, 2$. Since $\phi \in \mathbf{Aut}(G)$, by Lemma 5, it follows that $\phi(v_i) = v_i, i = 1, 2$. Hence $\phi$ is an isomorphism from $G \setminus e$ to $G \setminus e$. From Observation 11, $\phi_i = \{(v, u) \in \phi \mid v \in V(G_i)\} \in \mathbf{Aut}(G_i), i = 1, 2$. Observe that $\phi_i(v_i) = \phi(v_i) = v_i, i = 1, 2$. Since $G_i \in \mathcal{T}_n, i = 1, 2$, by the induction hypothesis, $\phi_i, i = 1, 2$ is the trivial automorphism of $G_i$. Therefore, $\phi = \mathbf{id}$. $\square$

Let $G$ be a graph and $v \in V(G)$. We denote by $\mathbf{tr}_G(v)$ the orbit of the automorphism group of $G$ that contains $v$, i.e. $\mathbf{tr}_G(v) = \{u \in V(G) \mid \exists \phi \in \mathbf{Aut}(G)$ such that $\phi(u) = v\}$.

**Lemma 7.** *Let $G_1, G_2$ be disjoint graphs such that $G_1, G_2 \in \mathcal{T}_k$, $v_2, v_2' \in V(G_2)$ such that $v_2 \in \mathbf{tr}_{G_2}(v_2')$ and $v_1 \in V(G_1)$. Then $G = \mathbf{j}(G_1, G_2, v_1, v_2)$ and $G' = \mathbf{j}(G_1, G_2, v_1, v_2')$ are isomorphic.*

*Proof.* Let $\mathbf{id} \in \mathbf{Aut}(G_1)$ and $\phi \in \mathbf{Aut}(G_2)$, such that $\phi(v_2) = v_2'$. Then $\mathbf{id} \cup \phi$ is an isomorphism from $G$ to $G'$. $\square$

**Lemma 8.** *Let $G_1, G_2$ be disjoint graphs such that $G_1, G_2 \in \mathcal{T}_k$, $v_2, v_2' \in V(G_2)$ such that $v_2 \notin \mathbf{tr}_{G_2}(v_2')$ and $v_1 \in V(G_1)$. Then $G = \mathbf{j}(G_1, G_2, v_1, v_2)$ and $G' = \mathbf{j}(G_1, G_2, v_1, v_2')$ are not isomorphic.*

*Proof.* Assume, in contrary, that $\phi$ is an isomorphism from $G$ to $G'$. Observation 13 implies that either $\phi(v_1) = v_1$ and $\phi(v_2) = v_2'$ or $\phi(v_1) = v_2'$ and $\phi(v_2) = v_1$. We first exclude the case where $\phi(v_1) = v_1$ and $\phi(v_2) = v_2'$. Indeed, by Observation 11, $\phi' = \{(x, y) \in \phi \mid x \in V(G_2)\} \in \mathbf{Aut}(G_2)$ and moreover $\phi'(v_2) = \phi(v_2) = v_2'$, a contradiction since $v_2 \notin \mathbf{tr}_{G_2}(v_2')$. Thererefore, $\phi(v_1) = v_2'$ and $\phi(v_2) = v_1$. By Observation 11, $\phi_i = \{(x, y) \in \phi \mid x \in V(G_i)\}$ is an isomorphism from $G_i$ to $G_{3-i}, i = 1, 2$. Then $\psi = \phi_1 \circ \phi_2 \in \mathbf{Aut}(G_2)$ and $\psi(v_2) = \phi_1(\phi_2(v_2)) = \phi_1(\phi(v_2)) = \phi_1(v_1) = v_2'$. It follows that $v_2 \in \mathbf{tr}_{G_2}(v_2')$, a contradiction. $\square$

Given a graph $G$ we say that $G$ is *asymmetric* if it has a trivial automorphism group. Moreover, we say that a graph $G$ is *2-asymmetric* if its only non-trivial automorphism is an involution without fixed points.

**Lemma 9.** *Let $k$ be a non-negative integer and let $G_1, G_2$ be two disjoint non-isomorphic graphs such that $G_1, G_2 \in \mathcal{T}_k$. Then the graph $G = \mathbf{j}(G_1, G_2, v_1, v_2)$ is asymmetric.*

*Proof.* Suppose that $\phi \in \mathbf{Aut}(G)$ and $\phi \neq \mathbf{id}$. From Lemma 6, $\phi(v) \neq v$ for all $v \in V(G)$ and from Observation 13, $\phi(v_i) = v_{3-i}, i = 1, 2$. From Observation 12, $G_1$ is isomorphic to $G_2$, a contradiction. ☐

**Lemma 10.** *Let $k$ be a non-negative integer and let $G_1, G_2$ two disjoint graphs such that $G_1, G_2 \in \mathcal{T}_k$. If $\phi$ is an isomorphism from $G_1$ to $G_2$ and $v_i \in V(G_i), i = 1, 2$ such that $\phi(v_1) \notin \mathbf{tr}_{G_2}(v_2)$, then $G = \mathbf{j}(G_1, G_2, v_1, v_2)$ is asymmetric.*

*Proof.* Suppose that $\psi \in \mathbf{Aut}(G)$ and $\psi \neq \mathbf{id}$. From Lemma 6, $\psi(v) \neq v$ for all $v \in V(G)$ and from Observation 13, $\psi(v_1) = v_2$ and $\psi(v_2) = v_1$. From Observation 12, $\chi = \{(x, y) \in \psi \mid x \in V(G_1)\}$ is an isomorphism from $G_1$ to $G_2$. Moreover, $\phi \circ \chi^{-1}$ is an automorphism of $G_2$ mapping $v_2$ to $\phi(v_1)$, contradicting the assumption that $\phi(v_1) \notin \mathbf{tr}_{G_2}(v_2)$. ☐

**Lemma 11.** *Let $k$ be a non-negative integer and let $G_1, G_2$ be two disjoint graphs such that $G_1, G_2 \in \mathcal{T}_k$. If $\phi : V(G_1) \to V(G_2)$ is an isomorphism from $G_1$ to $G_2$ and $v_i \in V(G_i), i = 1, 2$ are two vertices such that $\phi(v_1) \in \mathbf{tr}_{G_2}(v_2)$, then $G = \mathbf{j}(G_1, G_2, v_1, v_2)$ is 2-asymmetric.*

*Proof.* Since $\phi(v_1) \in \mathbf{tr}_{G_2}(v_2)$, there exists an isomorphism $\psi : V(G_1) \to V(G_2)$ such that $\psi(v_1) = v_2$. Observe that $\chi = \psi \cup \psi^{-1}$ is an automorphism of $G$, and that $\chi$ is an involution without fixed points. Consider an automorphism $\chi' \neq \mathbf{id}$ of $G$. By Lemma 6 and Observation 13, $\chi'(v_1) = v_2$ and by Observation 12, $\chi'_1 = \{(x, y) \in \chi' | x \in V(G_1)\}$ is an isomorphism of $G_1$ and $G_2$. Then $\chi'_1 \circ \psi^{-1}$ is an automorphism of $G_2$ that fixes $v_2$, and by Lemma 6, $\chi'_1 = \psi$. We conclude that $\chi' = \chi$, and thus $\mathbf{Aut}(G) = \{\mathbf{id}, \chi\}$ and $G$ is 2-asymmetric. ☐

From Theorem 2 and Lemmata 9, 10 and 11 follows directly that.

**Observation 14.** *If $G$ is a graph such that $G \in \mathcal{T}_k$ then $G$ is either asymmetric or 2-asymmetric.*

For every integer $k \geq 0$, we define for following partition of $\mathcal{T}_k$:

$$\mathcal{A}_k = \{G \in \mathcal{T}_k \mid \mathbf{Aut}(G) = \{\mathbf{id}\}\} \text{ and } \mathcal{B}_k = \{G \in \mathcal{T}_k \mid \mathbf{Aut}(G) \neq \{\mathbf{id}\}\}.$$

We denote $\alpha_k = |\mathcal{A}_k|$, $\beta_k = |\mathcal{B}_k|$ and $\tau_k = |\mathcal{T}_k| = \alpha_k + \beta_k$. We also set $\gamma_k = 2^{k-2}$. A direct consequence of Observations 10 and 14 is the following.

**Observation 15.** *Let $k \geq 2$ be an integer. Then the automorphism group of each graph $G \in \mathcal{A}_k$ (resp. $G \in \mathcal{B}_k$) has exactly $\gamma_{k+2}$ (resp. $\gamma_{k+1}$) orbits.*

**Observation 16.** $\beta_0 = \alpha_1 = \alpha_2 = 0$ *and* $\alpha_0 = \beta_1 = \beta_2 = 1$.

**Theorem 3.** *For every integer $k \geq 1$, $\tau_k = 2^{2^k-(2k+1)} + 2^{2^{k-1}-(k+1)}$.*

*Proof.* First observe that for $k = 1, 2$ the claim holds. Let $G$ be a graph. Recall that $G \in \mathcal{T}_k$ iff $G = \mathbf{j}(G_1, G_2, v_1, v_2)$ for some $G_i \in \mathcal{T}_{k-1}$, and $v_i \in V(G_i), i = 1, 2$. Therefore, in order to count $\tau_k$ it is sufficient to count the ways to choose $G_1, G_2 \in \mathcal{T}_{k-1}$ and $v_i \in V(G_i), i = 1, 2$ and not end up with isomorphic graphs. Let $G_1, G_2$ be graphs such that $G_i \in \mathcal{T}_{k-1}$ and $v_i \in V(G_i), i = 1, 2$. We define

$$\mathcal{A}_k^1 = \{G \mid G = \mathbf{j}(G_1, G_2, v_1, v_2), G_1 \not\simeq G_2, G_i \in \mathcal{A}_{k-1} \text{ and } v_i \in V(G_i), i = 1, 2\} \quad (1)$$
$$\mathcal{A}_k^2 = \{G \mid G = \mathbf{j}(G_1, G_2, v_1, v_2), G_1 \not\simeq G_2, G_i \in \mathcal{B}_{k-1} \text{ and } v_i \in V(G_i), i = 1, 2\} \quad (2)$$
$$\mathcal{A}_k^3 = \{G \mid G = \mathbf{j}(G_1, G_2, v_1, v_2), G_1 \not\simeq G_2, G_1 \in \mathcal{A}_{k-1}, G_2 \in \mathcal{B}_{k-1}, \text{ and } v_i \in V(G_i), i = 1, 2\} \quad (3)$$
$$\mathcal{A}_k^4 = \{G \mid G = \mathbf{j}(G_1, G_2, v_1, v_2), G_1 \simeq_\phi G_2, G_i \in \mathcal{A}_{k-1}, \text{ and } v_i \in V(G_i), i = 1, 2, \text{ such that } \phi(v_1) \notin \mathbf{tr}_{G_2}(v_2)\} \quad (4)$$
$$\mathcal{A}_k^5 = \{G \mid G = \mathbf{j}(G_1, G_2, v_1, v_2), G_1 \simeq_\phi G_2, G_i \in \mathcal{B}_{k-1}, \text{ and } v_i \in V(G_i), i = 1, 2, \text{ such that } \phi(v_1) \notin \mathbf{tr}_{G_2}(v_2)\} \quad (5)$$
$$\mathcal{B}_k^1 = \{G \mid G = \mathbf{j}(G_1, G_2, v_1, v_2), G_1 \simeq_\phi G_2, G_i \in \mathcal{A}_{k-1}, \text{ and } v_i \in V(G_i), i = 1, 2, \text{ such that } \phi(v_1) \in \mathbf{tr}_{G_2}(v_2)\} \quad (6)$$
$$\mathcal{B}_k^2 = \{G \mid G = \mathbf{j}(G_1, G_2, v_1, v_2), G_1 \simeq_\phi G_2, G_i \in \mathcal{B}_{k-1}, \text{ and } v_i \in V(G_i), i = 1, 2 \text{ such that } \phi(v_1) \in \mathbf{tr}_{G_2}(v_2)\}. \quad (7)$$

By their definitions, the above sets are a partition of $\mathcal{T}_k$. By Lemma 9 (for Relations (1)–(3)) and by Lemma 10 (for Relations (4) and (5)), the union of the first five is a subset of $\mathcal{A}_k$. Moreover, by Lemma 11 (applied to Relations (6) and (7)) the union of the last two is a subset of $\mathcal{B}_k$. We conclude that $\mathcal{A}_k = \bigcup_{i=1,\ldots,5} \mathcal{A}_k^i$ and $\mathcal{B}_k = \mathcal{B}_k^1 \cup \mathcal{B}_k^2$.

From Observation 15, Lemmata 7 and 8, and Relations (1)–(7) we derive that

$$
\begin{aligned}
|\mathcal{A}_k^1| &= \binom{\alpha_{k-1}}{2} \cdot \gamma_{k+1}^2, \\
|\mathcal{A}_k^2| &= \binom{\beta_{k-1}}{2} \cdot \gamma_k^2, \\
|\mathcal{A}_k^3| &= \alpha_{k-1} \cdot \gamma_{k+1} \cdot \beta_{k-1} \cdot \gamma_k, \\
|\mathcal{A}_k^4| &= \alpha_{k-1} \cdot \binom{\gamma_{k+1}}{2}
\end{aligned}
$$

$$
\begin{aligned}
|\mathcal{A}_k^5| &= \beta_{k-1} \cdot \binom{\gamma_k}{2} \\
|\mathcal{B}_k^1| &= \alpha_{k-1} \cdot \gamma_{k+1} \\
|\mathcal{B}_k^2| &= \beta_{k-1} \cdot \gamma_k
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\alpha_k &= \binom{\alpha_{k-1}}{2}\gamma_{k+1}^2 + \binom{\beta_{k-1}}{2}\gamma_k^2 + \alpha_{k-1}\binom{\gamma_{k+1}}{2} + \beta_{k-1}\binom{\gamma_k}{2} + \\
&\quad + \alpha_{k-1}\beta_{k-1}\gamma_k\gamma_{k+1} \qquad (8) \\
\beta_k &= \alpha_{k-1}\gamma_{k+1} + \beta_{k-1}\gamma_k \qquad (9)
\end{aligned}
$$

By simplifying (8),

$$
\alpha_k = \frac{1}{2}\left[\left(\gamma_{k+1}^2\alpha_{k-1}^2 + \gamma_k^2\beta_{k-1}^2 + 2\alpha_{k-1}\beta_{k-1}\gamma_k\gamma_{k+1}\right) - \left(\alpha_{k-1}\gamma_{k+1} + \beta_{k-1}\gamma_k\right)\right] =
$$

$$
= \frac{1}{2}\left(\beta_k^2 - \beta_k\right).
$$

It follows (using Relation (9)) that,

$$
\tau_k = \frac{1}{2}\left(\beta_k^2 + \beta_k\right) \text{ and } \beta_k = \gamma_k\beta_{k-1}^2
$$

Let $\delta_k = 2^{k-1} - k$ and observe that $\beta_k = 2^{\delta_k} = 2^{2^{k-1}-k}$, for every integer $k \geq 2$. Then $\tau_k = 2^{2^k-(2k+1)} + 2^{2^{k-1}-(k+1)}$, $k \geq 3$ and the theorem follows. □

# 7 Obstructions for $\mathcal{G}_k$, $k \leq 3$

It is easy to prove that

- $\mathbf{obs}_{\leq}(\mathcal{G}_0) = \mathbf{obs}_{\subseteq}(\mathcal{G}_0) = \mathbf{obs}_{\sqsubseteq}(\mathcal{G}_0) = \{K_1\}$,
- $\mathbf{obs}_{\leq}(\mathcal{G}_1) = \mathbf{obs}_{\subseteq}(\mathcal{G}_1) = \mathbf{obs}_{\sqsubseteq}(\mathcal{G}_1) = \{K_2\}$,
- $\mathbf{obs}_{\leq}(\mathcal{G}_2) = \mathbf{obs}_{\subseteq}(\mathcal{G}_1) = \{K_3, P_4\}$ and $\mathbf{obs}_{\sqsubseteq}(\mathcal{G}_2) = \{K_3, P_4, C_4\}$.

Let $\mathcal{C}$ be the set of the graphs that appear inside the outer polygon in Figure 1. In this section we prove that $\mathbf{obs}_{\subseteq}(\mathcal{G}_3) = \mathcal{C}$.

**Theorem 4.** *For any graph $G$, $\mathbf{td}(G) > 3$ if and only if $G$ contains one of the graphs in $\mathcal{C}$ as a subgraph.*

*Proof.* Since each of the graphs in $\mathcal{C}$ is connected and has tree-depth four, it suffices to show that any connected graph with tree-depth four contains one of them as a subgraph. Suppose for contradiction that this is not the case, and let $G$ be a connected graph with tree-depth four that contains none of the graphs in $\mathcal{C}$ as a subgraph. We may assume that $G$ is minimal, i.e., that $\mathbf{td}(G \setminus e) = 3$ and $\mathbf{td}(G \setminus v) = 3$ for any edge $e \in E(G)$ and any vertex $v \in V(G)$. The graph $G$ cannot contain any cycles of length greater than four, otherwise, it would contain $C_5, C_6, C_7$, or $P_8$ as a subgraph.

Let $G'$ be a 2-connected subgraph of $G$, and suppose that $|V(G')| \geq 5$. Observe that $G'$ contains a 4-cycle $C = v_1v_2v_3v_4$. Consider a vertex $v_5 \in V(G') \setminus V(C)$. Since $G'$ is 2-connected, there exists a path $P$ with distinct end-vertices in $C$ such that $v_5 \in V(P)$ and $|V(P) \cap V(C)| = 2$. Since $G$ does not contain cycles of length at least 5, $P$ has length two and joins two opposite vertices of $C$, say $v_1$ and $v_3$. If the subgraph induced by $V(C) \cup \{v_5\}$ contains any of the edges $\{v_2, v_4\}$, $\{v_2, v_5\}$ or $\{v_4, v_5\}$, then $G$ contains $C_5$ as a subgraph, hence we may assume that this is not the case. Also, none of $v_2$, $v_4$ and $v_5$ may be incident with any other vertex of $G$, otherwise $G$ would contain $K_4^2$. Consider the graph $H$ obtained from $G$ by removing the edge $\{v_1, v_5\}$. By the minimality of $G$, $\mathbf{td}(H) = 3$. The graph $H$ is connected, hence $H$ contains a vertex $v$ such that $H \setminus v$ is a star forest. If $v = v_1$ or $v = v_3$, then $G \setminus v$ is a star forest, which is contradiction with $\mathbf{td}(G) = 4$.

However, $H \setminus v$ for any other vertex $v$ contains $P_4$ as a subgraph. This is a contradiction, hence we may assume that any 2-connected subgraph of $G$ has at most four vertices.

Let us now consider the case where $G$ contains a 4-cycle $C = v_1v_2v_3v_4$. If both edges $\{v_1, v_3\}$ and $\{v_2, v_4\}$ are in $G$, then $G$ contains $K_4$ as a subgraph, thus we may assume this is not the case. Suppose first that $\{v_1, v_3\}$ is an edge (thus $\{v_2, v_4\}$ is not an edge). If $v_2$ or $v_4$ is adjacent to a vertex outside of $C$, then $G$ contains $K_4^1$ as a subgraph. Otherwise, consider the graph $H$ obtained from $G$ by removing the edge $\{v_1, v_3\}$. By the minimality of $G$, there exists a vertex $v$ such that $H \setminus v$ is a star forest. The vertex $v$ must belong to $C$. Since $G \setminus v$ is not a star forest, $v \neq v_1$ and $v \neq v_3$, hence we may assume that $v = v_2$. Since $H \setminus v_2$ is a star forest, $v_4$ is the only neighbor of $v_1$ and $v_3$ in $H \setminus v_2$. But then $H = C$, and tree-depth of $G$ would be only three, which is a contradiction; therefore, any 4-cycle in $G$ is induced.

Let $C = v_1v_2v_3v_4$ be an induced 4-cycle in $G$. Since $G$ does not contain $K_4^2$ as a subgraph, the vertices of $V(G) \setminus V(C)$ can only be adjacent to two non-adjacent vertices of $C$, say $v_1$ and $v_3$. Since $\mathbf{td}(G) = 4$, we have $G \neq C$ and we may assume that there exists a vertex $v_5 \in V(G) \setminus V(C)$ adjacent to $v_1$. Let us consider the graph $H$ obtained from $G$ by removing the edge $v_1v_4$. By the minimality of $G$, there exists a vertex $v$ such that $H \setminus v$ is a star forest. Since $v_5v_1v_2v_3v_4$ is a path, $v$ must be $v_1$, $v_2$ or $v_3$. If $v = v_1$ or $v = v_3$, then $G \setminus v$ is a star forest, hence $v = v_2$. However, this means that $G \setminus v_1$ is a star forest, which is a contradiction, thus $G$ does not contain any 4-cycle.

Consider now the case where $G$ contains a triangle $C = v_1v_2v_3$. The graph $G$ cannot contain another triangle disjoint from $C$, since otherwise it would contain $K_3P_4^1$ or $K_3K_3$ as a subgraph. Together with the fact that each nontrivial 2-connected subgraph of $G$ is a triangle, this implies that all the triangles in $G$ intersect in one vertex. We may assume that there is at least one vertex $v_4$ not belonging to $C$ adjacent to $v_1$, and that all triangles in $G$ contain the vertex $v_1$.

The vertex $v_1$ is a cut-vertex in $G$. The graph $G \setminus v_1$ is not a star forest, hence one of its components contains a triangle or $P_4$. All triangles in $G$ contain the vertex $v_1$, hence one of the components of $G \setminus v_1$ contains a path $P$ of length three.

If $P$ is disjoint with $C$, then $G$ contains a subgraph $K_3P_4^1$ or $K_3P_4^2$. It follows that $C$ is the only triangle in $G$ and that the path $P$ intersects $C \setminus v_1$. If the degree of both $v_2$ and $v_3$ is greater than two, then $G$ contains the subgraph $K_4^3$, thus we may assume that degree of $v_2$ is two and that $P = v_2v_3v_5v_6$ for some vertices $v_5$ and $v_6$. Similarly, $G \setminus v_3$ contains $P_4$ as a subgraph, hence we may assume that there is a vertex $v_7$ adjacent to $v_4$. However, the graph $G$ then would contain $K_2K_3K_2$ as a subgraph. Therefore, $G$ does not contain a triangle, and it must be a tree.

It is however easy to verify using Theorem 2 that the only tree-depth critical trees with tree-depth four are $P_8$, $P_4^1P_4^2$ and $P_4^2P_4^2$. It follows that any graph with $\mathbf{td}(G) > 3$ contains one of the graphs in $\mathcal{C}$ as a subgraph. □

**Corollary 3.** *The set* $\mathbf{obs}_{\leq}(\mathcal{G}_3)$ *contains exactly all the graphs depicted in the inner polygon in Figure 1.*

*Proof.* Follows directly from Observation 3 and the fact that $C_5 \leq C_6$ and $C_5 \leq C_7$. □

**Corollary 4.** *The set* $\mathbf{obs}_{\sqsubseteq}(\mathcal{G}_3)$ *contains exactly all the graphs in Figure 1.*

*Proof.* Follows by inspection, using Observation 4. □

Notice that the obstructions for $\mathcal{G}_k$ have at most $2^k$ vertices for $k \leq 3$. Hence Theorem 1 is not sharp even in this case (it only claims that the obstructions have at most 16 vertices). We conclude with the following conjecture.

**Conjecture 1.** *For every* $k \geq 1$*, the order of the graphs in* $\mathbf{obs}_{\sqsubseteq}(\mathcal{G}_k)$ *is bounded by* $2^k$*.*

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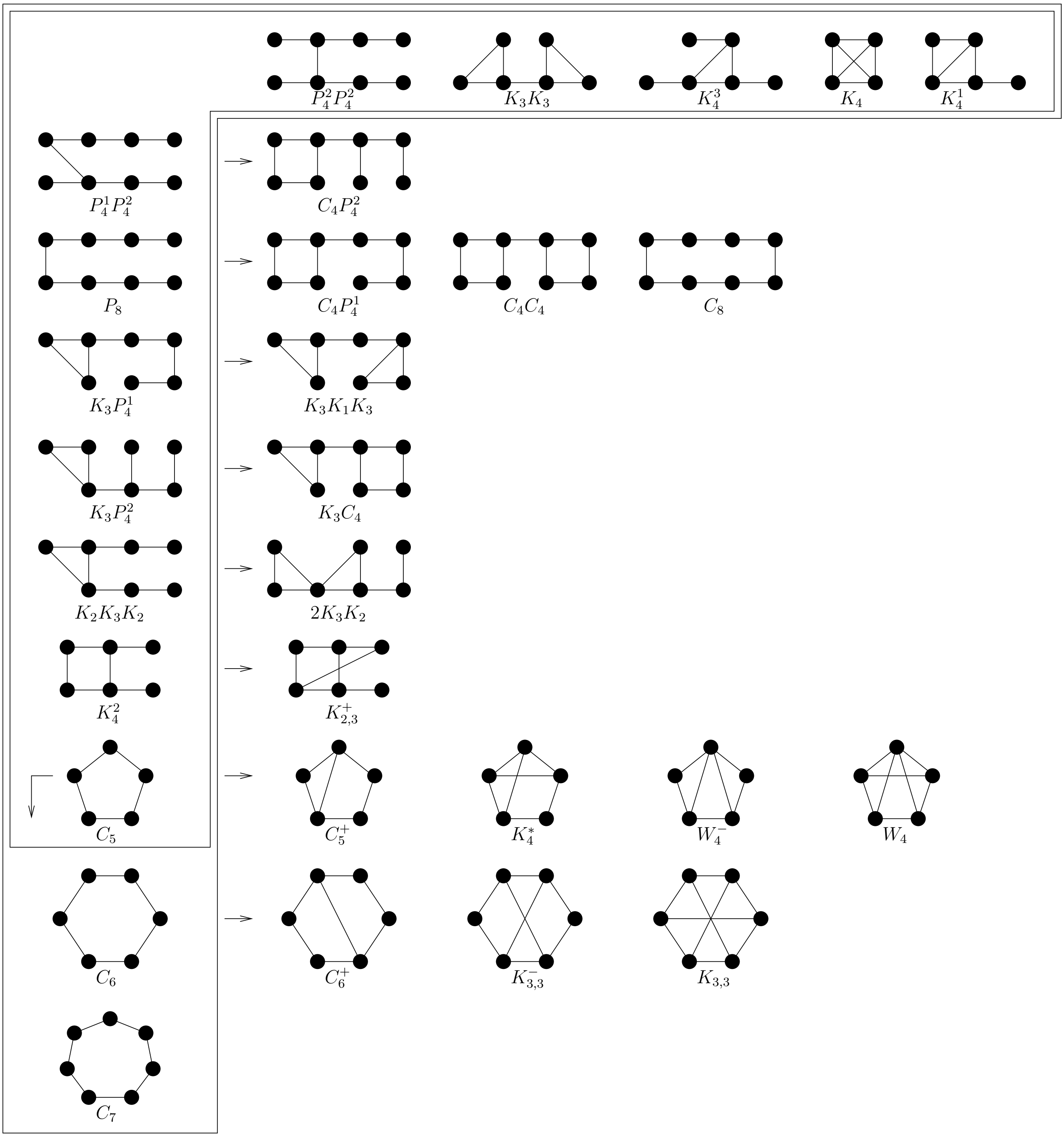


Figure 1: The forbidden graphs for $\mathcal{G}_3$