Edge-closure concept in claw-free graphs and stability of forbidden subgraphs

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Abstract.

Ryjáček introduced a closure concept in claw-free graphs based on local completion of a locally connected vertex. Connected graphs A, for which the class of (C, A)-free graphs is stable under the closure, were completely characterized. In this paper, we introduce a variation of the closure concept based on local completion of a locally connected edge of a claw-free graph. The closure is uniquely determined and preserves the value of the circumference of a graph. We show that the class of (C, A)-free graphs is stable under the edge-closure if $A \in \{H, P_i, N_{i,j,k}\}$.

Keywords: Closure concept, Edge-closure concept, Hamiltonicity, Stable property

AMS Subject Classification (2000): 05C45, 05C35

1 Introduction

In this paper, we consider only finite undirected graphs without loops and multiple edges. We use [1] for terminology and notations not defined here. The *circumference* of a graph G, denoted $c(G)$, is the length of a longest

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cycle in G. A cycle on n vertices is denoted by C_n. A *hamiltonian cycle* is a cycle in G on $|V(G)|$ vertices. A graph G is said to be *hamiltonian*, if $c(G) = |V(G)|$. For a nonempty set $A \subseteq V(G)$, the induced subgraph on A in G is denoted by $\langle A \rangle_G$. The line graph of a graph G is denoted by $L(G)$. We denote by P_i the path on i vertices and we say that the *length* of a path P is the number of edges of P. For any $A \subset V(G)$, G–A stands for the graph $\langle V(G) \setminus A \rangle_G$. An edge xy is *pendant* if $d_G(x) = 1$ or $d_G(y) = 1$.
For a connected graph H, a graph G is said to be *H-free*, if G does not contain a copy of H as an induced subgraph; the graph H will be also referred to in this context as a *forbidden subgraph*. The graph $K_{1,3}$ will be called the *claw* and in the special case $H = K_{1,3}$ we say that G is *claw-free*. The list of frequently used forbidden subgraphs is shown in Figure 1.

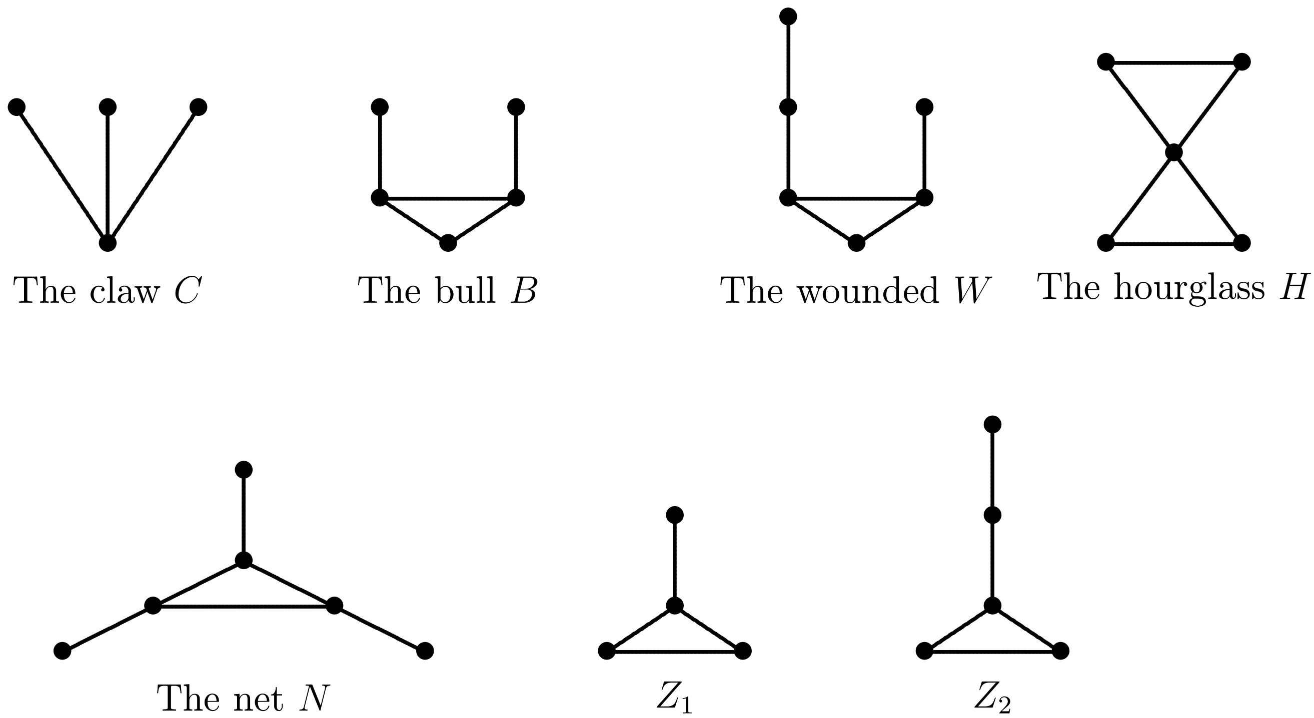


Figure 1: Frequently used forbidden subgraphs

Let $x \in V(G)$. The *neighbourhood* of x, denoted $N_G(x)$, is the set of all vertices adjacent to x. For a nonempty set $A \subset V(G)$, $N_G(A)$ denotes the set of all vertices of G–A adjacent to at least one vertex of A, and $N_G[A] = N_G(A) \cup A$. For an edge $xy \in E(G)$ we set $N_G(xy) = N_G(\{x, y\})$

and $N_G[xy] = N_G[\{x, y\}]$. A vertex $x \in V(G)$ is said to be *locally connected* if $\langle N_G(x) \rangle$ is connected. A graph G is *locally connected* if every vertex of G is locally connected. Analogously we define local connectivity of an edge of G. An edge $xy \in E(G)$ is *locally connected* if $\langle N_G(xy) \rangle$ is connected. A graph G is *edge-locally connected* if every edge of G is locally connected.
For an arbitrary vertex $x \in V(G)$, let $B_x = \{uv |\, u, v \in N_G(x),\, uv \notin E(G)\}$ and $G_x = (V(G),\, E(G) \cup B_x)$. The graph G_x is called the *local completion* of G at x. A locally connected vertex x with $B_x \neq \emptyset$ is called *eligible* (in G). We say that a graph F is a *closure* of G, denoted $F = \mathrm{cl}(G)$, if there is no eligible vertex in F and there is a sequence of graphs $G_1, \ldots, G_t$ and vertices $x_1, \ldots, x_{t-1}$ such that $G_1 = G$, $G_t = F$, x_i is an eligible vertex of G_i and $G_{i+1} = (G_i)_{x_i}$, $i = 1, \ldots t-1$ (equivalently, $\mathrm{cl}(G)$) is obtained from G by a series of local completions at eligible vertices, as long as this is possible). The following basic result was proved by Ryjáček.

Theorem A [9]. *Let G be a claw-free graph. Then*

(i) $\mathrm{cl}(G)$ *is well-defined (i.e., uniquelly determined),*

(ii) there is a triangle-free graph F such that $\mathrm{cl}(G) = L(F)$,

(iii) $c(G) = c(\mathrm{cl}(G))$.

Consequently, if G is claw-free, then so is $\mathrm{cl}(G)$, and G is hamiltonian if and only if so is $\mathrm{cl}(G)$. A claw-free graph G, for which $G = \mathrm{cl}(G)$, will be called *closed*. As an immediate consequence of Theorem A can be shown the following:

Theorem B [8]. *Let G be a connected locally connected claw-free graph. Then G is hamiltonian.*

Let $\mathcal{C}$ be a subclass of the class of claw-free graphs. We say that the class $\mathcal{C}$ is *stable under the closure* (or simply *stable*) if $\mathrm{cl}(G) \in \mathcal{C}$ for every $G \in \mathcal{C}$. Clearly, the class of CA-free graphs is trivially stable if A is not claw-free or if A is not closed. For the proofs of stability of several graph classes we use the following notation. For an induced subgraph A of G, we say that A is *permanent induced subgraph* of G (or simply permanent), if $\langle V(A)\rangle_{cl(G)} \simeq A$. The classes of CZ_2-free graphs, CB-free graphs and CN-free graphs were extended in [5] as follows. We denote by (see also Fig. 2):

Z_i, ($i \geq 1$) - the graph which is obtained by identifying a vertex of a triangle with an end vertex of a path of length i

$B_{i,j}$, ($i \geq j \geq 1$) - the generalized (i, j)-bull, i.e. the graph which is obtained by identifying each of some two distinct vertices of a triangle with an end vertex of one of two vertex-disjoint paths of lengths i, j

$N_{i,j,k}$, ($i \geq j \geq k \geq 1$) - the generalized (i, j, k)-net, i.e. the graph which is obtained by identifying each vertex of a triangle with an end vertex of one of three vertex-disjoint paths of lengths i, j, k.

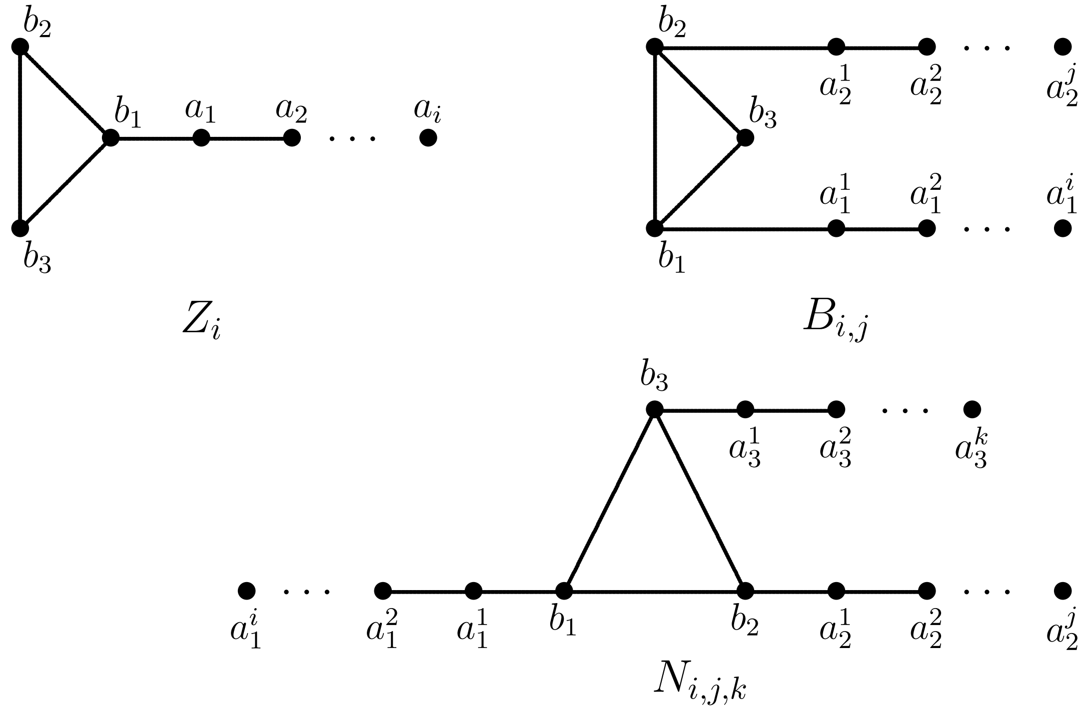


Figure 2: Generalized subgraphs Z_i, $B_{i,j}$ and $N_{i,j,k}$

Thus, $B_{1,1} \simeq B$ and $N_{1,1,1} \simeq N$. We will always keep the labeling of the vertices of the graphs Z_i, $B_{i,j}$ and $N_{i,j,k}$ as shown in Fig. 2.

Brousek, Ryjáček and Schiermeyer characterized all connected closed claw-free graphs A for which the CA-free class is stable.

Theorem C [6]. *Let A be a closed connected claw-free graph. Then G being CA-free implies* $\mathrm{cl}(G)$ *is CA-free if and only if*

$$A \in \{H, C_3\} \cup \{P_i | i \geq 3\} \cup \{Z_i | i \geq 1\} \cup \{N_{i,j,k} | i, j, k \geq 1\}.$$

The importance of the class of CH-free graphs is shown in the following theorem. Note that the hamiltonicity in general 4-connected claw-free graphs is still an open problem introduced by Thomassen [10] and Matthews and Sumner [7]. For more stable classes and stable properties we refer to [4].

Theorem D [2]. *Every 4-connected CH-free graph is hamiltonian.*

Let G' be a closed claw-free graph. The order of a largest clique in G' containing an edge e will be denoted by $\omega_{G'}(e)$. Let C be an induced cycle in G' of length k. We say that the cycle C is *eligible* in G' if $4 \leq k \leq 6$ and $\omega_{G'}(e) = 2$ for at least $k - 3$ nonconsecutive edges $e \in E(C)$. For an eligible cycle C in G' let $B_C = \{uv |\, u, v \in N_{G'}[C],\ uv \notin E(G')\}$. The graph $G_C = (V(G'), E(G') \cup B_C)$ is called the *C-completion* of G' at C. For a claw-free graph G, the graph F is a *cycle closure* of G, denoted $F = \mathrm{cl}_C(G)$, if there is a sequence of graphs $G_1, \ldots, G_t$ such that

(i) $G_1 = \mathrm{cl}(G)$,

(ii) $G_{i+1} = \mathrm{cl}((G_i)_C)$ for some eligible cycle C in G_i, $i = 1, \ldots, t-1$,

(iii) $G_t = F$ contains no eligible cycle.

Broersma and Ryjáček proved the following:

Theorem E [3]. *Let G be a claw-free graph. Then*

(i) $\mathrm{cl}_C(G)$ *is well-defined,*

(ii) $c(G) = c(\mathrm{cl}_C(G))$.

Consequently, a claw-free graph G is hamiltonian if and only if $\mathrm{cl}_C(G)$ is hamiltonian.

In Section 2 we will prove an analogous result for the edge-closure and, in Section 3, we will show that the classes of CH-free graphs, CP_i-free graphs and $CN_{i,j,k}$-free graphs are stable under the edge-closure for any $i, j, k \geq 1$. The class of $CB_{i,j}$-free graphs is not stable for any $i, j \geq 1$.

2 The concept of edge-closure in claw-free graphs

Let G be a claw-free graph and let $xy \in E(G)$. Let $B_{xy} = \{uv|\, u, v \in N_G[xy],\, uv \notin E(G)\}$. The edge xy is called *eligible* in G if xy is locally connected in G, $B_{xy} \neq \emptyset$ and xy is not a pendant edge. Let $G_{xy} = (V(G), E(G) \cup B_{xy})$. The graph G_{xy} is called the *local completion* of G at xy.
The following lemma shows that the circumference and the claw-freeness of a claw-free graph G are not affected by local completion of G at an eligible edge of G.

Lemma 1. *Let G be a claw-free graph and let xy be a locally connected edge of G such that $\langle N_G[xy] \rangle$ is not complete. Let $B_{xy} = \{uv|\, u, v \in N_G[xy],\, uv \notin E(G)\}$ and let G' be the graph with $V(G') = V(G)$ and $E(G') = E(G) \cup B_{xy}$. Then*

(i) the graph G' is claw-free,

(ii) $c(G) = c(G')$.

Proof.

1. Suppose that G' is not claw-free. Let H be a claw in G'. Since G is claw-free, $|E(H) \cap B_{xy}| \geq 1$. Since $\langle N_G[xy] \rangle$ is a clique, $|E(H) \cap B_{xy}| \leq 1$. Let $\{z, z_1, z_2, z_3\}$ denote the vertex-set of H, where $zz_1 \in B_{xy}$. Clearly $z_2, z_3 \notin N_G[xy]$, since otherwise $z_1z_2 \in E(G')$ or $z_1z_3 \in E(G')$, a contradiction. If $z_2z_3 \in E(G)$, then clearly $z_2z_3 \in E(G')$, a contradiction again. Since $zz_1 \in B_{xy}$, $z \in N_G[xy]$, and $xz \in E(G)$ or $yz \in E(G)$. But then $\langle \{z, x, z_2, z_3\} \rangle$ or $\langle \{z, y, z_2, z_3\} \rangle$ is a claw in G, which is a contradiction. Hence G' is claw-free.

2. Clearly $c(G) \leq c(G')$. Consider a locally connected edge xy with incomplete neighbourhood.

   - Suppose that both vertices x, y are locally connected in G. Then, using the Ryjáček's closure [9], we complete $N_G(x)$. Let G_1 denote the local completion of G at x. Clearly $N_G[x] \subset N_{G_1}[y]$ and $N_{G_1}[y] = N_G[xy]$. Let G_2 denote the local completion of G_1 at y. Clearly $N_{G_2}[x] = N_{G_2}[y] = N_G[xy]$ and $\langle N_{G_2}[xy] \rangle$ is complete. By Theorem A, $c(G) = c(G_2)$. Since $G' = G_2$, we obtain $c(G) = c(G')$.
   - Suppose that x is locally connected in G, but y not. Let G_1 denote the local completion of G at x. Clearly xy is locally connected in G_1. Since y is not locally connected in G and G is claw-free, the subgraph $\langle N_G(y) \rangle$ consists of exactly two cliques. Let C_1 denote the clique of $\langle N_G[y] \rangle$ such that $x \in V(C_1)$, let C_2 denote the other clique of $\langle N_G(y) \rangle$. Note that $V(C_2) \cap N_G(x) = \emptyset$, since y is not locally connected in G. Since $x \in N_G(y)$, $N_{G_1}[x] \subset N_{G_1}[y]$. Since xy is locally connected in G, there is an edge uv in G between C_2 and $\langle N_G(x) \rangle - y$, where $u \in V(C_2)$ and $v \in N_G(x) \setminus \{y\}$. Since $u, v \in N_{G_1}(y)$, the vertex y is locally connected in G_1. After local completion of G_1 at y and using the same argument as in the previous case we obtain $c(G) = c(G')$. Symmetrically we obtain $c(G) = c(G')$ if y is locally connected but x not.
   - Suppose that none of x, y is locally connected. Since G is claw-free, each of $\langle N_G(x) \rangle$ and $\langle N_G(y) \rangle$ consists of exactly two cliques. Let C_1 denote the clique of $\langle N_G(x) \rangle$ such that $y \in V(C_1)$, let C_2 denote the other clique of $\langle N_G(x) \rangle$. Let D_1 denote the clique of $\langle N_G(y) \rangle$ such that $x \in V(D_1)$, let D_2 denote the other clique of $\langle N_G(y) \rangle$.

     Now we show that $V(C_2) \cap V(D_2) = \emptyset$. Suppose that $z \in V(C_2) \cap V(D_2)$. Hence z is a neighbour of both x and y. But the edge zy is an edge between C_1 and C_2, which makes the vertex x locally connected in G, a contradiction.

Since C_1 is a clique, every vertex in $V(C_1) \setminus \{y\}$ is adjacent to both x and y, implying $(V(C_1) \setminus \{y\}) \subset V(D_1)$. Symmetrically $(V(D_1) \setminus \{x\}) \subset V(C_1)$.

Suppose that there is a vertex $z \in V(G)$ such that $z \in V(C_1) \cap V(D_1)$. Since xy is locally connected, there is an edge between z and a vertex $c \in (V(C_2) \cup V(D_2))$. But then x or y is locally connected, a contradiction.

Hence $V(C_1) = \{y\}$ and $V(D_1) = \{x\}$. Clearly $\omega_G(xy) = 2$. Since xy is locally connected, there is an edge ef between C_2 and D_2 such that $e \in V(C_2)$ and $f \in V(D_2)$. Now we consider the cycle $C = \langle\{x, y, f, e\}\rangle$. By the definition of an eligible cycle, the cycle C is eligible in G. Using the local completion of G at C we obtain a graph G_1. Clearly $G' \subset G_1$. By Theorem E, $c(G) = c(G_1)$ and hence $c(G) = c(G')$.

□

Now we define the main concept of this paper. Let G be a claw-free graph. We say that a graph F is an *edge-closure* of G, denoted $F = \mathrm{cl}'(G)$, if

(i) there is a sequence of graphs $G_1, \dots, G_t$ and edges $e_1, \dots e_{t-1}$ such that $G_1 = G$, $G_t = F$, e_i is an eligible edge in G_i and $G_{i+1} = (G_i)_{xy}$, $i = 1, \dots, t-1$,

(ii) there is no eligible edge in G_t.

Equivalently, $\mathrm{cl}'(G)$ is obtained from G by a series of local completions at eligible edges, as long as this is possible.

Theorem 1. *Let G be a claw-free graph. Then*

(*i*) *the closure* $\mathrm{cl}'(G)$ *is well defined,*

(*ii*) $c(G) = c(\mathrm{cl}'(G))$.

Proof.

1. Let G_1, G_2 be two edge-closures of G, suppose that $E(G_1) \setminus E(G_2) \neq \emptyset$. Let $H_1, \dots, H_t$ be the sequence of graphs that yields G_1. Let i be the smallest integer for which $E(H_i) \setminus E(G_2) \neq \emptyset$ and let $e = uv$ be an edge such that $e \in E(H_i) \setminus E(G_2)$. Then, since $e \in E(H_i)$, $u, v \in N_{H_{i-1}}[xy]$, where xy is eligible in H_{i-1}. But then, since $E(\langle N_{H_{i-1}}(xy)\rangle) \subset E(H_{i-1}) \subset E(G_2)$, $\langle N_{G_2}(xy)\rangle$ is connected. Hence $e = uv \in E(G_2)$, a contradiction

2. Immediately by Lemma 1, part (ii).

□

Corollary 1. *Let G be a claw-free graph. Then G is hamiltonian if and only if $\mathrm{cl}'(G)$ is hamiltonian.*

Corollary 2. *Let G be a connected edge-locally connected claw-free graph. Then G is hamiltonian.*

The following examples show the independence between the closure introduced by Ryjáček in [9] and the edge-closure (i.e., none of the closures can be obtained by using the other one). The graph G_1 shown in Fig. 3 (a) has no eligible vertex (i.e. $\mathrm{cl}(G_1) = G_1$) and exactly one eligible edge xy. Using the edge-closure we obtain a complete graph K (i.e. $\mathrm{cl}'(G_1) = K$). The class of graphs shown in Fig. 3 (b) contains graphs with no eligible vertices, but $\mathrm{cl}'(G)$ is complete again. Elliptical parts represent cliques of order at least two. Note that in both cases $\mathrm{cl}(G) = G$, but $\mathrm{cl}'(G)$ is complete.
The class of graphs shown in Fig. 4 (a) contains graphs such that each vertex of such graph is eligible, but not every edge of such a graph is eligible. Using the closures on such a graph we obtain that both closures of such a graph are cliques. The graph shown in Fig. 4 (b) has four eligible vertices, no eligible edge, and $\mathrm{cl}(G)$ is a clique. Elliptical part represents a clique of order at least two.

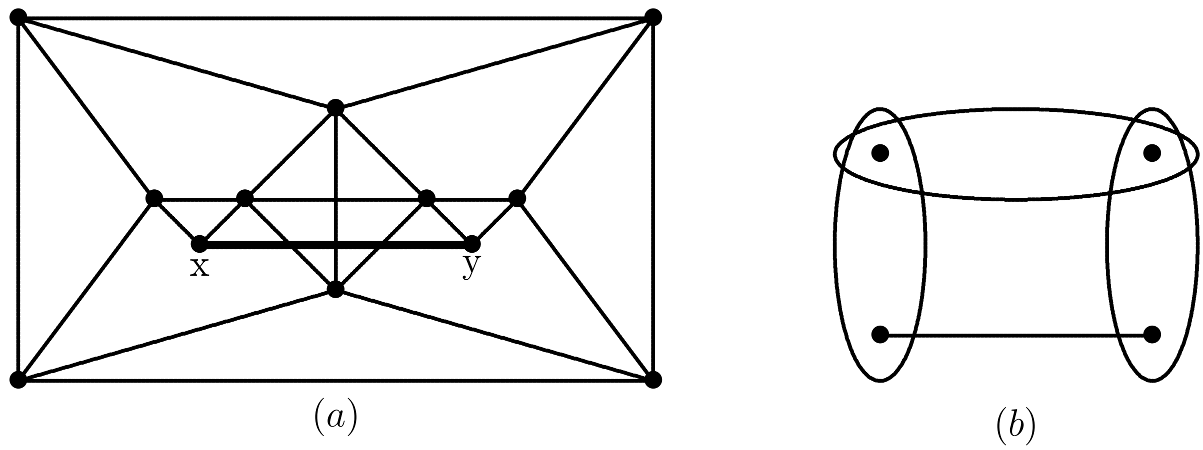


Figure 3: Closed graphs with eligible edges

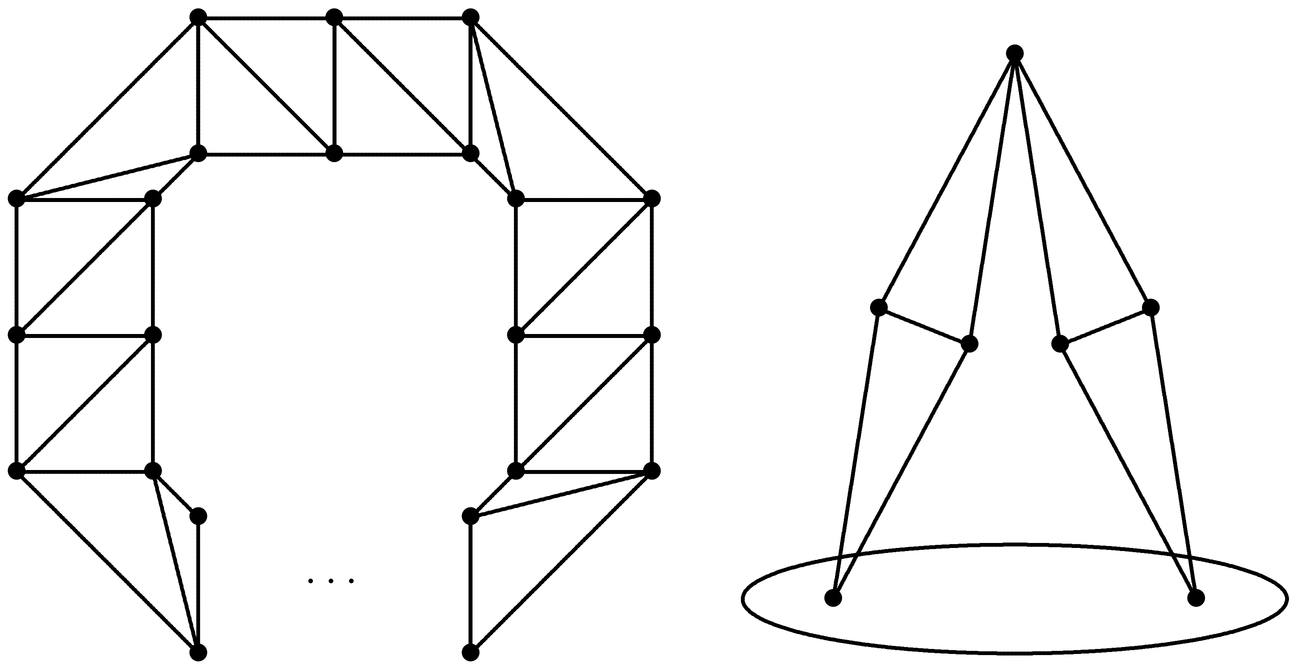
3 Stability of forbidden subgraphs

Theorem 2. *Let G be a CP_i-free graph $(i > 1)$ and let $xy \in E(G)$ be an eligible edge. Then the graph G_{xy} is CP_i-free.*

Proof. If G is CP_i-free, then, by Lemma 1, G_{xy} is claw-free. Suppose that $H = \langle\{a_1, a_2, \ldots, a_i\}\rangle_{G_{xy}}$ is an induced path in G_{xy} and let $B_{xy} = E(G_{xy}) \setminus E(G)$. Since H is induced in G_{xy} and G is P_i-free, $|E(H) \cap B_{xy}| \geq 1$. But $\langle N_G[x, y]\rangle_{G_{xy}}$ is a clique and H is an induced path, hence $|E(H) \cap B_{xy}| \leq 1$. Let $E(H) \cap B_{xy} = a_s a_{s+1}$, where $1 \leq s \leq i - 1$. If $a_t \in N_G[x, y] \cap V(H)$ for some $t \neq s, s + 1$, then $a_t a_s, a_t a_{s+1} \in E(G_{xy})$, which contradicts the fact that H is an induced path. Hence $V(H) \cap N_G[x, y] = \{a_s, a_{s+1}\}$. Consider the following cases:

Case 1: $\{x, y\} \in V(H)$. Then the only possibility (up to symmetry) is that $i = 2$, $x = a_1$ and $y = a_2$ since otherwise H is not an induced path. But $xy \in E(G)$, which contradicts the fact that G is P_i-free.

Case 2: $|\{x, y\} \cap V(H)| = 1$. Up to symmetry we can suppose that $x \in V(H)$. Then $x = a_1$ or $x = a_i$ since otherwise H is not an induced path in G_{xy}. Up to symmetry suppose that $x = a_1$. Clearly $a_s = x$, $a_{s+1} = a_2$. This yields $y, a_2, \ldots, a_i$ to be an induced path on i vertices in G. This



(a) (b)

Figure 4: Graphs with complete closure

is a contradiction.

Case 3: $\{x, y\} \cap V(H) = \emptyset$. If both vertices a_s, a_{s+1} have a common neighbour on the edge xy, say x, then the path $a_1, \ldots, a_s, x, a_{s+1}, \ldots, a_i$ is an induced path of length i in G, a contradiction. Symmetrically we obtain a contradiction if we consider the vertex y instead of x. Up to symmetry suppose that $a_s x \in E(G)$ and $a_{s+1} y \in E(G)$. Since $x a_{s+1} \notin E(G)$ and $y a_s \notin E(G)$, the path $a_1, \ldots, a_s, x, y, a_{s+1}, \ldots, a_i$ is an induced path of length $i+1$ in G, a contradiction.

□

Note that CP_i-free graphs for $i = 1, 2, 3$ are trivial.

Theorem 3. *Let G be a $CN_{i,j,k}$-free graph ($i \geq j \geq k \geq 1$) and let $xy \in E(G)$ be an eligible edge. Then the graph G_{xy} is $CN_{i,j,k}$-free.*

Proof. If G is $CN_{i,j,k}$-free, then, by Lemma 1, G_{xy} is claw-free. Suppose

that $H = \langle\{b_1, b_2, b_3, a_1^1, \ldots, a_1^i, a_2^1, \ldots, a_2^j, a_3^1, \ldots, a_3^k\}\rangle_{G_{xy}} \simeq N_{i,j,k}$ and let $B_{xy} = E(G_{xy}) \setminus E(G)$. Let $B_{xy}^H = B_{xy} \cap E(H)$. Clearly $|B_{xy}^H| \geq 1$.
Suppose that $\{x, y\} \subset V(H)$. But then, in any case, the subgraph $\langle N_G[xy]\rangle_{G_{xy}}$ is complete, implying H is not induced in G_{xy}, a contradiction.
Now suppose that $|V(H) \cap \{x, y\}| = 1$. Up to symmetry we can suppose that $x \in V(H)$ but y not. Clearly $N_G(xy) \cap V(H) = 1$, since otherwise H is not induced in G_{xy}. This implies that $x \in \{a_1^i, a_2^j, a_3^k\}$. Up to symmetry suppose that $x = a_3^k$. Clearly $|B_{xy}^H| = 1$, since otherwise H is not induced in G_{xy}. Suppose first that $k = 1$. Then y is adjacent to b_3 and $B_{xy}^H = \{b_3 a_3^1\}$, since otherwise H is not induced in G_{xy}. Then $\{b_1, b_2, b_3, a_1^1, \ldots, a_1^i, a_2^1, \ldots, a_2^j, y\}$ induces an $N_{i,j,k}$ in G, a contradiction. Now we suppose that $k \geq 2$. Clearly $a_3^{k-2}x \notin E(G)$ and $a_3^k y \notin E(G)$, since otherwise H is not induced in G_{xy}. Hence $B_{xy}^H = \{a_3^{k-1}a_3^k\}$ and $a_3^{k-1}y \in E(G)$. But then $\{b_1, b_2, b_3, a_1^1, \ldots a_1^i, a_2^1, \ldots, a_2^j, a_3^1, \ldots, a_3^{k-1}, y\}$ induces an $N_{i,j,k}$ in G, a contradiction.
Hence none of the vertices x, y belongs to H. Put $b_1 = a_1^0$, $b_2 = a_2^0$ and $b_3 = a_3^0$ and let $a_r^s a_t^u \in B_{xy}^H$. Suppose first that $s > 0$ or $u > 0$. Then clearly $|B_{xy}^H| = 1$, since otherwise H is not an induced $N_{i,j,k}$ in G_{xy}. Suppose that $B_{xy}^H = \{a_r^s a_t^u\}$, $r, t \in \{1, 2, 3\}$ and $s, u \in \{1, 2, \ldots, i\}$ if $r, t = 1$, $s, u \in \{1, 2, \ldots, j\}$ if $r, t = 2$, and $s, u \in \{1, 2, \ldots, k\}$ if $r, t = 3$. Clearly $r = t$ and $|s - u| = 1$, since otherwise H is not an induced $N_{i,j,k}$ in G_{xy}. Up to symmetry suppose that $r = 3$ and $u = s + 1$. If one of the vertices x, y, say x, is adjacent to both vertices a_3^s and a_3^{s+1}, then $\{b_1, b_2, b_3, a_1^1, \ldots, a_1^i, a_2^1, \ldots, a_2^j, a_3^1, \ldots, a_3^s, x, a_3^{s+1}, \ldots, a_3^k\}$, or $\{b_1, b_2, b_3, a_1^1, \ldots, a_1^i, a_2^1, \ldots, a_2^j, x, a_3^1, \ldots a_3^k\}$ for $s = 0$, induces an $N_{i,j,k+1}$ in G, a contradiction. Up to symmetry suppose that $xa_3^s \in E(G)$, $ya_3^{s+1} \in E(G)$, $xa_3^{s+1} \notin E(G)$ and $ya_3^s \notin E(G)$. But then $\{b_1, b_2, b_3, a_1^1, \ldots, a_1^i, a_2^1, \ldots, a_2^j, a_3^1, \ldots, a_3^s, x, y, a_3^{s+1}, \ldots, a_3^k\}$, or $\{b_1, b_2, b_3, a_1^1, \ldots, a_1^i, a_2^1, \ldots, a_2^j, x, y, a_3^1, \ldots a_3^k\}$ for $s = 0$, induces an $N_{i,j,k+2}$ in G, a contradiction. Hence we have $s = u = 0$, i.e., $B_{xy}^H \subset \{b_1b_2, b_1b_3, b_2b_3\}$. Moreover $xa_r^s, ya_r^s \notin E(G)$, $r = 1$ and $s = 1, \ldots, i$, $r = 2$ and $s = 1, \ldots, j$, $r = 3$ and $s = 1, \ldots, k$ since otherwise H is not an induced $N_{i,j,k}$. We consider the following cases.
Case 1: $|B_{xy}^H| = 1$. Without loss of generality we can suppose that $B_{xy}^H =$

$\{b_1b_2\}$. Then $\langle\{b_3, b_1, b_2, a_3^1\}\rangle$ is an induced claw in G, a contradiction.

Case 2: $|B_{xy}^H| = 2$. Without loss of generality suppose that $B_{xy}^H = \{b_1b_2, b_1b_3\}$. Now, up to symmetry, suppose that $b_2x \in E(G)$, $b_3y \in E(G)$, $b_3x \notin E(G)$, $b_2y \notin E(G)$. Then b_2 is a center of a claw in G, since otherwise H is not induced in G_{xy}. Hence one of the vertices x, y, say x, is adjacent to both b_2 and b_3, since otherwise there is an induced claw. If $xb_1 \in E(G)$, then there is an induced subgraph $H' \simeq N_{i+1,j,k}$ on $\{x, b_2, b_3, a_1^0, \ldots, a_1^i, a_2^1, \ldots, a_2^j, a_3^1, \ldots, a_3^k\}$. Thus we can suppose that $yb_1 \in E(G)$ but $xb_1 \notin E(G)$. If y is not adjacent to any of b_2, b_3, then there is an induced subgraph $H' \simeq N_{i+2,j,k}$ on $\{x, b_2, b_3, y, a_1^0, \ldots, a_1^i, a_2^1, \ldots, a_2^j, a_3^1, \ldots, a_3^k\}$ in G. If $yb_2 \in E(G)$, then $yb_3 \in E(G)$ too, since otherwise there is an induced claw on $\{b_2, b_3, y, a_2^1\}$ in G. But then there is an induced subgraph $H' \simeq N_{i+1,j,k}$ on $\{y, b_2, b_3, b_1, a_1^1, \ldots, a_1^i, a_2^1, \ldots, a_2^j, a_3^1, \ldots, a_3^k\}$ in G. Hence we obtain a contradiction.

Case 3: $B_{xy}^H = \{b_1b_2, b_1b_3, b_2b_3\}$. None of the end vertices of xy is adjacent to every vertex b_i, $i = 1, 2, 3$, since otherwise there is an induced claw in G. Let $B = \{b_1, b_2, b_3\}$. Let x be the end vertex of xy such that $|N_G(x) \cap B| \geq |N_G(y) \cap B|$. Since $B \subset N_G(x, y)$ and $|B| = 3$, x has exactly two neighbours in B. If y is adjacent to only one vertex of B, then x is a center of a claw in G, a contradiction. Thus both vertices x, y have exactly two neighbours in B, moreover they have exactly one common neighbour in B, since $B \subset N_G(x, y)$. Without loss of generality suppose that $xb_1, xb_2, yb_2, yb_3 \in E(G)$. This yields that there is an induced subgraph $H' \simeq N_{i+1,j+1,k}$ on $\{x, b_2, y, b_1, a_1^1, \ldots, a_1^i, a_2^1, \ldots, a_2^j, b_3, a_3^1, \ldots, a_3^k\}$ in G, a contradiction again.

$\square$

Theorem 4. *The classes of $CB_{i,j}$-free graphs are not stable under the edge-closure for any i, j, $i \geq j \geq 1$.*

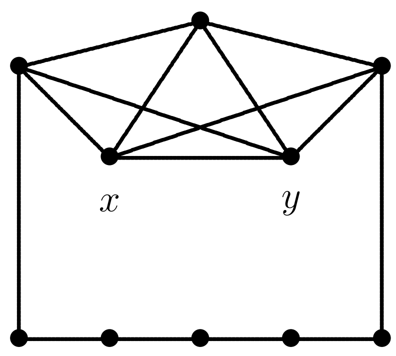


Figure 5: Class of $C, B_{i,j}$-free graphs is not stable under the edge-closure

Proof. Let $i, j \geq 1$, $i \geq j$ and let G be the graph obtained by identifying each of a pair of nonadjacent vertices of $K_5 - e$ with one end vertex of a path P of length at least $i + j + 2$. Let xy be an eligible edge of G (for $i = j = 2$ see Fig. 5). Then G is $C\,B_{i,j}$-free while $G_{xy} = \mathrm{cl}'(G)$ contains an induced subgraph isomorphic to $B_{i,j}$. □

As an immediate consequence of Theorems 2 and 3 we obtain the following theorem

Theorem 5. *The classes of CP_i-free graphs and $CN_{i,j,k}$-free graphs are stable under the edge-closure for any $i \geq j \geq k \geq 1$.*

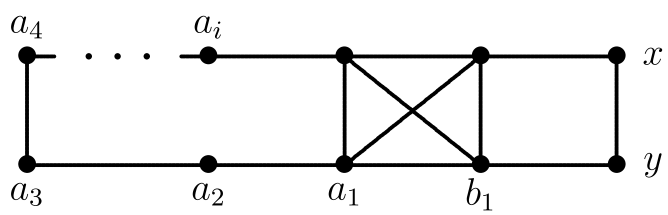


Figure 6: A graph G which is Z_i-free while G_{xy} contains an induced Z_i, $i \geq 3$

Consider the graph G shown in Fig. 6. When $i \geq 3$, the graph G is clearly Z_i-free, while $\{b_1, x, y, a_1, \ldots, a_i\}$ induces a Z_i in G_{xy}. Hence for the class of $C\,Z_i$-free graphs, $i \geq 3$, the analogue of Theorems 2 and 3 fails. Nevertheless, we believe that the analogue of Theorem 5 can be proved in this case.

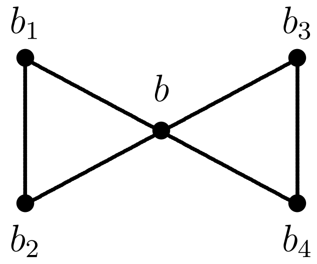


Figure 7: Graph H - hourglass

Conjecture 1. *The class of CZ_i-free graphs is stable under the edge-closure for any $i \geq 1$.*

Theorem 6. *The class of CH-free graphs is stable under the edge-closure.*

Proof. Suppose that $\mathrm{cl}'(G)$ contains an induced H. Let $G_1, \ldots, G_t$ be a sequence of graphs such that $G_1 = G$, $G_t = \mathrm{cl}'(G)$ and $G_{j+1} = (G_j)_{uv}$, where uv is an eligible edge in $(G_j)_{uv}$, $j = 1, \ldots, t-1$. Let s be the smallest number such that G_{s+1} contains a permanent H as an induced subgraph. Suppose that $s = 1$, i.e., $G_s = G$. Let $V(H) = \{b, b_1, b_2, b_3, b_4\}$ and we will always keep the labeling of the vertices of the graph H as shown in Figure 7.
Let xy be an eligible edge in G such that $G_2 = G_{xy}$, and let $B^H_{xy} = B_{xy} \cap E(H)$. Clearly $|B^H_{xy}| \geq 1$. Consider following cases:

Case 1: Both vertices x, y belong to $V(H)$. Clearly $x \neq b$, and $y \neq b$, since otherwise H is not induced in G_2. Up to symmetry suppose that $x = b_1$ and $y = b_2$. Then exactly one of the edges bx, by, say bx, does not belong to $E(H)$. Hence $B^H_{xy} = \{bx\}$. Since xy is not pendant, there is a vertex $a \in V(G) \setminus V(H)$ such that $ax \in E(G)$. Since xy is locally connected, there is an a,b-path P in $\langle N_G(xy) \rangle$. Choose P and a in such a way that P is shortest possible. Then every internal vertex of P is a neighbour of y but not of x (for the case $ab \in E(G)$, $ax \in E(G)$ since otherwise b is a center of a claw). Let z denote the neighbour of b on P. Since $yz \in E(G)$, there is at least one of the edges b_3z, b_4z in G_k for some $k > 1$, since otherwise $\{b, z, y, b_3, b_4\}$ is an induced H

is G, a contradiction. Choose k smallest possible with this property. If by is eligible in G_k, then $b_3y \in E(G_k)$, which contradicts the fact that H is permanent. Since by is not eligible in G_k, there is a vertex $c \in V(G) \setminus N_G(xy)$ such that $bc \in E(G)$ and c is not in the component of $N_{G_k}(by)$ containing b_3, b_4, x and $V(P) \setminus \{b\}$. Then $b_3c \in E(G_k)$ or $b_4c \in E(G_k)$, since otherwise b is a center of a claw in G. But then by is eligible in G_k, a contradiction.

Case 2: Exactly one of the vertices x, y, say x, belongs to $V(H)$. Clearly $x \neq b$, since otherwise H is not induced in G_2. Up to symmetry suppose that $x = b_1$. Then $B^H_{xy} \subset \{b_2b, b_2x, bx\}$ and $V(B^H_{xy}) \subset \{b, b_2, x\}$. If $|B^H_{xy}| = 3$, then clearly $by \in E(G)$, $b_2y \in E(G)$, which implies that $\{y, b, b_2, x\}$ induces a claw in G, a contradiction.

Subcase 2.1: $b_2b \in B^H_{xy}$. Since $b \in N_G(xy)$, exactly one of the edges bx, by belongs to $E(G)$, since otherwise there is an induced H in G. Suppose that $bx \in E(G)$. Since xy is locally connected, there is a b_2, b-path P in $\langle N_G(xy) \rangle$. Choose P shortest possible and let z be the neighbour of b on P.

- If $xz \notin E(G)$, then $b_3z \in E(G)$ and $b_4z \in E(G)$, since otherwise b is a center of a claw in G.
- If $xz \in E(G)$, then there is at least one of the edges b_3z, b_4z in G_k for some $k > 1$, since otherwise $\{b, z, y, b_3, b_4\}$ is an induced H is G, a contradiction. Choose k smallest possible with this property.

Hence there is at least one of the edges b_3z, b_4z in G_k. If by is eligible in G_k, then $b_3y \in E(G_k)$, which contradicts the fact that H is permanent. Since by is not eligible in G_k, there is a vertex $c \in V(G) \setminus N_G(xy)$ such that $bc \in E(G)$ and c is not in the component of $N_{G_k}(by)$ containing b_3, b_4, x and $V(P) \setminus \{b\}$. Then $b_3c \in E(G_k)$ or $b_4c \in E(G_k)$, since otherwise b is a center of a claw in G. But then by is eligible in G_k, a contradiction. Suppose that $bx \notin E(G)$. Hence $by \in E(G)$. But then we are in a symmetric situation interchanging the roles of x and y, a contradiction.

Subcase 2.2: $b_2b \in E(G)$. Suppose that $bx \in E(G)$. Then $b_2x \in E(G)$, since otherwise $\{b, x, b_2, b_3\}$ induces a claw in G. But then $\{b, x, b_2, b_3, b_4\}$ induces an H in G, a contradiction. If $by \in E(G)$, then we are in a symmetric situation interchanging the roles of x and y, a contradiction again.

Case 3: None of the vertices x, y belongs to $V(H)$. If, up to symmetry, b_1 or b_2 belongs to $N_G(xy)$, then none of the vertices b_3, b_4 belongs to $N_G(xy)$ since otherwise H is not induced in G_2. Up to symmetry we can suppose that $V(H) \cap N_G(xy) \subset \{b, b_1, b_2\}$. If $B^H_{xy} = \{b_1b_2\}$, then b is a center of a claw in G, a contradiction. Hence $b \in V(B^H_{xy})$, implying that $b \in N_G(xy)$. Since G is H-free, the vertex b is a neighbour of exactly one of the vertices x, y. Up to symmetry suppose that $bx \in E(G)$.

Subcase 3.1 $|B^H_{xy}| = 3$. If x is a neighbour of all of the vertices b, b_1, b_2, then $\{x, b, b_1, b_2\}$ induces a claw in G, a contradiction. Hence x is a neighbour of exactly one of the vertices b_1, b_2. Up to symmetry suppose that $b_1x \in E(G)$, implying that $yb_2 \in E(G)$. Then $b_1y \in E(G)$, since otherwise x is a center of a claw. Since xy is locally connected, there is a b_1, b-path P in $\langle N_G(xy)\rangle$. Choose P shortest possible and let z be the neighbour of b on P.

- If $xz \notin E(G)$, then $b_3z \in E(G)$ and $b_4z \in E(G)$, since otherwise b is a center of a claw in G.
- If $xz \in E(G)$, then there is at least one of the edges b_3z, b_4z in G_k for some $k > 1$, since otherwise $\{b, z, y, b_3, b_4\}$ is an induced H is G, a contradiction. Choose k smallest possible with this property.

Hence there is at least one of the edges b_3z, b_4z in G_k. If bx is eligible in G_k, then $b_3x \in E(G_k)$, which contradicts the fact that H is permanent. Since bx is not eligible in G_k, there is a vertex $c \in V(G) \setminus N_G(xy)$ such that $bc \in E(G)$ and c is not in the component of $N_{G_k}(by)$ containing b_3, b_4, x and $V(P) \setminus \{b\}$. Then $b_3c \in E(G_k)$ or $b_4c \in E(G_k)$, since otherwise b is a center of a claw in G. But then bx is eligible in G_k, a contradiction.

Subcase 3.2: $|B_{xy}^H| = 2$. Suppose that $b_1b_2 \in B_{xy}^H$. If, up to symmetry, $b_1b \in E(G)$, then $b_1x \in E(G)$ since otherwise b is a center of a claw. But then $\{b, x, b_1, b_3, b_4\}$induces an H is G, a contradiction. Hence $b_1b_2 \in E(G)$, implying that $B_{xy}^H = \{b_1b, b_2b\}$. Since xy is locally connected, there is a b_1, b-path P in $\langle N_G(xy)\rangle$. Choose P shortest possible and let z be the neighbour of b on P.

- If $xz \notin E(G)$, then $b_3z \in E(G)$ and $b_4z \in E(G)$, since otherwise b is a center of a claw in G.
- If $xz \in E(G)$, then there is at least one of the edges b_3z, b_4z in G_k for some $k > 1$, since otherwise $\{b, z, y, b_3, b_4\}$ is an induced H is G, a contradiction. Choose k smallest possible with this property.

Hence there is at least one of the edges b_3z, b_4z in G_k. If bx is eligible in G_k, then $b_3x \in E(G_k)$, which contradicts the fact that H is permanent. Since bx is not eligible in G_k, there is a vertex $c \in V(G) \setminus N_G(xy)$ such that $bc \in E(G)$ and c is not in the component of $N_{G_k}(by)$ containing b_3, b_4, x and $V(P) \setminus \{b\}$. Then $b_3c \in E(G_k)$ or $b_4c \in E(G_k)$, since otherwise b is a center of a claw in G. But then bx is eligible in G_k, a contradiction.

Subcase 3.3: $|B_{xy}^H| = 1$. If $B_{xy}^H = b_1b_2$, then b is a center of a claw in G, a contradiction. Up to symmetry suppose that $B_{xy}^H = b_1b$. Then $b_1x \in E(G)$, since otherwise b is a center of a claw in G. But then $\{b, b_1, x, b_3, b_4\}$ induces an H is G, a contradiction.

□

The following example shows a class of C, H-free nontrivial graphs. Consider a graph G consisting of two cliques C_1, C_2 of arbitrary orders and of a matching M such that each edge of M has one end-vertex in C_1 and the other end-vertex in C_2.

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