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Welcome to Logic, Algebra and Truth Degrees 2010!

Mathematical Fuzzy Logic (MFL) is the subdiscipline of mathematical logic devoted to the study of formal systems of fuzzy logic. It has been a fairly active research field for more than two decades, since scholars undertook the task of providing solid formal foundations for deductive systems arising from Fuzzy Set Theory by realizing that these systems could be seen as a special kind of many-valued logics. This approach turned out to be very fruitful when Petr Hájek collected the results of the first systematic study of fuzzy logics in his monograph *Metamathematics of Fuzzy Logic* (Kluwer, 1998), a true landmark of the field. This book, together with other influential works by prominent researchers, was the start of an ambitious scientific agenda aiming to the study of all aspects of fuzzy logics, including algebraic semantics, proof systems, game-theoretic semantics, functional representation, first-order and higher-order logics, decidability and complexity issues, model theory, philosophical issues and applications. Moreover, it was made clear that these systems of fuzzy logic constitute a particular family of substructural logics, a central topic for the broad community of researchers studying non-classical logics. Since substructural logics typically enjoy good proof systems and algebraic semantics based on classes of residuated lattices, this connection shed great light on MFL and led to further avenues of research.

In order to promote and organize research in the field at the international level, the Working Group on Mathematical Fuzzy Logic (MathFuzzLog) was established under the auspices of the European Society for Fuzzy Logic and Technology (EUSFLAT) in September 2007. Since then it has grown and now includes almost all scholars working on MFL across the globe, with more than 90 members from more than 20 countries. It has its own web site (`http://www.mathfuzzlog.org`) which can be edited by any member of the group to share information and resources of common interest. Moreover, the group has promoted special issues on MFL in mainstream logic journals and organized specific workshops and special sessions at broader international conferences.

Among them, a central activity of the group is to organize its official conference Logic, Algebra and Truth Degrees (LATD) every two years. It is intended to be the main event of our community: besides being the conference where we all can meet and share the latest developments of our field, it also aims to bring together prominent researchers from neighboring fields, and become an excellent opportunity to enjoy fruitful discussion between researchers with different and complementary background. These goals were already achieved in the first LATD in Siena, Italy in September 2008. That conference was dedicated to Franco Montagna on the occasion of his 60th birthday, was attended by 65 participants, and included 8 invited lectures and 29 contributed talks in a single track.

Now we meet again, in the very historical center of Prague, for the second LATD. This time the conference is dedicated to Petr Hájek in the year of his 70th birthday. The numbers show its consolidation as the major event in MFL: over 65 participants, 8 invited lectures, 2 tutorials, and 35 contributed talks (some in parallel tracks). Again, invited lecturers include prominent senior experts along with outstanding young researchers from our field and its neighboring areas. Contributed talks will quite exhaustively showcase the various current trends of research in MFL and related topics. Furthermore, this time we will enjoy two alternative tutorial lectures (3 hours each) intended to introduce the participants to some specific topics in MFL. Another special part of the program will be the session on Wednesday afternoon devoted to our working group, where its goals, activities and future will be discussed. Those participants who are not members of the group are kindly invited to attend the session and even to join MathFuzzLog if they think that part of their research activities lie within its scope. In the social aspect of the program, we will enjoy a welcome party on Tuesday evening and the conference banquet on Friday evening.

We thank all the members of the program and organizing committees for their effort in preparing the conference and all the participants to be here, and we wish everybody a pleasant time in Prague and a fruitful conference.

Petr Cintula and Carles Noguera
Coordinators of MathFuzzLog

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Truth Degrees, Relevance, and Paraconsistency

Arnon Avron[*]

The intuition and use of degrees of truth is usually associated with fuzziness, and so with the family of logics known as "fuzzy logics". However, the idea is relevant, and has successfully been used, also in connection with other notions and families of logics. Our main goal in this talk is to describe some of its applications for paraconsistent logics and for relevance logics.

Our starting point is the logic $RMI_{\overset{\sim}{\to}}$. This logic is most naturally obtained from the sequent calculus for $R_{\overset{\sim}{\to}}$, the purely intensional (or "multiplicative") fragment of the relevant logic R, by viewing sequents as consisting of *finite sets of formulas* on both side of $\Rightarrow$ (rather than as multisets or sequences of formulas). This is equivalent to adding the converse of contraction (also known as "expansion" or "anti-contraction") to the usual sequential formulation of $R_{\overset{\sim}{\to}}$ (the latter has exchange and contraction, but not weakening, as its structural rules, and the usual multiplicative rules for negation, conjunction, and implication as its logical rules). Thus the logical rules of $RMI_{\overset{\sim}{\to}}$ for its multiplicative conjunction (or "fusion") are:

$$(\otimes \Rightarrow) \quad \frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \otimes \psi \Rightarrow \Delta} \qquad \frac{\Gamma_1 \Rightarrow \Delta_1, \varphi \qquad \Gamma_2 \Rightarrow \Delta_2, \psi}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, \varphi \otimes \psi} \quad (\Rightarrow \otimes)$$

The most important property of $RMI_{\overset{\sim}{\to}}$ is that unlike $R_{\overset{\sim}{\to}}$, it has a very useful and effective semantics. In fact, already in [1] it was shown that $RMI_{\overset{\sim}{\to}}$ is *weakly* sound and complete with respect to a certain sequence $\mathcal{A} = \{\mathcal{A}_n \mid n \in N\}$ of finite-valued matrices (where the number of truth-values in $\mathcal{A}_n$ is $n+2$). A sentence containing exactly k propositional variables is provable in $RMI_{\overset{\sim}{\to}}$ iff it is valid in $\mathcal{A}_k$, iff it is valid in all elements of $\mathcal{A}$. Moreover: although $RMI_{\overset{\sim}{\to}}$ has no finite (weakly) characteristic matrix, it has an effective infinite-valued matrix $\mathcal{A}_\omega$, for which it is again weakly sound

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and complete, and in which the matrices in $\mathcal{A}$ can all be embedded. The set of truth-values of $\mathcal{A}_\omega$ is $A_\omega = \{\mathbf{t}, \mathbf{f}, I_1, I_2, \ldots\}$. The set D_ω of designated values is $A_\omega - \{\mathbf{f}\}$, and the operations are defined as follows: $\neg\mathbf{t} = \mathbf{f}$, $\neg\mathbf{f} = \mathbf{t}$, $\neg I_i = I_i$, and $x \otimes y = min_{\preceq_\mathcal{A}}\{x, y\}$, where $\preceq_\mathcal{A}$ is defined by: $\mathbf{f} \preceq_\mathcal{A} \mathbf{t} \preceq_\mathcal{A} I_i$. $\mathcal{A}_n$ is then simply the submatrix of $\mathcal{A}_\omega$ induced by $A_n = \{\mathbf{t}, \mathbf{f}, I_1, I_2, \ldots, I_n\}$.

Now $RMI_{\tilde{\to}}$ is still very much a relevant logic, enjoying major properties like the variable-sharing property, and a very natural version of the relevant deduction theorem. However, all these properties are lost once one adds to $RMI_{\tilde{\to}}$ the "additive" (or extensional) conjunction $\wedge$ and disjunction $\vee$, together with the axioms and rules of R for these connectives. The resulting logic is known as the semi-relevant system RM, the most crucial property of which is the validity of the *linearity* axiom $(\varphi \to \psi) \vee (\psi \to \varphi)$. In fact, already $RM_{\tilde{\to}}$, the purely multiplicative fragment of RM, is an unconservative extension of $RMI_{\tilde{\to}}$ in which the variable-sharing property fails. A sound and complete cut-free sequential system $GRM_{\tilde{\to}}$ for this system is obtained by adding to that for $RMI_{\tilde{\to}}$ the following *combining rule:* from $\Gamma_1 \Rightarrow \Delta_1$ and $\Gamma_2 \Rightarrow \Delta_2$ infer $\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$.

It is well known that $RM_{\tilde{\to}}$ is sound and *weakly* complete for the three-valued matrix $\mathcal{A}_1$ described above. However, $\mathcal{A}_1$ is not sufficient for characterizing the *consequence relation* of $RM_{\tilde{\to}}$. Thus φ follows from $\varphi \otimes \psi$ in $\mathcal{A}_1$, although $\varphi \otimes \psi \nvdash_{RM_{\tilde{\to}}} \varphi$. In order to get a truly adequate semantics for $RM_{\tilde{\to}}$, it is necessary to introduce the idea of *degrees*. Let $\langle deg, \leq\rangle$ be some *linearly* ordered set. The *consistent matrix* based on $\langle deg, \leq\rangle$ is the matrix in which the set of truth-values is $deg \times \{\mathbf{t}, \mathbf{f}\}$, the set $\mathcal{D}$ of designated elements is $deg \times \{\mathbf{t}\}$, and the operations are defined by: $\neg\langle a, x\rangle = \langle a, \neg x\rangle$, and $X \otimes Y$ is $min_{\preceq_\otimes}\{X, Y\}$, where $\langle a, x\rangle \preceq_\otimes \langle b, y\rangle$ if either $a > b$ or $a = b$ and $x \leq y$ (here we assume that $\mathbf{f} < \mathbf{t}$). The *inconsistent matrix* based on $\langle deg, \leq\rangle$ is obtained from the consistent one by the addition of one extra element I such that $\neg I = I$, and $X \otimes I = I \otimes X = X$ for every $X \in \{I\} \cup (deg \times \{\mathbf{t}, \mathbf{f}\})$. We shall call a the *degree* of $\langle a, x\rangle$, and we define the degree of I to be some new object 0 which we take to be less than any element of *deg*. Let $Deg = deg \cup \{0\}$. $RM_{\tilde{\to}}$ is *strongly* sound and complete with respect to the inconsistent matrix $\mathcal{M}_0$ in which $Deg = [0, 1]$, and it is sound for every consistent or inconsistent matrix which is based on some linearly ordered set. This easily implies that $RM_{\tilde{\to}}$ is paraconsistent: $\neg p, p \nvdash_{RM_{\tilde{\to}}} q$ (assign I to p and some non-designated element to q).

In order to get from the above semantics for $RM_{\tilde{\to}}$ an adequate semantics for the full system RM, it suffices to follow the usual procedure: Given a matrix $\mathcal{M}$ constructed as above, let $X \leq Y$ in $\mathcal{M}$ iff $X \to Y \in \mathcal{D}$, where $X \to Y = \neg(X \otimes \neg Y)$. Then $\leq$ is a linear order on $\mathcal{M}$ (which can be characterized by: $\langle b, f\rangle < \langle a, f\rangle < I < \langle a, t\rangle < \langle b, t\rangle$ for every a, b such that $a < b$ in deg). Moreover: $X \to Y = inf_{\leq}\{Z \mid X \otimes Z \leq Y\}$. The desired semantics for which RM is sound and strongly complete is obtained by associating the connectives $\vee$ and $\wedge$ of RM with the lattice operations induced by $\leq$.

For comparison with the semantics of standard fuzzy logics, it is interesting to note that the extended matrix $\mathcal{M}_0$, for which RM is already strongly complete, is isomorphic to a matrix in which the set of truth-values is $[-1, 1]$, the set of designated values is $[0, 1]$, $\neg X = -X$, $X \vee Y = max\{X, Y\}$, $X \wedge Y = min\{X, Y\}$, and $X \otimes Y = min_{\preceq}\{X, Y\}$, where $X \preceq Y$ if either $|X| > |Y|$ or $|X| = |Y|$ and $X < Y$. Note that a similar matrix, with identical interpretations of $\neg$, $\vee$ and $\wedge$, can be associated with *Lukasiwicz* logic. The only differences are that the set of designated values there is $\{1\}$, while the interpretation of the intensional conjunction & of that logic is given by $X\&Y = max\{X + Y - 1, -1\}$.

Another aspect of similarity between RM and standard fuzzy logics is the proof-theoretical one. The main tool in the proof theory of fuzzy logics (see [4]) is the use of *hypersequential* calculi. This is true for RM too. In fact, already in [2] a sound and complete hypersequential calculus GRM was provided. It is obtained by adding to the obvious hypersequential version of $GRM_{\stackrel{\sim}{\to}}$ the hypersequential versions of the standard rules for the additive connectives, as well as the *splitting* rule: from $\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2 \mid G$ infer $\Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2 \mid G$. That this system admits cut elimination was shown in [2] using a very complicated syntactic proof. Recently a much simpler (and modular) *semantic* proof was found for the following *strong* cut elimination theorem (where for $G = \{\Gamma_i \Rightarrow \Delta_i | 1 \leq i \leq m\}$ we denote $\bigcup_{i=1}^{m}(\Gamma_i \cup \Delta_i)$ by $\mathcal{F}_G$):

Theorem: Let $\mathcal{S} \cup \{G\}$ be a finite set of hypersequents in the language of RM. Then either there is a proof in GRM of G from $\mathcal{S}$ in which all cuts are on formulas from $\bigcup_{H \in \mathcal{S}} \mathcal{F}_H$, or there is a model of $\mathcal{S}$ in $\mathcal{M}_0$ which is not a model of G. (In particular: A hypersequent is valid in $\mathcal{M}_0$ iff it has a cut-free proof in GRM.)

It should be noted that the modular character of the new semantic proof of this theorem might open the door to modular, non-deterministic semantics of fragments of RM.

As we said above, RM is paraconsistent, but it is not a relevance logic. To fully connect the idea of relevance with the idea of degrees, let us return to $RMI_{\underset{\rightarrow}{\sim}}$. It is easy to see that the example given above for the case of $RM_{\underset{\rightarrow}{\sim}}$ and $\mathcal{A}_1$ also shows that the characteristic matrix $\mathcal{A}_\omega$ of $RMI_{\underset{\rightarrow}{\sim}}$ is again not sufficient for characterizing the *consequence relation* of this logic. Again this can be remedied using the idea of degrees. In fact, this can be done using either of the two approaches described below.

The first approach retains the linearity of the set Deg of degrees (there is no need to distinguish between deg and Deg here). However, instead of using $deg \times \{\mathbf{t}, \mathbf{f}\}$ as the set of truth-values, we use $deg \times A_\omega$. The set $\mathcal{D}$ of designated values is now $deg \times D_\omega$. The operations are defined almost exactly as in the case of $RM_{\underset{\rightarrow}{\sim}}$. The only difference is that we use $\preceq_{\mathcal{A}}$ in the definition of $\preceq_\otimes$ instead of the order $\leq$ on $\{\mathbf{t}, \mathbf{f}\}$. Note that this time the partial order $\leq$ defined by $\rightarrow$ and $\mathcal{D}$ is *not* linear, but it does induce a lattice. As usual, the corresponding lattice operations can be used for adding $\wedge$ and $\vee$ to the language. However, the adjunction rule for $\wedge$ (from φ and ψ infer $\varphi \wedge \psi$) fails for the resulting system RMI (see [3]), even though most other properties that $\wedge$ and $\vee$ have in R are preserved. In fact, the resulting $\wedge$ serves as a *relevant* additive conjunction.

It can be shown that $RMI_{\underset{\rightarrow}{\sim}}$ and the full system RMI are strongly sound and complete with respect to the resulting class of matrices (for the full system RMI this has been shown already in [3]). In fact, both are again already strongly complete with respect to the matrix $\mathcal{M}_\omega$ obtained by taking the interval $[0, 1]$ as the underlying set of degrees. In addition, we have:

Theorem: There exists a sequence $\mathcal{S}$ of finite submatrices of $\mathcal{M}_\omega$, for which $RMI_{\underset{\rightarrow}{\sim}}$ is *finitely* strongly sound and complete. More precisely: if Γ is a finite set of formulas, φ is a formula, and the number of propositional variables in $\Gamma \cup \{\varphi\}$ is k, then $\Gamma \vdash_{RMI_{\underset{\rightarrow}{\sim}}} \varphi$ iff there is no element of $\mathcal{S}$ having less then $3k$ elements, in which all formulas of Γ are valid, but φ is not. This bound of $3k$ cannot be improved. Similar results hold for RMI.

The second approach to providing strongly sound and complete semantics for $RMI_{\underset{\rightarrow}{\sim}}$ and RMI is to retain in the consistent case the structure of $deg \times \{\mathbf{t}, \mathbf{f}\}$ from the above semantics of RM, but allows $\langle deg, \leq\rangle$ to be an inverse tree (rather than a chain). The definitions of $\mathcal{D}$,$\neg$,$\otimes$,$\rightarrow$, $\leq$,$\wedge$, and $\vee$ are then as before, except that now $X \otimes Y$ is $inf_{\preceq_\otimes}\{X, Y\}$ (rather than $min_{\preceq_\otimes}\{X, Y\}$). In the more general case it is possible (but not necessary) to replace both $\langle a, \mathbf{t}\rangle$ and $\langle a, \mathbf{f}\rangle$ by the single element $\langle a, I\rangle$ in any case in which a is a minimal degree. $\langle a, I\rangle$ is then designated, and $\neg\langle a, I\rangle = \langle a, I\rangle$. See [3] for more details.

From the proof-theoretical point of view, $RMI_{\tilde{\rightarrow}}$ and RMI again have strongly sound and complete hypersequential calculi, both admitting strong cut-elimination. These calculi are very similar to $GRM_{\tilde{\rightarrow}}$ and $GRMI$. The main difference is that in applications of the combining rule the two premises should share a common formula. Another difference is that the rules $(\Rightarrow \wedge)$ and $(\vee \Rightarrow)$ may be applied only if the internal context is not empty (thus one cannot infer $G \mid\Rightarrow \varphi \wedge \psi$ from $G \mid\Rightarrow \varphi$ and $G \mid\Rightarrow \psi$). Since this restriction can be imposed on GRM too, only the first difference is really significant.

To conclude, the moral of the above results is as follows. First, degrees of truth and truth-values are not necessarily the same thing. Second (and more important): the principle of comparability of degrees of truth or of truth values should be given up if questions of relevance are taken into consideration. Third: the construction of the semantics of RM and RMI indicates how the ideas of relevance can be combined with the idea of fuzziness: just use $deg \times [0, 1]$ instead of $deg \times \{\mathbf{t}, \mathbf{f}\}$ or $deg \times A_\omega$. This might open the door for new promising line of research.

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Fuzzy Modal Logics: A First Approach

*Félix Bou**

Classical modal logics were initially designed to deal with philosophical problems, but since the discovery of Kripke semantics they have found a lot of applications in some other fields, mainly in computer science (because Kripke models are nothing less than state transition systems). Briefly, classical modal logics can be considered as certain fragments of theories of first order classical logic; these fragments only contain formulas $\varphi(x)$ with one (and only one) free variable and are built expanding classical propositional logic with *bounded quantifiers*:

Universal Bounded Quantifier ($\Box$):	$\forall y(R(x,y) \rightarrow \varphi(y))$
Existential Bounded Quantifier ($\Diamond$):	$\exists y(R(x,y) \wedge \varphi(y))$

In latest years there has been a growing interest on studying *fuzzy modal logics*, and most of the papers published on the topic [4, 6, 8, 7, 2, 5, 3, 1] follow the pattern explained in the previous paragraph, i.e., considering fuzzy modal logics as the bounded fragments of first order fuzzy logics.

In this talk we will introduce the fuzzy modal logics $\Lambda(\mathsf{K}, \mathbf{A})$ (where K is a class of Kripke frames, and $\mathbf{A}$ is a BL chain) [1], and we will summarize what it is known (and what it is unknown) about these fuzzy modal logics. The main concern will be about completeness and decidability issues, where there are a lot of open questions.

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SAT in Monadic Gödel Logics: (Un)Decidability Results and Applications

Agata Ciabattoni[*]

Gödel logics $\mathbf{G}_V$ are among the main formal systems providing a foundation for reasoning in presence of vagueness. Defined in general over sets of truth values V which are closed subsets of $[0,1]$ containing both 0 and 1, $\mathbf{G}_V$ are the only many-valued logics which are completely specified by the *order structure* of V. This fact characterizes Gödel logics as logics of comparative truth and renders them an important case of so-called *fuzzy logics*, see [4].

In this talk I will describe some recent results about satisfiability in monadic Gödel logics $\mathbf{G}_V$, henceforth denoted by SAT_V^m.

Following [2], I will show that simple conditions on the topological type of the set of truth values V determine the decidability or the undecidability of SAT_V^m. The problem is decidable when 0 is an isolated point in V (i.e., 0 has Cantor-Bendixon rank $|0|_{\mathrm{CB}} = 0$). Satisfiability in these logics, that include $\mathbf{G}_V$ over any finite set $(\{0,1\} \in)V$, turns out to be equivalent to satisfiability in classical logic. In the remaining Gödel logics the presence of at least three predicate symbols, one of which is a constant different from 0 or 1, makes SAT_V^m undecidable. Furthermore without this constant predicate, SAT_V^m remains undecidable for all Gödel logics in which 0 is a limit point of limit points in V (i.e., $|0|_{\mathrm{CB}} \geq 2$). Standard Gödel logic, that is Gödel logic with truth values set $V = [0,1]$, being a prominent example.

The addition of the projection operator Δ enhances the expressive power of Gödel logics and their applicability. However, it renders SAT_V^m undecidable for all $\mathbf{G}_V$ with V infinite.

In [1] we identified a suitable fragment of monadic Gödel logics extended with Δ that is powerful enough to formalize important properties of fuzzy rule-based systems and whose satisfiability problem is decidable. This fragment – we refer to it as FO^1_{mon} – lies between the monadic and the one-variable

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fragment. SAT_V^m in FO^1_{mon} is shown to be NP-complete for all $\mathbf{G}_V$, even in presence of an additional, involutive negation $\sim$.

As proved in [3] the fragment FO^1_{mon} of standard Gödel logic extended with $\sim$ captures important features of the system CADIAG-2 (Computer-Assisted DIAGnosis). Developed at the Medical University of Vienna, CADIAG-2 is a well performing fuzzy expert system, "MYCIN-like", assisting in the differential diagnosis in internal medicine. A satisfiability check on the formulas formalizing the 20.000 rules of CADIAG-2 allowed the detection of some errors in its representation of the medical knowledge.

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Weakly Locally Finite MV-Algebras and Real-Valued Multisets

*Roberto Cignoli**

In [1] a duality was established between locally finite MV-algebras and multisets. A multiset was defined as a continuous mapping from a Boolean space into the lattice of generalized natural numbers equipped with the Scott topology. This lattice is isomorphic to the lattice of the subgroups of **Q** with 1 as a strong order unit, where **Q** denotes the additive group of the rational numbers endowed with the usual order [2]. The continuous maps between such generalized multisets are to be multiplicity-decreasing with respect to the divisibility order of generalized natural numbers.

The aim of this talk is to report joint work with Enzo Marra on the extension of these results to the class of MV-algebras that are *locally weakly finite*, i. e., such that all their finitely generated subalgebras split into a finite direct product of simple MV-algebras. Using the Scott topology on the lattice of subalgebras of the real unit interval $[0, 1]$ (regarded with its natural MV-algebraic structure), we construct a 'real-valued multiset' over the (boolean) space of maximal ideals of a locally weakly finite MV-algebra. Building on this, we obtain a duality for locally weakly finite MV-algebras that includes as a special case the above-mentioned duality for locally finite MV-algebras. We give an example that shows that the established duality via the Scott topology cannot be extended, without non-trivial modifications, to larger classes of algebras.

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Linearity Issues in the Algebra of Łukasiewicz Logic

Ioana Leuştean[*]

MV-algebras, the algebraic structures of Łukasiewicz ∞-valued logic, were defined by Chang in 1958 [3]. Their theory was developed extensively after 1986, when Mundici proved that *MV-algebras are categorically equivalent with abelian lattice-ordered groups with strong unit.* Consequently, MV-algebras are twofold structures: generalizations of boolean algebras and unit intervals of lattice-ordered groups with strong unit (ℓu-groups, shortly).

An MV-algebra [4] is a structure $(A, \oplus, ^*, 0)$, where $(A, \oplus, 0)$ is an abelian monoid, $(x^*)^* = x$, $0^* \oplus x = 0^*$ and $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$ for all $x, y \in A$. The real unit interval $[0, 1]$ equipped with the operations $x^* = 1-x$ and $x \oplus y = \min(1, x + y)$ is the *standard* MV-algebra, i.e. an equation holds in any MV-algebra if and only if it holds in $[0, 1]$. If we set $x \odot y = (x^* \oplus y^*)^*$ then $\odot$ is the Łukasiewicz t-norm on $[0, 1]$. Note that every MV-algebra A is a bounded distributive lattice, where $x \vee y = x \oplus (x \oplus y^*)^*$ and $x \wedge y = (x^* \vee y^*)^*$ for any x, $y \in A$.

The linearity issues in MV-algebra theory come up from two directions of research: states and the product operation.

States were defined in [15] as *averaging measures.* If A is an MV-algebra, a *state* is a function $s\colon A \to [0, 1]$ which is *normalized* (i.e. $s(1_A) = 1$) and *linear* (i.e. $s(x \oplus y) = s(x) + s(y)$ whenever $x \leq y^*$ in A). They generalize boolean probabilities as well as states on lattice-ordered groups. We shall not concentrate on the probabilistic interpretation of states in this talk, we refer instead to [17] for a comprehensive insight on this topic.

Another research direction was to characterize the class of structures generated by $[0, 1]$ in the language of MV-algebras enriched with the real product. The investigations led to the definition of *PMV-algebras* (*product*

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MV-algebras) [6, 13]. The analogue of Mundici's theorem for these structures was obtained by Di Nola and Dvurečenskij [6]: there exists a categorical equivalence between PMV-algebras and lattice-ordered rings with strong unit (ℓu-rings). In [14], Montagna axiomatized the quasivariety generated by $[0, 1]$ in the language of PMV-algebras.

It seems quite natural to introduce "modules" over such algebras. In [7], *MV-modules* are defined as structures corresponding through a categorical equivalence to lattice-ordered modules with strong-unit (ℓu-modules) over ℓu-rings. MV-modules over $[0, 1]$ will be called *Riesz MV-modules*, since they are unit intervals in Riesz spaces with strong unit u. PMV-algebras and MV-modules are particular cases of *MV-algebras with operators*, studied in [9], where Mundici's categorical equivalence for these structures is also proved. A *linear function* between two MV-algebras A and B is a map $\omega\colon A \to B$ satisfying $\omega(x) \leq \omega(y)^*$ and $\omega(x \oplus y) = \omega(x) \oplus \omega(y)$ whenever $x \leq y^*$ in A. An *operator* on an MV-algebra A is a linear function $\omega\colon A \to A$. A normalized operator is also called an *internal state* by Flaminio and Montagna in [8], where they introduce *SMV-algebras* as MV-algebras endowed with an internal state satisfying some additional properties.

These considerations show that, beyond the probabilistic interpretation, linear functions between MV-algebras do have a role to play. Also, the relation between rings (modules) and MV-algebras has to be further analyzed. It is worth noting that the logical roots of MV-algebras provide a logical insight on such topics that are not traditionally related to logic.

In this talk, we present some topics involving linearity issues in MV-algebra theory.

1. Łukasiewicz rings, introduced in [1], are exactly those commutative rings with the property that their lattice of ideals becomes an MV-algebra, with the Łukasiewicz t-norm defined by the product of ideals and the MV-algebraic negation defined by the annihilator. It turns out that an MV-algebra obtained in this manner is always complete and atomic. The relation between a Łukasiewicz rings and the correspondent MV-algebra of ideals is also expressed in categorical terms. All these results from [1] bring a different perspective on the role of rings in the algebra of Łukasiewicz logic and open further research directions.

2. State completion, developed in [12], is a standard construction: for every MV-algebra A, a state $s\colon A \to [0,1]$ determines a pseudo-metric $\rho_s(x,y) = s(d(x,y))$ for any x, $y \in A$, where $d(x,y) = (x \odot y^*) \oplus (y \odot x^*)$ is the Łukasiewicz distance. We say that A is s-complete if (A, ρ_s) is a Cauchy complete metric space. If A is a Riesz MV-algebra, then its s-completion is isomorphic with the unit interval of an L-space with strong unit; L-spaces are Banach lattices with additive norms (see, e.g., [5]). Using Kakutani's representation for L-spaces [10], we infer that for any s-complete Riesz MV-algebra A there exists a measure space (X, Ω, μ) such that A is isomorphic with the unit interval of $L_1(\mu)$. This result leads to a categorical equivalence between state complete Riesz MV-algebras and a particular class of metric spaces.

3. Tensor product is a classical construction in linear algebra. In MV-algebra theory, the tensor product was defined by Mundici [16] for two MV-algebras and generalized in [11] to arbitrary families of MV-algebras. This allows us to define the *tensor algebra* T_A associated to a given MV-algebra A, i.e. T_A is the tensor product of the family $\{A_n\}$, where $A_n = A$ for any n. It is straightforward to define a product such that T_A becomes a PMV-algebra, in which A is embedded. Following the classical theory, this construction yields an adjunction between MV-algebras and PMV-algebras.

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n-Contractive BL-Logics

Franco Montagna[*]

The logic BL has been introduced by Hájek in his book [12] both as a common fragment of the most important fuzzy logics and as the logic of continuous t-norms. The continuity of the t-norm is reflected by the divisibility axiom $(\phi \wedge \psi) \to (\phi \& (\phi \to \psi))$. Although BL is a contraction-free logic, divisibility is implied by (but it is not equivalent to) contraction, and hence it may be regarded as a weak form of contraction. The axiom (S) of strict negation, $\neg(\neg\phi \wedge \neg\neg\phi)$, can also be regarded as a form of contraction, because it is equivalent to $\neg\phi \to (\neg\phi \& \neg\phi)$, i.e., contraction for negated formulas. An even more evident form of contraction is the schema $\phi^n \to \phi^{n+1}$, which will be called n-contraction and will denoted by C_n. Given any propositional fuzzy logic L, C_nL will denote the extension of L with the axiom schema C_n. Extensions of fuzzy logics by the schema C_n, and C_nMTL in particular, have been investigated first in [9]. Yet another principle, whose meaning is more algebraic than proof-theoretic, but which implies some additional interesting proof-theoretic properties is the principle

$$(D_n) \qquad (\phi^{m-1} \leftrightarrow (\phi \to \phi^n)) \to \phi^n$$

for all m which does not divide n. The logic C_nBL added with (D_n) will be denoted by DC_nBL.

While the totally ordered models of C_nBL are precisely the ordinal sums of MV-chains with $n+1$ elements at most, the totally ordered models of DC_nBL are precisely the ordinal sums of MV-chains with $m+1$ elements, where m divides n.

The finite embeddability property for BL easily implies that for every finite set $\Gamma \cup \{\phi\}$ of formulas, the following are equivalent:

(1) $\Gamma \vdash_{BL} \phi$.

(2) For every $n > 1$, $\Gamma \vdash_{C_nBL} \phi$.

(3) For every $n > 0$, $\Gamma \vdash_{DC_nBL} \phi$.

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Hence, in order to check provability of formulas of fixed complexity, it is sufficient to check provability in C_nBL or DC_nBL for n sufficiently large. However, C_nBL and especially SC_nBL have many other good properties which are not enjoyed by BL. Most of them have been obtained in two papers in collaboration with Matteo Bianchi [2] and [3], and are listed below.

- Decision problems, countermodels: for every n, k, there are models $\mathbf{C}^i_{n,k}$ $(i = 1, ..., n)$ and $\mathbf{S}_{n,k}$ with cardinality $(k+1)\frac{n(n+1)}{2}$ and $(k+1)(n+1)$ respectively, such that A formula ϕ with k variables is provable in C_nBL iff it is true in $\mathbf{C}^i_{n,k}$ for $i = 1, ..., n$ and ϕ is provable in DC_nBL iff it is true in $\mathbf{S}_{n,k}$. Hence, C_nBL and SC_nBL are co-NP complete. Similar (and stronger) results hold for SC_nBL and for SDC_nBL, that is, for C_nBL and DC_nBL added with the axiom (S) of strict negation.
- Strong completeness, universal chains: SC_nBL, DC_nBL and SDC_nBL have a universal chain, that is, a chain for the logic in which all countable chains for the logic embed. The same is not true of C_nBL with $n > 1$. Hence, SC_nBL, DC_nBL and SDC_nBL are strongly complete with respect to a single chain of the logic.
- Interpolation: for $n > 1$, none of C_nBL, DC_nBL, SDC_nBL, SDC_nBL has Craig interpolation. Moreover, DC_nBL and SDC_nBL have deductive intepolation, while C_nBL and SC_nBL don't.
- Completions: a variety $\mathcal{V}$ of BL-algebras is closed under completion if every algebra in $\mathcal{V}$ embeds into a complete algebra in $\mathcal{V}$. Then, as shown by Busanice and Cabrer [7], $\mathcal{V}$ is closed under completions iff, for some n it is a subvariety if the variety of BL-algebras satisfying the equation $x^n = x^{n+1}$.
- Complexity of the semantics by complete chains: there are no models on $[0, 1]$ for C_nBL or for extensions of C_nBL, with the exception of C_1BL which coincides with Gödel logic. Hence, if $n > 1$, none of the logic examined so far is sound and complete with respect to the standard semantics. However, it makes sense to investigate completeness of the first order version L$\forall$ of a logic L$\in \{C_nBL, DC_nBL, SC_nBL, SDC_nBL\}$ with respect to the semantics given by complete chains. It turns out that an extension L of BL is complete with respect to its complete chains iff it extends C_nBL. It follows: If L is a recursively axiomatizable extension of BL for some n, then the class of L$\forall$ tautologies for the class of complete L-chains is recursively enumerable iff L extends C_nBL for some n.

- Supersoundness: the interpretation of first order formulas of a first order fuzzy logic L∀ is restricted to *safe* structures, that is, to structures in which every formula has a truth value (this requirement is not automatically satisfied: quantifiers are interpreted as suprema and infima, which need not exist in the structure). One may wonder what happens if we also consider interpretations which are not safe, but in which the formula taken into consideration has a truth value. That is, a formula ϕ of L∀ is called *supersound* if it is true not only in every safe structure for L∀, but also in every structure in which the truth value of ϕ is defined. A logic L∀ is said to be *supersound* if every theorem of L∀ is supersound.
  Once again, it turns out that if L is an extension of BL, then L∀ is supersound iff L extends C_nBL.

Summing up, contractive BL-logics constitute good approximations of BL. These logics have important advantages with respect to BL, both from a computational and from an algebraic point of view.

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Complete Ideal Completions of Residuated Lattices and Completeness of Substructural Predicate Logics

*Hiroakira Ono**

Completions of lattice-ordered algebras have been studied in many literatures on lattices and order. From a logical point of view, *regular* completions of residuated lattices will sometimes play an important role, as they often provide us algebraic completeness of substructural predicated logics. Here, we say that a completion is regular, when it is a completion with a regular embedding, an embedding preserving all existing joins and meets. (See e.g. [7].)

MacNeille completions are examples of regular completions. By using them, we can show the algebraic completeness of minimum predicate extensions of standard substructural propositional logics. On the other hand, it is known that MacNeille completions do not always preserve distributivity.

Now we will focus our attention on *complete ideal completions* of residuated lattices. The complete ideal completion $\mathbf{A}^K$ of a given residuated lattice $\mathbf{A}$ is an algebra consisting of all complete ideals of $\mathbf{A}$. Here an ideal I is said to be complete when the join $\bigvee S$, if exists, belongs to I for every nonempty subset S of I (see [1, 2]). It can be shown that complete ideal completions are always regular and moreover preserve infinite join distributivity. As a logical consequence, the algebraic completeness is shown for some of *distributive* substructural predicate logics with the axiom scheme $(\wedge, \exists):\ \exists x\varphi(x) \wedge \psi \to \exists x(\varphi(x) \wedge \psi)$.

In the present talk, we will show a preservation theorem under the mapping h from the set of all ideals to the set of all complete ideals such that $h(I) = K(I)$, where $K(I)$ denotes the complete ideal generated by I. The theorem implies a general result on the algebraic completeness of minimum

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predicate extensions with the axiom scheme $(\wedge, \exists)$ of distributive substructural propositional logics.

We will discuss also a close connection between ideal completions and Kripke-type semantics for substructural propositional logics, introduced in [10] and [3, 4], and also between complete ideal completions and Kripke-type semantics for substructural predicate logics, introduced in [6]. The former is already mentioned in [8], and the latter will be demonstrated by using Kripke-Joyal semantics in [5].

Complete ideal completions and their logical consequences will be discussed further in our forthcoming paper [9].

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Admissible Rules and the Leibniz Hierarchy

James Raftery[*]

Many researchers have asked: to what extent can we interpret a logic plausibly in its own meta-language? Disjunction properties are one manifestation of this concern. A problem in the converse spirit is the derivability of admissible rules. A rule is *admissible* in a formal system if its addition to the system produces no new theorems [7, 16]. A simple example is the rule of necessitation, $x \vdash \Box x$, which is admissible (and not derivable) in quasi-normal modal logics. More profoundly, the process of cut elimination shows that underivable cut rules are admissible in suitable sequent calculi.

Semantic treatments of admissibility cannot rely entirely on the theory of *algebraizable* logics [3]. For already, the quasi-normal modal systems and the cut-free subsystems of substructural logics are not algebraizable. Instead, in this talk, we analyze admissibility for systems at various levels of the *Leibniz hierarchy*, using the tools of abstract algebraic logic [4, 6, 10]. A sample of results follows.

Fortunately, *every* deductive system $\vdash$ has a nontrivial semantics, $\mathsf{Mod}^*(\vdash)$, comprising its *reduced matrix models*,[1] and we can already prove:

Theorem 1. *The following conditions on $\vdash$ and $\langle \Gamma, \alpha \rangle$ are equivalent.*

(i) *$\langle \Gamma, \alpha \rangle$ is an admissible rule of $\vdash$.*

(ii) *Every matrix model of $\vdash$ is a homomorphic image of one that validates $\langle \Gamma, \alpha \rangle$.*

(iii) *Every reduced matrix model of $\vdash$ is a homomorphic image of a subdirect product $\mathcal{A}$ of reduced matrix models of $\vdash$, where $\mathcal{A}$ validates $\langle \Gamma, \alpha \rangle$.*

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[1] A matrix $\langle \boldsymbol{A}, F \rangle$ is *reduced* if no congruence of $\boldsymbol{A}$ turns F into a union of congruence classes, except for the identity congruence [18].

The surjective homomorphisms in (ii) and (iii) are understood to preserve the set of designated elements, but they need not reflect this set. Of course, (iii) looks better in systems where the reduced matrix models are closed under subdirect products. These are exactly the *protoalgebraic* systems, i.e., the ones for which the *Leibniz operator* $\boldsymbol{\Omega}$ is *isotonic* in the following sense: whenever $\langle \boldsymbol{A}, F\rangle$ and $\langle \boldsymbol{A}, G\rangle$ are matrix models of $\vdash$, with $F \subseteq G$, then $\boldsymbol{\Omega}^{\boldsymbol{A}} F \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}} G$. [2] In sentential logics, this amounts to the existence of a set $\boldsymbol{\rho}$ of binary formulas such that $\vdash\ \boldsymbol{\rho}(x,x)$ and $x,\ \boldsymbol{\rho}(x,y) \vdash y$ [2].

Corollary 2. *A rule* $\langle \Gamma, \alpha\rangle$ *is admissible in a protoalgebraic system* $\vdash$ *iff every reduced matrix model of* $\vdash$ *is a homomorphic image of one that validates* $\langle \Gamma, \alpha\rangle$.

Example 3. The quasi-normal modal system $\mathbf{S4}^{\mathrm{MP}}$ is not algebraizable, but it is protoalgebraic. If we add the inference rule $x \vdash \Box x$ to $\mathbf{S4}^{\mathrm{MP}}$, we get a familiar system for $\mathbf{S4}$, whose reduced matrix models are just the pairs $\langle \boldsymbol{A}, \{1\}\rangle$ where $\boldsymbol{A}$ is an interior algebra with greatest element 1. The reduced matrix models of $\mathbf{S4}^{\mathrm{MP}}$ itself are the pairs $\langle \boldsymbol{A}, F\rangle$ where $\boldsymbol{A}$ is an interior algebra and F a lattice filter of $\boldsymbol{A}$ containing no $\Box$–closed lattice filter other than $\{1\}$. Thus, the identity map $a \mapsto a$ makes $\langle \boldsymbol{A}, F\rangle$ a homomorphic image of $\langle \boldsymbol{A}, \{1\}\rangle$, witnessing Corollary 2's criterion for admissibility of the necessitation rule in an extremely simple way. (It is much harder to see Theorem 1 at work in proofs of cut elimination.)

In the Leibniz hierarchy, there are two independent ways of strengthening protoalgebraicity. We confine the present discussion to sentential systems.

(I) *Weak algebraizability* [5] adds the demand that the *truth predicate* of $\vdash$ be *equationally definable* in $\mathsf{Mod}^*(\vdash)$, i.e., for a suitable set $\boldsymbol{\tau}$ of pairs $\tau = \langle \tau_\ell(x), \tau_r(x)\rangle$ of unary formulas, every reduced matrix model $\langle \boldsymbol{A}, F\rangle$ of $\vdash$ satisfies $x \in F \iff \&_{\tau\in\boldsymbol{\tau}}\ \tau_\ell(x) = \tau_r(x)$.

(II) *Equivalentiality* [13] asks that the Leibniz operator be definable, i.e., for a suitable set $\boldsymbol{\rho}$ of binary formulas, every matrix model $\langle \boldsymbol{A}, F\rangle$ of $\vdash$ satisfies $\langle x, y\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}} F \iff \boldsymbol{\rho}^{\boldsymbol{A}}(x,y) \subseteq F$. This facilitates a smooth generalization of the Lindenbaum-Tarski construction, with $\boldsymbol{\rho}$ in the role of the bi-conditional $\leftrightarrow$. We therefore call $\boldsymbol{\rho}$ a set of *equivalence formulas* for $\vdash$, and $\vdash$ is said to be *finitely equivalential* if it has a finite set of equivalence formulas.

[2] $\boldsymbol{\Omega}^{\boldsymbol{A}} F$ is the largest congruence of $\boldsymbol{A}$ for which F is a union of congruence classes.

A further specialization of (II) is

(III) *Order algebraizability* [14, 15]: This is characterized by the existence of binary formulas $\boldsymbol{\rho}$ such that the relation $x \leqslant_F y$ iff $\boldsymbol{\rho}(x, y) \subseteq F$ defines a partial order on every reduced matrix model $\langle \boldsymbol{A}, F\rangle$ of $\vdash$, and

$$x \dashv\vdash \bigcup \{\boldsymbol{\rho}(\tau_\ell(x), \tau_r(x)) : \tau = \langle \tau_\ell, \tau_r\rangle \in \boldsymbol{\tau}\}$$

for a suitable $\boldsymbol{\tau}$. The displayed rule ensures that the designated elements of a reduced matrix model are also definable in terms of the partial order, and that the two definitions essentially invert each other.

In (III), by the $\boldsymbol{\rho}$*–ordered algebras* of $\vdash$, we mean the structures $\langle \boldsymbol{A}, \leqslant_F\rangle$ obtained from all reduced matrix models $\langle \boldsymbol{A}, F\rangle$ of $\vdash$. Then Corollary 2 becomes

Theorem 4. *In a* $\boldsymbol{\rho}$*–order algebraizable system* $\vdash$*, a rule* $\langle \Gamma, \alpha\rangle$ *is admissible iff every* $\boldsymbol{\rho}$*–ordered algebra of* $\vdash$ *is a homomorphic image of one in which*

$$\Big(\&_{\gamma\in\Gamma,\ \tau\in\boldsymbol{\tau}}\ \tau_\ell(\gamma) \leqslant \tau_r(\gamma)\Big) \implies \Big(\&_{\tau\in\boldsymbol{\tau}}\ \tau_\ell(\alpha) \leqslant \tau_r(\alpha)\Big)$$

is valid, where $\boldsymbol{\tau}$ *is as in* (III).

Here, the surjective homomorphism is understood to preserve the partial order (as well as all operations), but it need not reflect the order.

Example 5. Let $\mathbf{R}_S$ denote the S–fragment of the relevance logic $\mathbf{R}$, formulated with the Ackermann truth constant t. Slaney and Meyer [17] proved that every finite rule admissible in $\mathbf{R}_{\rightarrow,\wedge}$ is derivable there, i.e., $\mathbf{R}_{\rightarrow,\wedge}$ is *structurally complete* in the sense of [11]. They asked whether the same could be said of $\mathbf{R}_{\rightarrow,\mathsf{t}}$ or $\mathbf{R}_{\rightarrow}$. It is convenient to work with $\mathbf{R}_{\rightarrow,\circ,\mathsf{t}}$ (where $\circ$ is fusion). For, although $\mathbf{R}_{\rightarrow,\circ,\mathsf{t}}$ is not algebraizable, it is $\boldsymbol{\rho}$–order algebraized by the easily described class of Church monoids, where $\boldsymbol{\rho}$ is $\{x \rightarrow y\}$. The corresponding $\boldsymbol{\tau}$ is $\{\langle \mathsf{t}, x\rangle\}$ (to be replaced by $\{\langle x \rightarrow x, x\rangle\}$ in the t–free fragments). A *Church monoid* is a commutative residuated po-monoid $\langle A; \circ, \rightarrow, \mathsf{t}, \leqslant\rangle$ satisfying $x \leqslant x\circ x$. It can be shown that every Church monoid is a homomorphic image of one that satisfies $\mathsf{t} \leqslant x \rightarrow \mathsf{t}\ \&\ \mathsf{t} \leqslant (x \rightarrow \mathsf{t}) \rightarrow \mathsf{t} \implies \mathsf{t} \leqslant x$. So, by Theorem 4, the rule $\langle\{x \rightarrow \mathsf{t},\ (x \rightarrow \mathsf{t}) \rightarrow \mathsf{t}\},\ x\rangle$ is admissible in each of $\mathbf{R}_{\rightarrow,\circ,\mathsf{t}}$ and $\mathbf{R}_{\rightarrow,\mathsf{t}}$. It is underivable even in the stronger system of classical logic. These conclusions were drawn in [9], using a detour through an algebraizable conservative extension of $\mathbf{R}_{\to,\circ,\mathsf{t}}$. But no such detour is needed, as Theorem 4 prescribes nothing more than order algebraizability.

Using a suitable Gentzen system with cut elimination, we can easily show that the rule $\langle\{x \circ y\},\, x\rangle$ is admissible in $\mathbf{R}_{\to,\circ,\mathsf{t}}$ (where it is not derivable). This, with Theorem 4, yields an un-obvious algebraic result:

Corollary 6. *Every Church monoid is a homomorphic image of one that satisfies* $\mathsf{t} \leqslant x \circ y \implies \mathsf{t} \leqslant x$.

These observations leave open the question of structural completeness for $\mathbf{R}_{\to}$, a.k.a. **BCIW**. A partial result can be stated, concerning the rule

(†) $\quad\langle\{(x \to (x \to x)) \to x\},\, x\rangle,$

which is underivable in **BCIW**.

Theorem 7. *Either* (†) *is admissible in* **BCIW** *or every admissible one-variable rule of* **BCIW** *is derivable in* **BCIW**.

As in the algebraic theory of (quasi-) varieties, every reduced matrix model of a finitary deductive system $\vdash$ is isomorphic to a subdirect product of reduced matrix models of $\vdash$ that are *relatively subdirectly irreducible* (RSI) with respect to $\mathsf{Mod}^*(\vdash)$. The RSI reduced matrix models $\langle \boldsymbol{A}, F\rangle$ are characterized by the demand that F be completely meet-irreducible in the $\vdash$*–filter* lattice of $\boldsymbol{A}$, i.e., the lattice of subsets G for which $\langle \boldsymbol{A}, G\rangle$ is still a matrix model of $\vdash$ [19].

Theorem 8. *If* $\vdash$ *is finitary and protoalgebraic, and if every finitely generated RSI reduced matrix model of* $\vdash$ *is projective (in the category* $\mathsf{Mod}^*(\vdash)$*, equipped with all matrix homomorphisms), then* $\vdash$ *is hereditarily structurally complete.*

(This means that $\vdash$ and each of its finitary extensions is structurally complete.)

When $\vdash$ is weakly algebraizable, its reduced matrix models are determined by their algebra reducts. Let $\mathsf{Alg}(\vdash)$ be the class of all these reducts. If $\vdash$ is algebraizable, then $\mathsf{Alg}(\vdash)$ will be its *equivalent algebraic semantics* in the sense of [3], but we do not need the full force of algebraizability to infer:

Corollary 9. *Suppose that* $\vdash$ *is finitary and weakly algebraizable. If every finitely generated relatively subdirectly irreducible algebra in* $\mathsf{Alg}(\vdash)$ *is projective in* $\mathsf{Alg}(\vdash)$*, then* $\vdash$ *is hereditarily structurally complete.*

Example 10. The extension of $\mathbf{R}$ by the *mingle* axiom $x \to (x \to x)$ is not structurally complete, but its negation-less fragment $\vdash$ (i.e., its $\to, \circ, \wedge, \vee, \mathsf{t}$ fragment) is hereditarily structurally complete. This cannot be proved by generalizing 'Prucnal's trick' from [12]. But the algebraic criterion of Corollary 9 can be applied. Indeed, $\vdash$ is algebraized by the locally finite variety of *positive Sugihara monoids* (PSMs) and it is proved in [8] that every finite subdirectly irreducible PSM is projective in this variety.

Andrzej Wroński [20] has drawn attention to the significance of *overflow rules*. For present purposes, we take these to be finite rules of the form $\langle \{\alpha_1, \dots, \alpha_n\}, y \rangle$, where y is a variable not occurring in any of $\alpha_1, \dots, \alpha_n$. The main theorem of [20] (and an earlier result of Bergman [1]) can be generalized as follows:

Theorem 11. *Let $\vdash$ be a finitary, finitely equivalential deductive system with at least one theorem and at least one non-theorem.*

Then every admissible overflow rule of $\vdash$ is derivable iff, in the first order language (with equality) of $\mathsf{Mod}^*(\vdash)$, *every existential positive sentence holds either in* all *of the nontrivial reduced matrix models of $\vdash$, or in* none *of them.*

Example 12. Let $\vdash$ be an axiomatic extension of the S–fragment of $\mathbf{FL_{ew}}$ (a.k.a. 'BCK–logic' or affine linear logic), where S includes at least $\to$ and $\bot$. If $\vdash$ is structurally complete, then $\mathsf{Alg}(\vdash)$ contains no simple algebra on more than 2 elements that is n–contractive for a finite n. In particular, $\mathsf{Alg}(\vdash)$ contains no *finite* simple algebra other than the 2–element Boolean algebra. The proof uses Theorem 11 and the existential positive sentence

$$\exists\, x\, (x \to^n \bot = \top \;\;\&\;\; (x \to \bot) \to x = \top)$$

where $\top$ is $\bot \to \bot$. This rules out structural completeness—and even 'overflow-completeness'—for a large class of fuzzy logics.

Let $\boldsymbol{Fm}$, T and *Var* be the algebra of formulas, the set of theorems, and the infinite set of variables of $\vdash$, and let $\boldsymbol{\mathcal{L}}^*$ abbreviate the *free* reduced matrix model $\langle \boldsymbol{Fm}/\boldsymbol{\Omega}T, T/\boldsymbol{\Omega}T \rangle$. The next theorem strengthens a result of Prucnal and Wronski [13].

Theorem 13. *A* [*finitely*] *equivalential finitary system $\vdash$ has the property that all of its admissible* [*finite*] *rules are derivable iff* $\mathsf{Mod}^*(\vdash)$ *is exactly* $\mathrm{UISP}(\boldsymbol{\mathcal{L}}^*)$ [*resp.* $\mathrm{ISPP_U}(\boldsymbol{\mathcal{L}}^*)$ *in the finite case*].

Here, I, S, P and P_U stand, as usual, for closure under isomorphisms, substructures, products and ultraproducts, while U is defined by

$$U(\mathsf{K}) := \{\mathcal{A} : \text{ every substructure of } \mathcal{A} \text{ on} \leqslant |\mathit{Var}| \text{ generators belongs to } \mathsf{K}\}.$$

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Tutorials

Proof Theory for Fuzzy Logics

*George Metcalfe**

Fuzzy logics are motivated primarily by semantic considerations, in particular, by the need to represent and reason about truth degrees. However, as logics, they clearly also have something to do with syntax and proof. This much is apparent already from their Hilbert-style axiomatizations, which (with enough patience) produce every theorem of a logic using a few simple rules and a possibly long list of axiom schema. But while Hilbert systems are convenient for presenting a wide range of logics directly related to classes of algebras, they are not so flexible when it comes to searching for, analyzing, and reasoning about proofs. At each step in a Hilbert system proof, the next axiom or instance of a rule like modus ponens must be guessed. Much better are proof systems with more restrictions on how to proceed – ideally, systems where proofs are analytic: built from the raw material (subformulas) of the formula proved. Such systems and their applications are the subject of this tutorial.

Surprisingly perhaps, most fuzzy logics investigated in the literature have a natural proof-theoretic formulation. They occur (alongside relevant logic, linear logics, and the full Lambek calculus) as substructural logics in the framework of Gentzen systems. Typically, Gentzen systems gain flexibility by dealing with sequents: ordered pairs of sequences (or sets or multisets) of formulas interpreted as implications. Sequent calculi provide a natural home for a diverse selection of logics spanning linguistics, philosophy, computer science, and mathematics, as well as corresponding directly to important classes of algebras. For fuzzy logics, however, sequents are not quite enough. A step up in complexity is required to hypersequents: multisets of sequents interpreted as disjunctions of implications. Many fuzzy logics are then obtained simply by transferring sequent calculi to the hypersequent level and adding an extra rule (or two). For example, a hypersequent calculus for Gödel logic is obtained by allowing hypersequent contexts in Gentzen's sequent calculus for intuitionistic logic and adding a rule permitting "communication" between sequents.

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The goal of this tutorial is not only to show how fuzzy logics can be presented proof-theoretically, but also to explain why proof theory matters for this field. What can be done with analytic proof systems that cannot be achieved with algebraic approaches? For many substructural logics, a standard answer is that such systems are essential to establishing algorithmic properties like decidability, complexity, and interpolation. In particular, the only known proofs of decidability for the full Lambek calculus (even at the first-order level) and, equivalently, the equational theory of residuated lattices, make use of Gentzen systems. For hypersequents, the situation is not so straightforward. Decidability and complexity results can be obtained in certain cases, but often the mere existence of an analytic calculus does not help much. So why bother? This tutorial will focus on two important applications. First, proof theory can be used to tackle one of the central problems in the field: standard completeness, showing that a logic is complete with respect to the intended fuzzy semantics, or, equivalently, to showing that a variety of algebras is generated by certain distinguished members. Roughly, the idea is to add a special density rule to a logic that guarantees standard completeness and then to show proof-theoretically that it can be eliminated. For example, this method provides the only known proof of the standard completeness of uninorm logic, or, equivalently, the generation of the variety of semilinear bounded commutative residuated lattices by its members with lattice reduct $[0,1]$. The second area of application is the extension of propositional fuzzy logics to the first-order level, a step that is somewhat problematic algebraically but completely natural for Gentzen systems. The proof-theoretic presentation facilitates investigation of key topics such as Herbrand's theorem and Skolemization, as well as providing a means for tackling (fragments of) non recursively axiomatizable cases such as first-order Łukasiewicz logic.

The tutorial will assume some familiarity with fuzzy logics but no knowledge of proof theory. Rather, the need for particular structures in proof systems will be motivated incrementally, starting with simple examples such as lattices where inequations suffice, then continuing to distributive lattices, classical logic, and substructural logics like the full Lambek calculus where sequents are required, and finally to the case of fuzzy logics and hypersequents. The focus throughout will be on what the proof-theoretic approach has to offer a researcher working in the field of mathematical fuzzy logic. Proofs of key theorems such as cut elimination and density elimination will be given for simple but instructive cases, many more technical and historical details, may be found in the monograph [1] and the forthcoming handbook chapter [2].

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Mathematical Fuzzy Logic in Linguistic Semantics

Vilém Novák[*]

In this paper, we will focus on possible ways how mathematical fuzzy logic (MFL) can be utilized when dealing with linguistic semantics. We believe that MFL can bring new tools both to formal as well as computational semantics [1, 12] because it enables to join classical logical tools used in linguistics with models of vagueness that is inherently present in the meaning of natural language expressions.

It should be emphasized that MFL is first of all the *logic* and so, it is important to apply its power in the development of a mathematical model of linguistic semantics on the basis of *logical analysis* of the latter. One of nice possibilities are methods of the logical analysis of concepts, as developed, e.g. by P. Materna in [3, 4]. An interesting source of inspiration can be found also in Montague grammar (cf. [2]).

A useful linguistic system is *Functional Generative Description* of natural language developed by P. Sgall, E. Hajičová and J. Panevová in [13] and elsewhere. According to this theory, natural language is described on the basis of functional approach. Namely, the description of a sentence is understood as a sequence of its representations on certain ordered levels. The lowest level stresses the outer (sound) form of the sentence while the highest one represents its meaning. Five levels are differentiated, namely *phonetic* (PH), *phonemic* (PM), *morphemic* (MR), *surface syntactic* (SS) and *tectogrammatical* TR). We will show how MFL can be applied in connection with the latter.

The meaning of sentence on tectogrammatical level is represented as a dependency tree the root of which represents verb, nodes represent word shapes and edges represent dependency relations. Labels of nodes in a dependency tree are word forms (complex units) that consist of elements of the following categories:

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1. Lexical unit; the basic lexical units are *nouns, adjectives, adverbs* and *verbs*.
2. Information about membership in the topic or focus.
3. *Grammatemes* (morphological meanings) of several categories for each class of words.

Using mathematical fuzzy logic, we can develop mathematical model of the meaning of some special lexical units and special expressions (syntagms). Till now, the most elaborated is the theory of the meaning of *evaluative linguistic expressions* which are expressions such as *small, medium, big, twenty five, roughly one hundred, very short, more or less strong, not very tall, about twenty five, roughly small or medium, very roughly strong*, etc. They are interesting for MFL because their meaning is fundamental bearer of the vagueness phenomenon and, at the same time, a consequence of the indiscernibility between objects which can be modeled by a fuzzy equality. Evaluative expressions are related with *evaluative predications* which are linguistic syntagms such as *weight is small, pressure is very high, extremely rich person*, etc. The verb "is" is here usually taken as a copula joining an object with its property. In MFL, we can construct semantics of evaluative predications either as special expressions having its intension, or as propositions having a truth value in a given model.

The most convenient constituent of MFL suitable for construction of models mentioned above is higher-order fuzzy logic, namely the *fuzzy type theory* (FTT) because it has great explicative power and relatively simple syntax which is in linguistic theory widely used. In general, FTT can be seen as extension of classical type theory which has many valued semantics. This is accomplished by replacing the basic axiom saying "there are just two truth values" by a set of axioms characterizing the assumed structure (algebra) of truth values. Though there are several kinds of FTT developed depending on the algebra of truth values (IMTL, BL, Łukasiewicz, EQ), the most suitable for modeling the linguistic semantics seems to be the Łukasiewicz FTT (Ł-FTT).

The fundamental concepts of the model of semantics are context (=possible world), intension, and extension. Concerning evaluative expressions, a formal theory T^{Ev} is developed as a special theory of Łukasiewicz fuzzy type theory.

The *intension* of an evaluative expression (or predication) $\mathcal{A}$ is obtained as interpretation of a formula $\lambda w\, \lambda x\, (Aw)x$ of Ł-FTT in a special model $\mathcal{M}$: Let U be a linearly ordered set U (we usually put $U = \mathbb{R}$).

- In the general theory of semantics, we mean by a *context* (a *possible world*) a state of the world at a given point in time and space. It is very difficult to formalize such a definition. In case of evaluative expressions, the situation is simpler. It is argued in [8] that their extensions are classes of elements taken from some scale. Thus, we identify possible worlds by *contexts* which are nonempty, linearly ordered and bounded sets. Each context is determined by three distinguished limit points: *left bound* v_L, *right bound* v_R, and a *central point* v_S. Thus we define context as a triple

  $$w = \langle v_L, v_S, v_R\rangle, \qquad v_L, v_S, v_R \in U.$$

  A simple example is the predication "$\mathcal{A}$ town", for example "small town", "very big town", etc. Then, the corresponding context for the Czech Republic can be $\langle 3\,000, 50\,000, 1\,000\,000\rangle$, while for the USA it can be $\langle 30\,000, 200\,000, 10\,000\,000\rangle$. We introduce a set W of contexts. Each element $w \in W$ gives rise to an interval $w = [v_L, v_R] \subset U$.

- Let $\mathcal{A}$ be an evaluative expression. Its *intension* $\mathrm{Int}(\mathcal{A})$ is modeled as a function $\mathrm{Int}(\mathcal{A}) : W \to \mathcal{F}(U)$ where $\mathcal{F}(U)$ is a set of all fuzzy sets over U.

- The *extension* $\mathrm{Ext}_w(\mathcal{A})$ of $\mathcal{A}$ in the context $w \in W$ is a fuzzy set of elements

  $$\mathrm{Ext}_w(\mathcal{A}) = \mathrm{Int}(w)(\mathcal{A}) \underset{\sim}{\subseteq} w.$$

  In T^{Ev}, the *extension* of an evaluative expression is obtained as a shifted horizon where the shift corresponds to a linguistic hedge, which is thus modeled by a function $L \to L$.

In our example, the truth value of a "small town having 30 000 inhabitants" could be, for example, 0.7 in the Czech Republic and 1 in the USA. The full formal theory including its informal justification, can be found in [8].

Other class of lexical units whose meaning can be modeled using MFL are *intermediate quantifiers* analyzed logically in detail in [11]. In general, these are special adverbs. In MFL, however, we can hardly model the meaning of them separately but better as parts of special syntagms of the form

$$QA \text{ are } B$$

where Q is an intermediate quantifier such as *many, most, almost all, few*, etc. and A, B can be more complex syntagms. A formal theory of their meaning and also 105 of generalized syllogisms can be found in [5, 9].

Future development of the theory of semantics of natural language using MFL can be seen in several respects. First, it is possible to continue the development of the model of meaning of further lexical units and special syntagms. A special place is taken by the model of meaning of verbs. These are characterized by variable complementation (roughly speaking, the number of dimensions) and many kinds of further aspects. We can continue with formation of the semantics of sentences, e.g., in the direction initiated in [6]. Another possibility is to extend MFL to become a tool modeling the natural human reasoning for which it is characteristic to use natural language. This was started under the name *Fuzzy Logic in Broader Sense* (FLb) (see [7, 10]).

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Contributed talks

Some Categorical Equivalences Involving Gödel Algebras

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The aim of this work is to investigate three categorical equivalences of Gödel algebras (cf. [2]): the first one involves *Idempotent Involutive Uninorm* (IIU in symbols)-algebras (cf. [4]), the second is between Gödel algebras and the subcategory of Nilpotent Minimum algebras NM^+ of those algebras whose involutive negation has a fix point, and finally the third one is the subcategory of those Nilpotent Minimum algebras NM^- whose negation has not a fix point.

Recall that a IIU-algebra is a bounded commutative residuated lattices $\langle A, *, \to, \leq, \mathbf{e}, \bot, \top\rangle$ satisfying $(x \to \mathbf{e}) \to \mathbf{e} = x$ (for all $x \in A$), and $x * x = x$ for all $x \in A$. To simplify the notation, we write $\neg x$ instead of $x \to \mathbf{e}$. The standard example of IIU-algebra is the system $\langle [0,1], *, \to, \frac{1}{2}, 0, 1\rangle$, where for every $x, y \in [0,1]$,

$$x * y = \begin{cases} \max\{x,y\} & \text{if } x + y > 1 \\ \min\{x,y\} & \text{otherwise.} \end{cases}$$

and

$$x \to y = \begin{cases} \max\{1-x,y\} & \text{if } x \leq y \\ \min\{1-x,y\} & \text{otherwise.} \end{cases}$$

A NM-algebra is a any algebra in the signature $\langle \odot, \to, \wedge, \vee, \bot, \top\rangle$ of type $(2,2,2,2,0,0)$. The variety of NM-algebras is generated by the *standard* NM-algebra $\langle [0,1], \odot, \Rightarrow, 0, 1\rangle$, where for all $x, y \in [0,1]$,

$$x \odot y = \begin{cases} \min\{x,y\} & \text{if } x + y > 1 \\ 0 & \text{otherwise.} \end{cases}$$

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and

$$x \Rightarrow y = \begin{cases} 1 & \text{if } x \leq y \\ \max\{1-x, y\} & \text{otherwise.} \end{cases}$$

The *negation* of any NM-algebra is defined as $\neg x = x \Rightarrow 0$, and the equation $\neg\neg x = x$ holds in any NM-algebra. An algebra A in the signature of NM, is said to be a NM$^-$-algebra if the following is satisfied:

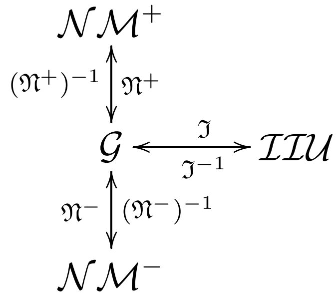
$$\neg((\neg(x \odot x) \odot \neg(x \odot x)) = (\neg(\neg x \odot \neg x)) \odot (\neg(\neg x \odot \neg x)).$$

Consider the signature of NM-algebras, extended by a fresh symbol for a constant $\mathbf{f}$. In this extended signature, we say that an algebra A is an NM$^+$-algebra if A satisfies the fix point equation:

$$\neg\mathbf{f} = \mathbf{f}.$$

We respectively denote by $\mathcal{G}$, $\mathcal{IIU}$, $\mathcal{NM}^+$, and $\mathcal{NM}^-$ the categories whose objects are Gödel, IIU, NM$^+$, and NM$^-$-algebras, and having homomorphisms as morphisms. Functors between the subdiretly irreducible elements of any of the above categories can be defined by adapting the Jenei [3] constructions of connected and disconnected rotations (to respectively define subdirectly irreducible $\mathcal{NM}^+$ and $\mathcal{NM}^-$ algebras by subdirectly irreducible Gödel algebras), and an analogous rotation-like construction to define a subdirectly irreducible IIU-algebra, by a subdirectly irreducible Gödel algebra. On the other way round, a Gödel algebra can always be defined by restricting a IIU, NM$^+$, or NM$^-$-algebra on the domain.

The following diagram summarizes the main equivalences:



Once the functor $\mathfrak{I}$, $\mathfrak{N}^+$ and $\mathfrak{N}^-$ are defined on subdirectly irreducible algebras, and on morphisms accordingly to the chosen rotation-like construction, the equivalence follows by the subdirect representation theorem.

Our investigation is now about the following directions:

(i) We firstly explore if the above introduced functors preserve basic logical and algebraic properties we know hold for $\mathcal{G}$, $\mathcal{NM}^+$, $\mathcal{NM}^-$, and $\mathcal{IIU}$.

(ii) Then we want to establish if the proposed categorical equivalence allows to define additional structure to IIU, NM^+ and NM^--algebras. In particular we are interested in showing what *states on Gödel algebras* (cf. [1]) correspond once the functors $\mathfrak{I}$, $\mathfrak{N}^+$ and $\mathfrak{N}^-$ are applied. Moreover we address the problem of showing whether de Finetti's theorem for states on Gödel algebras extends to IIU, NM^+ and NM^--algebras as well.

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On Similarity in Fuzzy Description Logics

*Eva Armengol**   *Pilar Dellunde**,†   *Àngel García-Cerdaña**

In this paper we explore the possibility of introducing the equality symbol in the languages of Fuzzy Description Logics (FDLs) interpreted as a similarity relation. In the talk we will present a state of the art concerning the notion of similarity in some fields of artificial intelligence, and we analyze some variants of the languages for the FDLs introduced in [10]. The goal is twofold: dealing with attribute-value representations at the domain objects level, and integrating the treatment of similarities inside the description languages and their corresponding knowledge bases.

Description Logics (DLs) are knowledge representation languages built on the basis of classical logic. DLs allow the creation of knowledge bases and provide ways to reason on the contents of these bases. A full reference manual of the field is [1]. The vocabulary of DLs consists of *concepts*, which denote sets of individuals, and *roles*, which denote binary relations among individuals. From atomic concepts and roles and by means of *constructors*, DL systems allow us to build complex descriptions of both concepts and roles. These complex descriptions are used to describe a domain through a knowledge base (KB) containing the definitions of relevant domain concepts or some hierarchical relationships among them (*Terminological Box* or *TBox*), and a specification of properties of the domain instances (*Assertional Box* or *ABox*). One of the main issues of DLs is the fact that in both the *TBox* and the *ABox* can be identified with formulas in first-order logic or an extension of it; therefore we can use reasoning to obtain implicit knowledge from the explicit knowledge in the KB.

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Fuzzy Description Logics (FDLs) are natural extensions of DLs for dealing with vague concepts, commonly present in real applications (see for instance [21, 22, 20, 2] or [16] for a survey). Hájek [13] proposed to deal with FDLs taking as basis t-norm based fuzzy logics with the aim of enriching the expressive possibilities in FDLs and to capitalize on recent developments in the field of mathematical fuzzy logic. From this perspective, in [10] a family of DLs was defined. These languages include truth constants for representing truth degrees, thus allowing the definition of the axioms of the knowledge bases as sentences of a fuzzy predicate language in much the same way as in classical DLs.

Similarity has been a central issue for decades in different disciplines, ranging from philosophy (Leibniz's Principle of the Identity of Indiscernibles) and psychology (Tversky's stimuli judged similarity) to natural sciences (taxonomy) and mathematics (geometric similarity). Also in artificial intelligence similarity plays an important role since the analogy reasoning is behind some of the early machine learning methods. Particularly, case-based reasoning methods are based on the principle that "similar problems have similar solutions" where the notion of similarity has a capital importance (see [14]).

In the *fuzzy framework*, the notion of similarity was introduced by Zadeh in [24] as a generalization of the notion of equivalence relation (see [17] for a historical overview on the notion of t-norm based similarity). As Zadeh pointed out, one of the possible semantics of fuzzy sets is in terms of similarity: the degree of membership of an object to a fuzzy set can be seen as the degree of resemblance between this object and prototypes of the fuzzy set. Thus, an important logical issue is to define a logic of similarity that can account for the proximity between the boolean interpretations of a propositional language. Dubois and Prade developed this idea in [8]. Ruspini [18] presents a formal characterization of the notions of both implication and consequence of propositional fuzzy logic in terms of the similarity notion between pairs of possible worlds. Based on his work, Esteva et al. [9] consider graded entailments between sets of propositional formulas induced by a similarity relation on the set of interpretations. Hájek studies similarities in fuzzy predicate logics and applies the obtained results to the analysis of fuzzy control [12]. Bělohlávek [4] presents a general theory of fuzzy relational systems. Gerla [11] proposes a fuzzy predicate theory whose fuzzy models are plausible candidates for the notion of approximate similarity. For a reference about model-theoretic properties of algebras with fuzzy equalities see [6, 4].

Similarity in DLs has been studied by Borgida et al. [3] and D'Amato et al. [7] among others, focusing on similarity measures between DL concepts, and from the logical point of view by Sheremet et al. [19]. D'Amato et al. take as starting point the idea that measures for estimating concept similarity have to be able to appropriately consider concept semantics in order to correctly assess their similarity value. In accordance with this goal the authors propose a set of properties that a semantic similarity measure should have, analyze different extensional-based and intensional-based similarity measures proposed in the literature, and show that these approaches lack some of the needed properties. Finally, they define a measure for complex descriptions in some DL languages that is compliant with all of these criteria. Sheremet et al. propose an integration of logic-based and similarity-based approaches in classical DLs. They use concept constructors such as 'in the a-neighborhood of C' where a is a positive rational number; or the operator $C \leftleftarrows D$ which is interpreted by the set of all points in the similarity space that are closer to the instances of C than to the instances of D. For example, it can be used to model statements like 'X resembles C more than D'. Since we want to study the similarity in FDLs, we explore the possible generalization of both approaches, [19] and [7], to the fuzzy framework.

Our approach. In this paper we introduce the equality symbol in the language of FDLs and in their knowledge bases interpreted as a similarity relation. We also define variants of the languages for the FDLs introduced in [10]. These variants allow to deal with attribute-value representations at the domain objects level. The attribute-value representation is commonly used in artificial intelligence. In this representation, domain objects are sets of pairs attribute-value, where the value of an attribute may be qualitative or quantitative (see [23] to clarify the classes of data types). The *global* similarity between objects has to be seen as an aggregation of the *local* similarities of the features (see [15] and [5] for a collection of similarity and aggregation measures, respectively). To integrate the treatment of similarities inside knowledge bases, we can take profit of the presence of truth constants in FDL languages, as it is done in [10]. This fact allows to state graded notions at the syntactical level for both similarity between objects and similarity between concepts. As Ruspini suggests in [18], the degree of similarity between two objects A and B may be regarded as the degree of truth of the vague proposition "A is similar to B". Thus, similarity among objects can be seen as a phenomena essentially fuzzy. Following this idea, the capabilities of a

FDL language to deal with truth degrees at the syntactical level are specially relevant in which concerns to a useful treatment of the notion of *similarity* in the DLs framework. Therefore, the basic lines of our approach are:

- to represent domain objects by means of sets of pairs attribute-value as it is commonly done in artificial intelligence. Our goal is to integrate such representation in the conceptualization of the domain by means of FDLs.
- to explore the possibility of defining a Similarity Fuzzy Description Logic, introducing similarity roles and new concept constructors, the former interpreted as similarities between objects, the later as similarities between concepts. From our point of view the similarity between objects can be seen as an aggregation of the local similarities between attributes. Also, the similarity between concepts could be defined from similarities among objects.
- the use of the truth degrees included in the languages as similarity degrees between both objects and concepts.

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Kripke-Style Semantics for Normal Systems

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In this paper we describe a generalized framework of sequent calculi, called *normal systems*, which is suitable for a wide family of propositional logics. This family includes classical logic, several modal logics, intuitionistic logic, its dual, bi-intuitionistic logic, and various finite-valued logics. We present a general method for providing a non-deterministic finite-valued Kripke-style semantics for every normal system, and prove a corresponding general soundness and completeness theorem.

As observed in [3], there are two dual standpoints regarding signed calculi (or many-sided sequent calculi). In the "positive" interpretation a signed formula consisting of a sign i and a formula ψ is true, iff i is the truth value assigned to ψ. In the "negative" interpretation the same signed formula is true, iff i is *not* the truth value assigned to ψ (The two approaches practically coincide in the two-valued case). As in [4], it turns out that the "negative" interpretation is more suited for our purposes, since it leads to a natural Kripke-style semantics, that generalizes the well-known two-valued semantics for the above-mentioned logics. The definitions below (and the notation we use) reflect this choice.

Definition A *signed formula* for a propositional language $\mathcal{L}$ and a finite set of signs $\mathcal{I}$ is an expression of the form $i \div \psi$, where $i \in \mathcal{I}$ and ψ is an $\mathcal{L}$-formula. A *sequent* is a finite set of signed formulas. The usual two-sided notation $\Gamma \Rightarrow \Delta$ is interpreted as $\{t \div \psi \mid \psi \in \Gamma\} \cup \{f \div \psi \mid \psi \in \Delta\}$, i.e. a sequent over $\mathcal{I} = \{t, f\}$.

Definition

1. A *context-restriction* is a set of signed formulas. A *normal premise* is an expression of the form $\langle s, \pi \rangle$, where s is a sequent and π is a context-restriction.

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2. A *normal rule* is an expression of the form: S/C, where S is a finite set normal premises, and C is a sequent. An *application* of the normal rule S/C is any inference step which derives $\sigma(C) \cup s'$ from the set of sequents $\{\sigma(s) \cup (s' \cap \pi) \mid \langle s, \pi \rangle \in S\}$, where s' is a sequent and σ is an $\mathcal{L}$-substitution.

3. The *identity axioms* are normal rules of the form $\emptyset/\{i \div p_1, j \div p_1\}$, for every $i, j \in \mathcal{I}$ such that $i \neq j$. The *cut rule* is the normal rule $\{\langle i \div p_1, \pi \rangle \mid i \in \mathcal{I}\}/\emptyset$, where π is the set of all signed formulas. The *weakening rules* are the normal rules of the form $\{\langle \emptyset, \pi \rangle\}/i \div p_1$ for every $i \in \mathcal{I}$, where π is the set of all signed formulas.

4. A sequential system is called *normal* iff it consists of normal rules only, and the identity axioms, cut, and weakening are among its rules.

We give some examples of the variety of derivation rules included in our normal rule definition:

Classical Implication. Assume $\mathcal{I} = \{t, f\}$. The following normal rules are the rules for implication in Gentzen's LK (see [6])[1] :

$$\{\langle \{t \div p_1, f \div p_2\}, \pi_1 \rangle\}/f \div p_1 \supset p_2$$
$$\{\langle \{f \div p_1\}, \pi_2 \rangle, \langle \{t \div p_2\}, \pi_2 \rangle\}/t \div p_1 \supset p_2$$

where π_1 and π_2 are the sets of all signed formulas for $\mathcal{L}$ and $\mathcal{I}$. Using a more usual notation, applications of these rules have the form:

$$\frac{\Gamma, \psi \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \psi \supset \varphi} \qquad \frac{\Gamma \Rightarrow \Delta, \psi \quad \Gamma, \varphi \Rightarrow \Delta}{\Gamma, \psi \supset \varphi \Rightarrow \Delta}$$

Intuitionistic Implication. Assume $\mathcal{I} = \{t, f\}$. Consider the previous two rules for $\supset$ where π_2 is the same, and π_1 is the set of all signed formulas of the form $t \div \psi$. This gives the rules for implication in LJ' (see [7]). Using a more usual notation, applications of the second rule have the form:

$$\frac{\Gamma, \psi \Rightarrow \varphi}{\Gamma \Rightarrow \Delta, \psi \supset \varphi}$$

[1] More precisely, we mean the version of LK, in which the two sides of a sequent consists of sets of formulas. The same applies to the other systems that we mention.

S4's Box. Assume $\mathcal{I} = \{t, f\}$. The following normal rules are the rules for box in $S4^*$ (see [5]):

$$\{\langle\{f \div p_1\}, \pi_1\rangle\}/f \div \Box p_1 \quad \{\langle\{t \div p_1\}, \pi_2\rangle\}/t \div \Box p_1$$

where π_1 is the set of all signed formulas of the form $t \div \Box\psi$, and π_2 is the set of all signed formulas for $\mathcal{L}$ and $\mathcal{I}$. Using a more usual notation, applications of these rules have the form:

$$\frac{\Box\Gamma \Rightarrow \psi}{\Gamma', \Box\Gamma \Rightarrow \Delta, \Box\psi} \qquad \frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma, \Box\psi \Rightarrow \Delta}$$

3-Valued Lukasiewicz's Implication. Assume $\mathcal{I} = \{t, I, f\}$. The following normal rules are the rules for implication in $NL3$ (see [8]):

$$\{\langle\{f \div p_1\}, \pi\rangle, \langle\{f \div p_2, t \div p_1\}, \pi\rangle\}/f \div p_1 \supset p_2$$
$$\{\langle\{f \div p_1, I \div p_1\}, \pi\rangle, \langle\{I \div p_2, t \div p_1\}, \pi\rangle\}/I \div p_1 \supset p_2$$
$$\{\langle\{f \div p_1\}, \pi\rangle, \langle\{t \div p_2\}, \pi\rangle, \langle\{I \div p_1, I \div p_2\}, \pi\rangle\}/I \div p_1 \supset p_2$$

where π is the set of all signed formulas for $\mathcal{L}$ and $\mathcal{I}$. Using a more usual notation, applications of these rules have the form:

$$\frac{\Gamma|\Delta|\Sigma, \psi \quad \Gamma, \varphi|\Delta|\Sigma, \psi}{\Gamma, \psi \supset \varphi|\Delta|\Sigma} \quad \frac{\Gamma, \psi|\Delta, \psi|\Sigma \quad \Gamma|\Delta, \varphi|\Sigma, \psi}{\Gamma|\Delta, \psi \supset \varphi|\Sigma} \quad \frac{\Gamma, \psi|\Delta|\Sigma \quad \Gamma|\Delta|\Sigma, \varphi \quad \Gamma|\Delta|\Sigma}{\Gamma|\Delta, \psi, \varphi|\Sigma, \psi \supset \varphi}$$

Now we define the Kripke semantics induced by any given normal system. Non-deterministic Kripke-frames for single-conclusion proper sequential calculi were first introduced in [2].

Definition Let $\mathbf{G}$ be a normal system. Denote by $\Pi_{\mathbf{G}}$ the set of context-restrictions which appear in $\mathbf{G}$.

1. A $\Pi_{\mathbf{G}}$-*frame* is a tuple $\langle W, \mathcal{R}, v\rangle$, where:
   (a) W is a set of elements, and $\mathcal{R}$ is a set of preorders on W, consisting of a preorder $\lesssim_\pi$ for every $\pi \in \Pi_{\mathbf{G}}$. If π is the set of all signed formulas, then $\lesssim_\pi$ is the identity relation.
   (b) $v : W \times Frm_{\mathcal{L}} \to \mathcal{I}$ is a *persistent* function, i.e. for every $\pi \in \Pi_{\mathbf{G}}$: if $i \div \psi \in \pi$ and $v(a, \psi) = i$, then $v(b, \psi) = i$ for every $b \in W$ such that $a \lesssim_\pi b$.

2. Let $\mathcal{W} = \langle W, \mathcal{R}, v \rangle$ be a $\Pi_{\mathbf{G}}$-frame.

   (a) A sequent s is *true* in $a \in W$ iff there exists $i \div \psi \in s$ such that $v(a, \psi) \neq i$. $\mathcal{W}$ is a *model* of s iff s is true in every $a \in W$.

   (b) $\mathcal{W}$ *respects* a normal rule of $\mathbf{G}$, $r = S/C$, iff for every $a \in W$ and every $\mathcal{L}$-substitution σ: If $\sigma(s)$ is true in every $b \in W$ such that $a \lesssim_\pi b$ for every $\langle s, \pi \rangle \in S$, then $\sigma(C)$ is true in a.

   (c) $\mathcal{W}$ is $\mathbf{G}$-*legal* iff it respects all the rules of $\mathbf{G}$.

We give some examples of the sets of all $\mathbf{G}$-legal frames for important logics:

Classical Logic. Consider the normal system LK. LK includes one sort of context-restriction, π, which is the set of all signed formulas. Thus we consider frames with one preorder $\lesssim_\pi$, which is the identity relation. The persistence condition then vacuously holds. LK-legality reduces to usual Tarski-style semantics in every element of W.

Intuitionistic Logic. Consider the full normal system LJ' from [7]. It has two sorts of context-restriction: π_1 and π_2 from the example of intuitionistic implication given above. π_1 (the set of all signed formulas of the form $t \div \psi$) is used in the right rule for implication, π_2 (the set of all signed formulas) in all the rest. Accordingly, the persistence condition means here that if $v(a, \psi) = t$ then $v(b, \psi) = t$ for every $b \in W$ such that $a \lesssim_{\pi_1} b$. Now it is easy to see (using persistence) that a frame is LJ'-legal accoding to our definitions iff: (1) $v(a, \psi \supset \varphi) = t$ iff there does not exist an element $b \in W$ such that $a \lesssim_{\pi_1} b$, $v(b, \psi) = t$ and $v(b, \varphi) = f$; and (2) it respects the usual truth-tables of the other connectives ($\wedge$,$\vee$ and $\perp$) in every element of W.

S4. Consider the normal system $S4^*$ from [5]. $S4^*$ again has two sorts of context-restriction: π_1 and π_2 from the example of $S4$'s Box given above. Now $\lesssim_{\pi_2}$ is again the identity relation, so the persistence condition means that if $v(a, \Box\psi) = t$ then $v(b, \Box\psi) = t$ for every $b \in W$ such that $a \lesssim_{\pi_1} b$. Using the fact that $\lesssim_{\pi_1}$ is a preorder, it is easy to see that a frame is $S4^*$-legal iff: (1) $v(a, \Box\psi) = t$ iff $v(b, \psi) = t$ for every b such that $a \lesssim_{\pi_1} b$; and (2) it respects the usual truth-tables of the other connectives in every element of W.

Our main result is the following:

Theorem (Strong Soundness and Completeness) Let $\mathbf{G}$ be a normal system. There exists a proof in $\mathbf{G}$ of a sequent s from a set of sequents $\mathcal{S}$, iff every $\mathbf{G}$-legal $\Pi_{\mathbf{G}}$-frame which is a model of $\mathcal{S}$ is also a model of s.

A Compactness property of our semantic consequence relation (for any given normal system) is an easy corollary of this theorem. Moreover, the proof of this theorem shows that it is sufficient to consider a narrower class of frames. We point out two properties of the frames in this class:

1. If $\{j \div \psi \mid j \neq i\} \subseteq \pi_1$ whenever $i \div \psi \in \pi_2$, then $\lesssim_{\pi_2}$ is the inverse of $\lesssim_{\pi_1}$. In particular, if $i \div \psi \in \pi$ for every $i \in \mathcal{I}$ whenever $i \div \psi \in \pi$ for some $i \in \mathcal{I}$, then $\lesssim_{\pi}$ is symmetric.

2. If for every formula ψ, $i \div \psi \notin \pi$ for at most one $i \in \mathcal{I}$, then $\lesssim_{\pi}$ is anti-symmetric.

In particular:

1. For $S5^*$ (a calculus for $S5$, see [5]), it suffices to consider frames with one equivalence relation.

2. For LJ', it suffices to consider frames with an order relation. The same is true for dual-intuitionistic logic and for bi-intuitionistic logic.

Following this method, one obtains a finite-valued Kripke-style semantics for any normal system. The semantics is modular, allowing to separately investigate the effect of every normal rule. Note that for many normal systems, the resulting semantics is *non-deterministic*, and the truth-functionality principle does not hold. This happens, for example, for any normal system whose set of rules is a proper subset of any of the above-mentioned normal systems.

We believe that the current work is a good starting point for developing and investigating sequent calculi for various sorts of combinations of many valued, intuitionistic, modal, and multi-modal logics. However, it should be noted that in order to obtain a decision procedure for a normal system using this semantics, one have to ensure *analycity* ([1]) of the semantics, i.e. that every legal frame which is defined on some set of formulas can be extended to a legal frame defined on all formulas. It is interesting to look for a general characterization of normal systems which admit this property. Another important question is how (and in what cases) can we extract a semantic proof of cut-elimination from our completeness proof. We leave these questions for a future work.

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Gödel Logics with an Operator that Shifts Truth Values

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We consider an extension of [0,1]-Gödel logic by a unary operator o that adds a constant $r \in [0,1]$ fixed for every interpretation. The set of formulas in propositional logic valid for all r is known to be axiomatizable by a simple Hilbert-Frege system that extends BL and IPL and consists only of finitely many schemata. We answer Hájek's question of the axiomatization of the formulas valid in a logic enhanced by o where one takes instead of [0,1] a truth-value set V, $\{0,1\} \subseteq V \subseteq [0,1]$ such that $\mathrm{o}\,A$ is defined for all formulas A.

We consider a language $\mathcal{L}_{\mathrm{o}}^{\mathrm{P}}$ that comprises a countably infinite set of propositional variables, connectives $\bot$, $\supset$, $\wedge$, $\vee$ with their usual arities as well as a unary connective o. The semantics of *propositional Gödel logics with* o in $\mathcal{L}_{\mathrm{o}}^{\mathrm{P}}$ is determined by Gödel *r-interpretations* $\mathfrak{I}$; here $r \in [0,1]$ and $\mathfrak{I}$ maps formulas to $[0,1]$ such that

$$\mathfrak{I}(A \wedge B) = \min\{\mathfrak{I}(A), \mathfrak{I}(B)\}, \qquad \mathfrak{I}(\bot) = 0,$$

$$\mathfrak{I}(A \vee B) = \max\{\mathfrak{I}(A), \mathfrak{I}(B)\}, \qquad \mathfrak{I}(A \supset B) = \begin{cases} 1 & \text{if } \mathfrak{I}(A) \le \mathfrak{I}(B), \\ \mathfrak{I}(B) & \text{if } \mathfrak{I}(A) > \mathfrak{I}(B). \end{cases}$$

$$\mathfrak{I}(\mathrm{o}(A)) = \min\{1, r + \mathfrak{I}(A)\},$$

A formula A is *valid* if $\mathfrak{I}(A) = 1$ holds for all Gödel r-interpretations $\mathfrak{I}$, $r \in [0,1]$. We introduce the well-known abbreviations $\top := \bot \supset \bot$, $\neg A := A \supset \bot$, $A \prec B := (B \supset A) \supset B$, $A \leftrightarrow B := (A \supset B) \wedge (B \supset A)$ and define the o-powers $\mathrm{o}^0 A := A$, $\mathrm{o}^{n+1} A := \mathrm{o}^n\,\mathrm{o}\,A$. We have then, e. g.,

$$\mathfrak{I}(A \prec B) = \begin{cases} 1 & \text{if } \mathfrak{I}(A) < \mathfrak{I}(B) \text{ or } \mathfrak{I}(A) = \mathfrak{I}(B) = 1, \\ \mathfrak{I}(B) & \text{if } \mathfrak{I}(A) \ge \mathfrak{I}(B). \end{cases}$$

Dummett [1] proved that the set of valid formulas in propositional Gödel logic (without o) is axiomatized by the Hilbert-Frege style proof system (GPL)

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that consists of the axiom schema of linearity (LIN) $A \supset B \vee B \supset A$ added e. g. to the system (IPL), which axiomatizes propositional intuitionistic logic:

$$\begin{array}{ll} A \supset (B \supset A), & \text{modus ponens: } \frac{A \quad A \supset B}{B}, \\ (A \wedge B) \supset A, & (A \supset (B \supset C)) \supset (A \supset B) \supset (A \supset C), \\ (A \wedge B) \supset B, & (A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C)), \\ A \supset (A \vee B), & A \supset (B \supset (A \wedge B)), \\ B \supset (A \vee B), & \bot \supset A. \end{array}$$

One can prove that the set of valid formulas in propositional Gödel logics with o over [0,1] is given by (GPL) plus the following simple axiom schemata:

$$(\bot \prec \mathrm{o}\,\bot) \supset (A \prec \mathrm{o}\,A)$$
$$(\bot \leftrightarrow \mathrm{o}\,\bot) \supset (A \leftrightarrow \mathrm{o}\,A)$$
$$(\mathrm{o}(A \prec B)) \leftrightarrow (\mathrm{o}\,A \prec \mathrm{o}\,B)$$

The proof employs Dummett's idea to use chains, i. e. linear orderings of propositional variables and their o-powers w. r. t. $\prec$ and $\leftrightarrow$, to 'evaluate' the given formula; cf. also [2]. In the proof, a finite counter-model is constructed for each formula that is not valid.

We will show how the above system of (GPL) plus the three ring axiom schemata must be extended to obtain an axiomatization of the propositional Gödel logics with o over a given finite truth-value set V, $\{0, 1\} \subseteq V \subseteq [0, 1]$, where all o-operations are defined. An important axiom will be the pigeon hole principle

$$p_1 \vee (p_1 \supset p_2) \vee (p_2 \supset p_3) \vee \ldots \vee (p_{n-2} \supset p_{n-1}) \vee (p_{n-1} \supset \bot).$$

We can even specify a sequence of truth-value sets $(V_i)_i$ such that a formula is valid in propositional Gödel logics with o over [0,1] if and only if it is valid over all V_i. Moreover, we will characterize the logics arising from infinite V.

We will also indicate the relation between the above logics, where o shifts truth-values by a constant, and the logics, where o adds certain monotonous functions.

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Semantics of Counterfactuals in Higher-Order Fuzzy Logic

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First-order fuzzy logics are expressive enough to support non-trivial axiomatic theories of fuzzy mathematics. In the framework of Henkin-style higher-order fuzzy logic [6], several axiomatic theories of fuzzy mathematics have been developed to an extent which enables a smooth mathematical work with fuzzy structures (esp. fuzzy sets and fuzzy relations [6, 5, 8]). The apparatus has already been applied to formal semantics of erotetic logic [1] and propositional dynamic logic [2]; here we propose its application in Lewis-style semantics of counterfactual conditionals [11].

Counterfactual conditionals are propositions of the form "if it were the case that A, then it would be the case that B", where A is an irreal condition (i.e., is presupposed to be false in the actual world). Interpreting counterfactual conditionals as classical material implication $A \rightarrow B$ would clearly be counter-intuitive, as it would render all counterfactuals as true. Intuitively, however, some counterfactuals are perceived as true (e.g., "if kangaroos had no tail, they would topple over") while others as false (e.g., "if kangaroos had no tail, they would be able to fly"). Logical analysis of counterfactual conditionals therefore requires intensional rather than extensional implication, which is usually modeled by possible-world semantics. Several proposals of formal semantics of counterfactuals have been offered; the most influential analysis was provided independently by David Lewis [11] and Robert Stalnaker [12]; we shall follow (several variants of) Lewis' account here.

Lewis' semantics of counterfactuals is based on comparing the 'distance' (in the sense of subjective similarity) between possible worlds: the counterfactual "if it were the case that A, then it would be the case that B" is

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considered true iff the closest worlds in which both A and B are true are closer to the actual world than closest worlds in which A is true and B is false. This idea can be rendered in a very natural way by means of higher-order fuzzy logic: the notion of 'distance' can be represented by a fuzzy similarity (of the worlds), i.e., a fuzzy relation which is reflexive, symmetric, and transitive (in the sense of fuzzy logic of our choice); the larger the truth degree of the fuzzy similarity relation, the closer are the worlds to each other. The apparatus of higher-order fuzzy logic (MTL or stronger) then easily and naturally accommodates all operations that are needed for internalization of Lewis' semantics for counterfactuals.

The reconstruction of Lewis-style semantics can be carried out in (any extension of) higher-order logic MTL [9], which is arguably the weakest fuzzy logic suitable for the enterprise of formal fuzzy mathematics [3] (unless generalizations without the rules of exchange or weakening are considered). Besides the formal theory of fuzzy relations (esp. theorems on fuzzy similarities, fuzzy orderings, and fuzzified maxima and minima) as developed in [5], some theorems on fuzzified metric notions (such as the distance from a fuzzy set) are necessary for the fuzzy semantics of counterfactuals; these theorems (which form a part of number-free metric theory, cf. [4]) are, nevertheless, straightforwardly derivable in the framework of higher-order MTL. (Technical details of the reconstruction of Lewis' semantics are omitted here due to space restrictions, but will be presented in the talk.)

Apart from providing a new perspective on Lewis' semantics, the formalization within the framework of higher-order fuzzy logic offers several advantages compared to the classical definition. First, it automatically accommodates *graded* counterfactuals (e.g., "If John were *tall,* then…"—where *tall* is a graded predicate whose truth value reflects a person's height measured in cm's). Arguably, most propositions of natural language are gradual in this sense; and classical rendering of such propositions as bivalent is an idealization susceptible to the sorites-style paradoxes. Second, the many-valuedness of fuzzy logics automatically accommodates the gradual truth of counterfactuals, making it possible to capture the intuition that some counterfactuals are perceived as *truer* than others. And third, the structure of truth degrees makes it possible not only to compare, but also 'measure' the 'distance' between possible worlds. Even though measuring the distance by real numbers would hardly be justifiable (and therefore was rejected by Lewis), the semantics of formal fuzzy logics provides abstract degrees of distance in various algebras of truth values yielded by algebraic semantics of fuzzy logics.

The price paid for the formalization of Lewis' semantics in fuzzy logic is the necessity of using non-classical reasoning in its description (including certain peculiar features that are due to the non-idempotent conjunction, cf. [7]). Nevertheless, the deductive apparatus of fuzzy logic is by now sufficiently developed and all necessary theorems on fuzzy orderings and similarities either are already at our disposal [5] or are easily derivable; the non-classical theory can therefore be developed at a rather low cost.

In the talk, a reconstruction of Lewis' semantics and an independent analysis of both graded and bivalent counterfactuals in formal fuzzy logic will be presented (including the technical details omitted here). Both systems will then be compared, and the merits of the fuzzy approach will be discussed.

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Gödel Rings

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Given a distributive lattice one can try to find a representation of this lattice by relating it to the lattice of ideals of a ring. Von Neumann's coordinatization theorem is a prime example of this (see [13]) showing that if a complemented modular lattice L has a finite homogeneous spanning sequence with at least four elements, then it is *coordinatizable*, that is, there exists a von Neumann regular ring R (see [12]) whose lattice of principal right ideals is isomorphic to L. Somewhat similarly, Bergman [1] showed that an algebraic distributive lattice with at most $\aleph_0$ compact elements is isomorphic to the lattice of ideals of a Von Neumann regular ring. Wehrung proved in [14] that $\aleph_0$ in Bergman's result can be replaced by $\aleph_1$ (i.e.: every algebraic distributive lattice with at most $\aleph_1$ compact elements is isomorphic to the ideal lattice of a von Neumann regular ring) and has also shown that $\aleph_1$ cannot be replaced by $\aleph_2$.

In this work we take a different point of view and try to address the following question: *What properties does a ring have if we require its semiring of ideals to be a Gödel algebra?*

Let R be a ring, not necessarily commutative nor necessarily with identity. We assume that R satisfies the condition

$$(\star) \qquad \text{for all } x \in R \text{ there are } s, t \in R \text{ such that } sx = x = xt.$$

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We want to study a subclass of the above rings whose semiring of ideals $Id(R)$ can be given the structure of a Gödel algebra. The lattice of ideals and the semiring of ideals of a ring R are identical as *sets* but not as algebraic structures. We choose to force equality so that the semiring product agrees with the lattice meet. Such rings are called *Gödel rings*.

A prominent example of Gödel rings are Von Neumann regular rings (VNR) (see [12, 7]). Recall that a ring R is a VNR ring provided that for all $x \in R$ there is a $y \in R$ such that $x = xyx$. Indeed, the set of ideals of a VNR can be easily seen to be a Heyting algebra. Moreover, as shown by Kaplansky [10], if R is the full ring of linear transformations on a vector space over a division ring, then R is VNR, and its set of ideals forms a chain. Then $Id(R)$ is indeed a Gödel algebra.

Examples of Gödel rings that are not VNR can be found, for instance, in the class of Weyl algebras. Indeed (see [8, pg.33]), let F be a field of characteristic 0, and let $R = F[x, y]$ be the ring of polynomials $p(x, y) = \Sigma_{ij}\alpha_{ij}x^i y^j$ where x, y commute with the elements of F but $xy - yx = 1$. Such a ring can be shown to be a Gödel ring.

Our main result states that every Gödel ring R is a subdirect product of prime Gödel rings R_i, and the Gödel algebra $Id(R)$ associated to R is subdirectly embeddable as an algebraic lattice into $\prod_i Id(R_i)$, where each $Id(R_i)$ is the algebraic lattice of ideals of R_i that can be equipped with the structure of a Gödel algebra. Moreover, the mapping associating to each Gödel ring its Gödel algebra of ideals is functorial from the category of Gödel rings with epimorphisms into the full subcategory of frames whose objects are Gödel algebras and whose morphisms are complete epimorphisms.

Representation Theorem If R is a Gödel ring, then

1. R is a subdirect product of subdirectly irreducible prime Gödel rings R_i;
2. $Id(R) \hookrightarrow \prod_i Id(R_i)$;
3. each $Id(R_i)$ is an algebraic Gödel algebra with a unique atom.

Furthermore, we study the connection between prime ideals of a Gödel ring and prime complete ideals of its related Gödel algebra and we show that the related spectra are homeomorphic.

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Leibniz Interpolation Properties

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In the framework of Abstract Algebraic Logic (see [2] for a general survey) a number of different interpolation properties have been considered: Maehara Interpolation Property, Robinson Interpolation Property, and Deductive Interpolation Property among others (see [1] as well as the references therein).

In [3], Kihara and Ono introduce some new notions of interpolation for substructural logics. For instance, one of the interpolation properties that they study is the following: for any arbitrary pair of formulas φ, ψ, if $\vdash (\varphi\backslash\psi) \wedge (\psi\backslash\varphi)$, then there exists a formula δ with variables $\mathrm{var}(\delta) \subseteq \mathrm{var}(\varphi) \cap \mathrm{var}(\psi)$ such that

$$\vdash (\varphi\backslash\delta) \wedge (\delta\backslash\varphi) \quad \text{and} \quad \vdash (\delta\backslash\psi) \wedge (\psi\backslash\delta).$$

In the same paper they present algebraic characterizations for all these notions and investigate their relation with the usual interpolation properties. The authors also remark: "only a few properties specific to FL-algebras or residuated lattices are used in our discussion," and they propose the study of these interpolation properties in a more general setting. The objective of the present paper is to give an answer to that remark by exhibiting a general framework in which these interpolation properties can be expressed and characterized by means of algebraic tools.

First, we note that $\nabla(x, y) = \{(x\backslash y) \wedge (y\backslash x)\}$ is a set of equivalence formulas (in the sense of Abstract Algebraic Logic) for the subestructural logics, that is, it defines the Leibniz congruences of their models. In particular, for every theory Γ and every pair of formulas φ, ψ,

$$\Gamma \vdash \nabla(\varphi, \psi) \quad \Leftrightarrow \quad \langle \varphi, \psi \rangle \in \boldsymbol{\Omega}\Gamma.$$

This motivates the choice of the equivalential logics, i.e., logics having a set of equivalence formulas, as our general setting.

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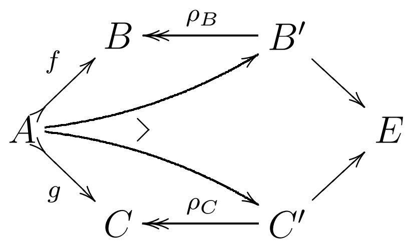
We introduce a family of what we call 'Leibniz' interpolation properties, in which we are mainly interpolating pairs of elements in the Leibniz congruence of a particular free algebra. Applied to the case of substructural logics, some of them coincide with those introduced in [3].

The first notion of interpolation that we introduce is the *Leibniz Interpolation Property* (LIP). A sentential logic $\mathcal{S}$ has the LIP if for every set of formulas Γ and every pair of formulas φ, ψ, if $\langle \varphi, \psi \rangle \in \boldsymbol{\Omega}\overline{\Gamma}$, then there exists a formula δ with variables $\mathrm{var}(\delta) \subseteq \mathrm{var}(\Gamma, \varphi) \cap \mathrm{var}(\Gamma, \psi)$ such that $\langle \varphi, \delta \rangle, \langle \delta, \psi \rangle \in \boldsymbol{\Omega}\overline{\Gamma}$, where $\overline{\Gamma} = Cn_{\mathcal{S}}(\Gamma)$ is the theory of $\mathcal{S}$ generated by Γ. By adding new conditions we can strengthen this property in several different ways.

In the context of equivalential logics, we find characterizations for these interpolations properties in terms of categorical properties of their classes of reduced models, reduced algebras, or both. We say that a family of matrices **C** has *approximating pullbacks for (pairs of) strict morphisms* if whenever $f : A \rightarrowtail B$ and $g : A \rightarrowtail C$ is a pair of strict matrix morphisms, there exists a pullback and onto matrix morphisms ρ_B, ρ_C rendering commutative the following diagram:



(1)

We prove that every equivalential logic $\mathcal{S}$ satisfies the LIP, and this implies that the category $\mathbf{Mod}^*\mathcal{S}$ of its reduced models has approximating pullbacks for strict matrix morphisms. For the others interpolations properties we find characterizations, by strengthening the conditions in Diagram (1).

Three of the notions that we introduce and deserve a special attention are the *Relative Leibniz*, *Robinson-Leibniz*, and *Maehara-Leibniz Interpolation Properties*. They are the natural generalizations of three of the interpolation properties considered in [3] for substructural logics. We investigate the relation of these properties with those studied by Czelakowsky and Pigozzi in [1] and obtain the following result for the equivalential logics satisfying the G-rule, that is, the regularly algebraizable logics.

Theorem. For every regularly equivalential logic $\mathcal{S}$, we have the following implications:

(i) The Relative Interpolation Property implies Interpolation Property.

(ii) The Robinson-Leibniz Interpolation Property implies the Robinson Interpolation Property.

(iii) The Maehara-Leibniz Interpolation Property implies the Maehara Inerpolation Property.

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Compatible Operators on Residuated Lattices

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This work provides a possible generalization to the context of residuated lattices of frontal operators in the sense of the article [5].

Let L be a residuated lattice (RL for short) and $f\colon L \to L$ a function. It is possible to give a complete characterization for the compatibility of f with respect to every congruence on L. This characterization generalizes that given in [2] for commutative RLs and allow us to prove that some functions generalizing frontal operators are compatible.

Frontal operators in Heyting algebras were studied in [3, 5, 6]. They are always compatible, but not necessarily new or implicit in the sense of [1]. Classical examples of new implicit frontal operators are the functions γ, (Example 3.1 of [1]), the successor (Example 5.2 of [1] and [4]), and Gabbay's operation (Example 5.3 of [1]).

Let L be a RL. We say that $\tau : L \to L$ is a *T-left pre-frontal operator* (Tlp for short) if there exists a binary term T in the language of residuated lattices such that for every Heyting algebra H, we have that $T^H(x, y) = y \to x$ and for every $x, y \in L$ the following equations hold:

(l1) $\tau(x) \leq y \vee T(x, y)$,

(l2) $e \leq \tau(e)$,

(l3) $(x\backslash y) \wedge e \leq \tau(x)\backslash\tau(y)$.

If τ satisfies the additional equation

(f) $\tau(x) \wedge \tau(y) \leq \tau(x \wedge y)$,

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we say that τ is a *T-left frontal operator* (Tlf for short).

Similarly we define *T-right pre-frontal operators* (Trp for short) and *T-right frontal operators* (Trf for short).

It can be seen that both Tlp and Trp functions are compatible. In any commutative residuated lattice (CRL for short), τ is Tlp (resp. Tlf) if and only if it is Trp (resp. Trf). In this case we simply say that τ is T-pre-frontal (T-frontal respectively). We write Tp and Tf as shorthands for them.

Theorem: Let H be a Heyting algebra, $\tau : H \to H$ a function.

The following conditions are equivalent:

(a) τ is Tf.

(b) τ is a frontal operator.

Let L be a RL. For every $y \in L$ we define $y^0 = e$ and for $k \geq 1$, $y^k = y^{k-1}.y$. Fix a natural number n. We define the functions $\overleftarrow{S}_n : L \to L$ and $\overrightarrow{S}_n : L \to L$ through the following equations:

$(LS1n)$ $\overleftarrow{S}_n(x)^n \backslash x \leq \overleftarrow{S}_n(x)$. $(RS1n)$ $x/\overrightarrow{S}_n(x)^n \leq \overrightarrow{S}_n(x)$.
$(LS2n)$ $\overleftarrow{S}_n(x) \leq y \vee (y^n \backslash x)$. $(RS2n)$ $\overrightarrow{S}_n(x) \leq y \vee (x/y^n)$.

Proposition: Let L be a RL. Then the following conditions hold:

(a) $\overleftarrow{S}_n$ is a Tlp taking $T(x,y) = y^n \backslash x$. Moreover, $\overleftarrow{S}_n(e) = e$.

(b) If the underlying lattice of L is distributive then $\overleftarrow{S}_n$ is a Tlf.

(c) $\overleftarrow{S}_n$ is characterized by $\overleftarrow{S}_n(x) = min\{y \in L : y^n \backslash x \leq y\}$.

Similarly for the case $\overrightarrow{S_n}$.

If L is a CRL, note that there exists S_n (see Section 5 of [2]) iff there exist $\overrightarrow{S}_n$ and $\overleftarrow{S}_n$, being $S_n = \overrightarrow{S}_n = \overleftarrow{S}_n$. For $n = 1$ we write S. This function is called successor function.

Let L be a RL with first element and fix a natural number n. We define the functions $\overleftarrow{\gamma_n} : L \to L$ and $\overrightarrow{\gamma_n} : L \to L$ through the following equations:

$(Lg1n)$ $\overleftarrow{\gamma_n}(0)^n \backslash 0 \leq \overleftarrow{\gamma_n}(0)$. $(Rg1n)$ $0/\overrightarrow{\gamma_n}(0)^n \leq \overrightarrow{\gamma_n}(0)$.
$(Lg2n)$ $\overleftarrow{\gamma_n}(0) \leq y \vee (y^n \backslash 0)$. $(Rg2n)$ $\overrightarrow{\gamma_n}(0) \leq y \vee (0/y^n)$.
$(Lg3n)$ $\overleftarrow{\gamma_n}(x) = x \vee \overleftarrow{\gamma_n}(0)$. $(Rg3n)$ $\overrightarrow{\gamma_n}(x) = x \vee \overrightarrow{\gamma_n}(0)$.

Proposition: Let L be a RL with first element and n a natural number. Then the following conditions hold:

(a) $\overleftarrow{\gamma_n}$ is a Tlp taking $T(x,y) = x \vee (y^n \backslash x)$.

(b) If the underlying lattice of L is distributive then $\overleftarrow{\gamma_n}$ is a Tlf. Moreover, in this case $\overleftarrow{\gamma_n}$ preserves $\wedge$.

(c) Function $\overleftarrow{\gamma_n}$ is characterized by $\overleftarrow{\gamma_n}(x) = min\{y \in L : (y^n \backslash 0) \vee x \leq y\}$.

There is a similar proposition for the case $\overrightarrow{\gamma_n}$. When L be a CRL with first element we write γ_n in place of $\overleftarrow{\gamma_n}$ or $\overrightarrow{\gamma_n}$; for $n = 1$ we write γ. This function will be called gamma function.

Let L be a RL with first element. For every $x \in L$ we define $l(x) = x\backslash 0$ and $r(x) = 0/x$. Fix a natural number n. We define the functions $\overleftarrow{G_n} : L \to L$ and $\overrightarrow{G_n} : L \to L$ through the following equations:

$(LG1n)$ $(\overleftarrow{G_n}(x)^n \backslash x) \wedge rl(x) \leq \overleftarrow{G_n}(x)$. $(RG1n)$ $(x\backslash \overrightarrow{G_n}(x)^n) \wedge lr(x) \leq \overrightarrow{G_n}(x)$.
$(LG2n)$ $\overleftarrow{G_n}(x) \leq y \vee ((y^n \backslash x) \wedge rl(x))$. $(RG2n)$ $\overrightarrow{G_n}(x) \leq y \vee ((x/y^n) \wedge lr(x))$.

Proposition: Let L be a RL with first element and fix a natural number n. Then the following conditions hold:

(a) $\overleftarrow{G_n}$ is a Tlp taking $T(x,y) = y^n \backslash x$. Moreover, $\overleftarrow{G_n}(e) = e$.

(b) Function $\overleftarrow{G_n}$ is characterized by

$$\overleftarrow{G_n}(x) = min\{y \in L : (y^n \backslash x) \wedge rl(x) \leq y\}.$$

There is a similar proposition for the case $\overrightarrow{G_n}$.

When L be a CRL with first element we write G_n in place of $\overleftarrow{G_n}$ or $\overrightarrow{G_n}$; for $n = 1$ we write G. This function will be called Gabbay's function.

Let V be a variety of algebras of type F and let $\epsilon(C)$ be a set of identities of type $F \cup C$ where C is a family of new function symbols. We say that $\epsilon(C)$ defines *implicitly* C, if in each algebra $A \in V$ there is at most one family $\{f_H : H^n \to H\}_{f \in C}$ such that $(A, f_A)_{f \in C}$ satisfies the universal closure of the equations in $\epsilon(C)$ (in this case we say that each f is given by equations).

In [2] (sections 4 and 5) a condition is given on a function $P(x, y)$ in a commutative residuated lattice L that imply that the function $x \mapsto min\{y \in L : P(x, y) \leq y\}$ is equational and compatible when defined. Inspirited by [2] one can ask whether conditions on functions $P(x, y)$ and $Q(x, y)$ in a residuated lattice L imply that the function $x \mapsto min\{y \in L : P(x, y) \leq Q(x, y)\}$ is equational and compatible when defined. We present a sufficient condition for equationally and compatibility; finally we give some examples of these functions.

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Implicational Logics vs. Order Algebraizable Logics

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Although many logical systems have an equivalence connective which is either a primitive symbol or definable by a single formula in two variables, there is a long-established tradition of Abstract Algebraic Logic which considers equivalence connectives definable by means of (possibly infinite and parameterized) sets of formulae. This allows for a well known classification of logics with a good algebraic semantics partly based on their (definable) equivalence connectives: the Leibniz hierarchy of protoalgebraic logics (see e.g. [2]). This classification has been refined in the paper [1] where we have presented the hierarchy of implicational logics. The main idea consists in shifting the focus from equivalence to implication connectives. Indeed, we say that a logic L is weakly p-implicational if there is a set of formulae in two variables (and possibly with parameters) such that L proves its reflexivity, modus ponens, transitivity and congruence property w.r.t. all primitive connectives. By requiring additional properties to this connective we obtain a number of subclasses of logics. Since the symmetrization of a weak p-implication is an equivalence, weakly p-implicational logics coincide with protoalgebraic logics and our classification turns out to be a refinement of Leibniz hierarchy. Furthermore, each weak p-implication induces a preorder in the matrices for the logic which becomes an order in reduced matrices. We have characterized which logics in the hierarchy are complete w.r.t. the matrices where the induced order is linear.

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On the other hand, the recent paper [3] studies the notion of order algebraizable logics based on previous works by Pigozzi. In this approach, instead of considering a built-in notion of order (as that given by implication) one adds a partial order relation as a primitive extra-logical element to algebraic semantics. Then a logic L is order algebraizable if it is equivalent to the inequational consequence given by a class of partially ordered algebras.

In this talk we will compare both approaches and present a classification of logical systems based on their combination.

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Probabilities on Nuanced MV-Algebras

Lavinia Corina Ciungu[*]

The n-nuanced MV-algebras were introsuced by G. Georgescu and A. Popescu as an answer of the problem: starting from a given logical system and using the idea of nuance, how can one construct an n-nuanced logical system based on the given one?

This concept extends both MV-algebras and n-valued Łukasiewicz-Moisil algebras. In the new structure the authors put together two approaches to multiple-valued-ness: that of infinitely-valued Łukasiewicz logic and that of Moisil's n-nuanced logic.

First of all we present the basic definitions and properties of n-nuanced MV-algebras. For the rest of this paper we denote $J = \{1, \cdots, n-1\}$ with $n \in \mathbb{N}$, $n \geq 2$.

Definition 1. A *generalized De Morgan algebra* is a structure $(L, \oplus, \odot, N, 0, 1)$ of the type $(2, 2, 1, 0, 0)$ such that the following conditions are satisfied:
(GM_1) $(L, \oplus, 0)$, $(L, \odot, 1)$ are commutative monoids;
(GM_2) $N(x \oplus y) = Nx \odot Ny$ and $NNx = x$ for all $x, y \in L$.

Consider the structure L of the form $(L, \oplus, \odot, N, \varphi_1, \cdots, \varphi_{n-1}, 0, 1)$ where $(L, \oplus, \odot, N, 0, 1)$, is a generalized De Morgan algebra and $\varphi_1, \cdots, \varphi_{n-1}$ are unary operations on L. For this structure we consider the following axioms:
(nMV_0) $\varphi_i x \oplus (N\varphi_i x \odot \varphi_i y) = \varphi_i y \oplus (N\varphi_i y \odot \varphi_i x)$ for all $i \in J$;
(nMV_1) $\varphi_i(x \oplus y) = \varphi_i x \oplus \varphi_i y$,
$\varphi_i(x \odot y) = \varphi_i x \odot \varphi_i y$,
$\varphi_i(0) = 0$,
$\varphi_i(1) = 1$ for all $i \in J$;
(nMV_2) $\varphi_i x \oplus N\varphi_i x = 1$, $\varphi_i x \odot N\varphi_i x = 0$ for all $i \in J$;
(nMV_3) $\varphi_i \circ \varphi_j = \varphi_j$ for all $i, j \in J$;
(nMV_4) $\varphi_i \circ N = N \circ \varphi_{n-i}$ for all $i \in J$;
(nMV_5) If $\varphi_i x = \varphi_i y$ for all $i \in J$, then $x = y$ (*Moisil's determination principle*).

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Define $M(L) = \{x \in L \mid \varphi_i x = x \text{ for all i} \in \text{J}\}$, called the *MV-center* of L because of the next Proposition.

Definition 2. An *n-nuanced MV-algebra* ($NMVA_n$ for short) is a structure $(L, \oplus, \odot, N, \varphi_1, \cdots, \varphi_{n-1}, 0, 1)$ such that $(L, \oplus, \odot, N, 0, 1)$ is a generalized De Morgan algebra and $\varphi_1, \cdots, \varphi_n$ satisfy the axioms $(nMV_0) - (nMV_5)$ and the axiom:

$$(nMV_6) \qquad \varphi_1 x \leq \varphi_2 x \leq \cdots \leq \varphi_{n-1} x.$$

Let L be an $NMVA_n$, $x \in L$ and $n \in \mathbb{N}$. We introduce the following notations:

$$0x = 0, \quad nx = x \oplus (n-1)x \quad \text{for any } n \geq 1,$$
$$x^0 = 1, \quad x^n = x \odot x^{n-1} \quad \text{for any } n \geq 1.$$

Definition 3. An $NMVA_n$ L is called *centered* if there exist the elements $c_0, \cdots, c_{n-1} \in L$ such that, for all $i, j \in J$,

$$\varphi_i c_j = \begin{cases} 0, & \text{if } i + j < n \\ 1, & \text{if } i + j \geq n. \end{cases}$$

The elements $c_0, \cdots, c_{n-1}$ are called *centers* of L.

Example 4. Let $(A, \oplus, \odot, ^-, 0, 1)$ be an MV-algebra and we define $T(A) = \{(x_1, \cdots, x_{n-1}) \in A^{n-1} \mid x_1 \leq \cdots \leq x_{n-1}\}$. We denote by 0_1 and 1_1 the usual constant vectors. Then A^{n-1} is an MV-algebra with component-wise operations induced from A and $T(A)$ is closed under the operations $0_1, 1_1, \oplus, \odot$. Define $N, \varphi_1, \cdots, \varphi_{n-1}$ by:

$$N(x_1, \cdots, x_{n-1}) = (x_{n-1}^-, \cdots, x_1^-)$$

$$\varphi_i(x_1, \cdots, x_{n-1}) = (x_i, \cdots, x_i), \text{ for i} \in \text{J}.$$

Then $(T(A), \oplus, \odot, N, \varphi_1, \cdots, \varphi_{n-1}, 0_1, 1_1)$ is an $NMVA_n$.

Remark 5. It was proved that n-valued Łukasiewicz-Moisil algebras are connected with Boolean algebras through an adjunction which allows the transfer of many properties from the Boolean case. For any arbitrary $NMVA_n$ L, we consider the function $\psi_L : L \to T(M(L))$ defined by

$$\psi_L(x) = (\varphi_1 x, \varphi_2 x, \cdots, \varphi_{n-1} x) \text{ for any x} \in \text{L}.$$

One can easily check that ψ_L is an injective $NMVA_n$-morphism. If A is an MV-algebra, then the constant vectors are the only elements of $M(T(A))$. For any MV-algebra A, we consider the function $\varphi_A : M(T(A)) \to A$ defined by

$$\varphi_A(x, x, \cdots, x) = x \text{ for all x} \in \text{A}.$$

By the determination principle, it follows that φ_A is an MV-isomorphism.

Let L be an $NMVA_n$. In what follows we introduce the notion of a state on L and we study some of its properties.

Definition 6. A *state* on L is a function $s\colon L \longrightarrow [0,1]$ satisfying:
$(nmv - s_1)$ $s(0) = 0$, $s(1) = 1$;
$(nmv - s_2)$ $s(x \oplus y) = s(x) + s(y) - s(x \odot y)$ for all $x, y \in L$;
$(nmv - s_3)$ $s(x) = \frac{1}{n-1}\sum_{i=1}^{n-1} s(\varphi_i x)$ for all $x \in L$.

Remark 7. If s is a state on L, then $s\mid_{M(L)}$ is a state on $M(L)$.

Proposition 8. *If s is a state on L, then the following hold:*
(1) $s(Nx) = 1 - s(x)$;
(2) *If $x \le y$, then $s(x) \le s(y)$.*

Proposition 9. *If L is a centered lattice-ordered $NMVA_n$ and s is a state on L, then*

$$s(x \vee y) + s(x \wedge y) = s(x) + s(y).$$

Lemma 10. Let L be an $NMVA_n$, s a state on L and $x, a \in L$. Then

$$\sum_{i=1}^{n-1} s(\varphi_i x \odot a) = \sum_{i=1}^{n-1} s(x \odot \varphi_i a).$$

Theorem 11. *Every state $s : M(L) \longrightarrow [0,1]$ can be uniquely extended to a state $s^* : L \longrightarrow [0,1]$.*

Corollary 12. *There is a one-to-one correspondence between the set of states on L and the set of states on the MV-algebra $M(L)$.*

Corollary 13. *Every n-nuanced MV-algebra L admits a state on it.*

Corollary 14. *Let s be a state on L. Then there exists a unique state S on $T(M(L))$ such that $S \circ \psi_L = s$.*

We introduce a general notion of a conditional state on nuanced MV-algebras.

Definition 15. Let s be a state on L and $a \in L$ such that $s(a) > 0$. The *conditional state* $s(\cdot \mid a) : L \longrightarrow [0,1]$ is defined by:

$$s(x \mid a) = \frac{1}{(n-1)s(a)} \sum_{i=1}^{n-1} s(\varphi_i x \odot a) = \frac{1}{(n-1)s(a)} \sum_{i=1}^{n-1} s(x \odot \varphi_i a).$$

The above definition is correct, taking into consideration Lemma 10.

Remark 16. Let s be a state on L and $a \in L$ such that $s(a) > 0$. Then:
(1) $s(0 \mid a) = 0$,
(2) $s(1 \mid a) = 1$,
(3) $s(x \mid 1) = s(x)$ for all $x \in L$,
(4) If $a \in M(L)$ such that $s(a) > 0$, then $s(x \mid a) = \frac{s(x \odot a)}{s(a)}$ for all $x \in L$.

Proposition 17. *Let s be a state on L and $a, b \in L$ such that $s(a), s(b) > 0$. Then*

$$s(a) \cdot s(b \mid a) = s(b) \cdot s(a \mid b).$$

Proposition 18. *If s is a state on L, then*

$$(n-1)s(x \mid a) = \sum_{i=1}^{n-1} s(\varphi_i x \mid a),$$

for every $x, a \in L$ with $s(a) > 0$.

Proposition 19. *If s is a state on L, then*

$$(n-1)s(a) \cdot s(x \mid a) = \sum_{i=1}^{n-1} s(\varphi_i a) \cdot s(x \mid \varphi_i a),$$

for every $x, a \in L$ with $s(\varphi_1 a) > 0$.

Theorem 20. *If $r \in J$ such that $s(\varphi_i a) = 0$ for $i < r$ and $s(\varphi_r a) > 0$ for $i \geq r$, then*

$$(n-1)s(x \mid a) = \sum_{i=r}^{n-1} s(\varphi_i a) \cdot \frac{\sum_{j=1}^{n-1} s(\varphi_j x \mid \varphi_i a)}{\sum_{j=r}^{n-1} s(\varphi_j a)}$$

for every $x, a \in L$ with $s(a) > 0$.

Corollary 21. *If $s(\varphi_i a) = 0$ for $i < n-1$ and $s(\varphi_{n-1} a) > 0$, then*

$$(n-1)s(x \mid a) = \sum_{j=1}^{n-1} s(\varphi_j x \mid \varphi_{n-1} a),$$

for every $x \in L$.

These results show that the computation of a conditional state on L can be reduced to the computation of the state obtained by restriction to MV-algebra $M(L)$.

On MTL-Algebras with an Internal Possibilistic State

Pilar Dellunde[*,†] *Lluís Godo*[†] *Enrico Marchioni*[†]

In some recent works, different notions of graded necessity (in the sense of Possibility theory [3]) for Łukasiewicz and Gödel logic formulas have been explored, see [4] and [2] respectively. These can be considered as possibilistic counterparts of (probabilistic) states, originally defined over algebras of Łukasiewicz logic [5, 6] and later on also on Gödel algebras [1].

Given a t-norm based logic L, by a possibilistic counterpart of a state we understand in this paper an assignment to formulas $N : Fm(L) \to [0,1]$ satisfying at least the following three properties:

1. $N(\top) = 1, N(\bot) = 0$.
2. $N(\varphi \wedge \psi) = \min(N(\varphi), N(\psi))$.
3. $N(\varphi) = N(\psi)$ whenever $\vdash_L \varphi \equiv \psi$.

corresponding to those of a necessity measure respecting logical equivalence.

In [4, 2] modal logics expanding the base logic with a modal operator N over a restricted language (not allowing nested applications of the modal operator) have been introduced and Kripke style semantics have been defined. Inspired by [5], preliminary steps are taken in [2] to define a full modal logic over Gödel logic, called PNG. Its algebraic semantics consists of the class of NG-algebras, which are expansions of Gödel algebras with a unary operator capturing the main properties of necessity measures, that can be understood as a internal possibilistic state.

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In this paper we plan to generalize this approach to the weaker framework of MTL logic. The axioms and rules of the basic fuzzy modal logic over MTL we will consider, named N-MTL, will be those of MTL plus the Necessitation rule for N (from ψ derive $N\psi$) and the same modal axioms of PNG:

1. $N(\varphi \wedge \psi) \leftrightarrow (N\varphi \wedge N\psi)$.
2. $N\psi \leftrightarrow NN\psi$.
3. $\neg N\overline{0}$.

The aim of this paper is to study the algebraic semantics of the N-MTL logic given by the variety of N-MTL algebras, where an N-MTL algebra is a structure (A, N) where A is a MTL-algebra and $N : A \to A$ is a monadic operator satisfying the equations corresponding to the above properties. In particular we will focus on faithful N-MTL algebras (those satisfying the property $Nx > 0$ for $x > 0$) and prove some results on satisfiability, specially for some subvarieties of interest.

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Loomis-Sikorski Theorem and Stone Duality Theorems for MV-Algebras with Internal State

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In the last decade, the interest to probabilistic uncertainty in many valued logic increased. A new approach to states on MV-algebras was recently presented by T. Flaminio and F. Montagna in [6]; they added a unary operation, τ, (called as an inner state or a state-operator) to the language of MV-algebras, which preserves the usual properties of states. It presents a unified approach to states and probabilistic many valued logic in a logical and algebraic settings.

We recall that a *state MV-algebra* is a couple (A, τ), where τ is a mapping from A into itself such that satisfying, for each $x, y \in A$:

(i) $\tau(0) = 0$,

(ii) $\tau(x^*) = (\tau(x))^*$,

(iii) $\tau(x \oplus y) = \tau(x) \oplus \tau(y \odot (x \odot y)^*)$,

(iv) $\tau(\tau(x) \oplus \tau(y)) = \tau(x) \oplus \tau(y)$;

the operator τ is said to be a *state-operator*.

In [1, 2, 3], the authors studied a subvariety of state MV-algebras, called *state-morphism MV-algebras* as state-MV-algebras (A, τ) such that τ is an

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MV-homomorphism from A into itself such that $\tau \circ \tau = \tau$, called a *state-morphism-operator*. In the talk, we show how subdirectly irreducible elements can be described:

Theorem 1. *Let (M, σ) be a subdirectly irreducible state-morphism MV-algebra. Then (M, σ) is one of the following three possibilities.*

(i) *M is linear, $\sigma = \mathrm{id}_M$, and the MV-reduct M is a subdirectly irreducible MV-algebra.*

(ii) *The state-morphism operator σ is not faithful, M has no nontrivial Boolean elements, and the MV-reduct M of (M, σ) is a local MV-algebra.*

(iii) *The state-morphism operator σ is not faithful, M has a nontrivial Boolean element. There are a linearly ordered MV-algebra A, a subdirectly irreducible MV-algebra B, and an injective MV-homomorphism $h : A \to B$ such that (M, σ) is isomorphic as a state-morphism MV-algebra with the state-morphism MV-algebra $(A \times B, \sigma_h)$, where $\sigma_h(x, y) = (x, h(x))$ for any $(x, y) \in A \times B$.*

We show that any state-operator on the variety $\mathrm{V}(S_1, \ldots, S_n)$ is a state-morphism-operator, [3]. We describe an analogue of the Loomis-Sikorski theorem, [4], for a state-morphism MV-algebra (A, τ), where A is a σ-complete MV-algebra and τ is a σ-endomorphism: We show that any such state-morphism MV-algebra is a σ-epimorphic image of $(\mathcal{T}, \tau_{\mathcal{T}})$, where $\mathcal{T}$ is a tribe defined on a totally disconnected compact Hausdorff topological space and $\tau_{\mathcal{T}}$ is a σ-endomorphism generated by a continuous function.

Finally we show Stone Duality Theorems, [5], for (i) the category of Boolean algebras with a fixed state-operator and the category of compact Hausdorff topological spaces with a fixed idempotent continuous function, and for (ii) the category of weakly divisible σ-complete state-morphism MV-algebras and the category of Bauer simplices whose set of extreme points is basically disconnected and with a fixed idempotent continuous function.

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On the Hyperreal State Space

Antonio Di Nola[*] *Tommaso Flaminio*[†]

The aim of this work is to study a class of non-archimedean valued measures on MV-algebras. We call them *hyperreal states* and their definition naturally arise from (the uniform version of) Di Nola representation theorem for MV-algebras (cf [5, 6]): for any MV-algebra $A = (A, \oplus, \neg, \top, \bot)$ there exists a ultrafilter $\mathfrak{U}$ on the cardinality of A such that A embeds into $({}^*[0,1]_{\mathfrak{U}})^{Spec(A)}$ (where as usual $Spec(A)$ denotes the space of prime ideals of A). Therefore, if A is any MV-algebra, there exists a non-archimedean extension ${}^*[0,1]_{\mathfrak{U}}$ of the real unit interval $[0,1]$ such that every element a of A can be regarded as a function $f_a : Spec(A) \to {}^*[0,1]_{\mathfrak{U}}$. Since MV-algebras are the equivalent algebraic semantics for Łukasiewicz logic, Di Nola's theorem states that formulas of Łukasiewicz calculus are visualized as black and white pictures printed by a palette of *infinitesimals* grey levels ([9, §2]).

As it is well known [3], the proper subclass of *semisimple* MV-algebras can be characterized, by Chang and Belluce theorem (cf. [2, 1]), as algebras of *real-valued* functions: up to isomorphisms every semisimple MV-algebra A is an algebra of $[0,1]$-valued functions defined over the space of maximal ideals $\mathcal{M}(A)$ of A. Therefore, following the above metaphor, if we interpret formulas of Łukasiewicz calculus into a semisimple MV-algebra, then by Chang and Belluce theorem there are no infinitesimals grey levels, and hence the palette used to print (i.e. interpret) the propositions, only has *real* grey levels (cf. [9, §2]).

States on MV-algebras have been introduced by Mundici in [9]: for every MV-algebra A, *state* on A is a map $s : A \to [0,1]$ that is, normalized and additive[1]. Therefore states are real valued maps defined on (possibly) hyperreal-valued functions, and hence they do no preserve, the non-archimedean structure of the MV-algebra they are defined over.

Therefore we introduce *hyperreal states* to support the intuition that, if an MV-algebra A do have infinitesimal elements (and hence, is not semisimple),

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[1]We refer the reader to [9, Definition 2.1].

then a *state* on A should preserve the same grey-level palette also to *measure* its elements. Those are defined as follows: let A be an MV-algebra, then a *hyperreal state* on A is a map $s : A \to {}^*[0,1]_{\mathfrak{U}}$ satisfying:

(i) $s(\top) = 1$ (normalization),

(ii) whenever $x \odot y = \bot$, $s(x \oplus y) = s(x) + s(y)$ (additivity).

We denote by $\mathcal{HS}(A)$ the set of all hyperreal states on A. Consider the following: for every A, let $\mathfrak{p}$ be a prime filter in $Spec(A)$. The quotient $A/\mathfrak{p}$ is linearly ordered (call f the canonical homomorphism of A into $A/\mathfrak{p}$), and $A/\mathfrak{p}$ embeds into ${}^*[0,1]_{\mathfrak{U}}$ via a map g (where $\mathfrak{U}$ is referred to A). Then the map $s_{\mathfrak{p}} = g \circ f : A \to {}^*[0,1]_{\mathfrak{U}}$ is a hyperreal state of A, whence, $\mathcal{HS}(A)$ is non empty.

Let ${}^*[0,1]$ be any non-archimedean extension of the real unit interval $[0,1]$. Let us denote by sh is the *shadow map* from ${}^*[0,1]$ to $[0,1]$ mapping every $x \in {}^*[0,1]$ into that unique real $sh(x)$ such that the distance $|x - sh(x)|$ is infinitesimal. Then hyperreal states provide a generalization of states [9] in the following sense:

Theorem 1. If A is a semisimple MV-algebra (and hence A is an MV-algebra of $[0,1]$-valued functions), every hyperreal state on A actually is a state. Moreover if A is not semisimple and s is a hyperreal state on A, then the composition $sh \circ s$ is a state on A.

In order to study the geometric properties of the hyperreal state space $\mathcal{HS}(A)$ as a subspace of ${}^*[0,1]_{\mathfrak{U}}^A$, we noticed that there is no a standard way to extend the usual interval topology of reals to every unit interval ${}^*[0,1]$ and keeping the space to be locally convex. The S-topology (cf. [8]) suffices our purposes of making ${}^*[0,1]$ a locally convex space, but unfortunately it does not preserve the property of being Hausdorff. The hyperreal state space $\mathcal{HS}(A)$ can hence be regarded as a compact subspace of the locally convex space ${}^*[0,1]_{\mathfrak{U}}^A$. Then Krein-Milman theorem can be applied to show that $\mathcal{HS}(A)$ coincides with the closure of its extremal points $\mathrm{ext}(\mathcal{HS}(A))$. As it is well known (see [9, Theorem 2.5]) the extremal states of A are homomorphisms of A into $[0,1]$. As regards to hyperreal states we proved the following:

Theorem 2. Let A be an MV-algebra, and let $s \in \mathcal{HS}(A)$. Then s is an extremal hyperreal state iff $sh \circ s$ is extremal state.

The above theorem, together with [9, Theorem 2.5] states that a hyperreal state s on an MV-algebra A is extremal iff $sh \circ s$ is a homomorphism of A into $[0,1]$. Therefore the extremal hyperreal states on A and the space $Spec(A)$ of prime filters of A, are related as follows: for every prime filter $\mathfrak{p} \in Spec(A)$, the map

$$a \in A \overset{f}{\mapsto} a/\mathfrak{p} \in A/\mathfrak{p} \overset{g}{\mapsto} {}^*\alpha \in {}^*[0,1]_{\mathfrak{U}}, \qquad (1)$$

where f and g are as above, is a homomorphism, and hence is an extremal hyperreal state of A because $sh \circ f \circ g$ is a homomorphism of A into $[0,1]$. Conversely, for every homomorphism $s : A \to {}^*[0,1]_{\mathfrak{U}}$ (and hence $sh \circ s \in \mathrm{ext}\mathcal{S}(A)$), $\mathfrak{p}_s = \{x \in A : h(x) = 1\}$ is a prime filter of A. In fact, since s is a homomorphism, its kernel is an ideal of A, whence $\mathfrak{p}_s$ is a filter. Moreover, if $x \wedge y \in \mathfrak{p}_s$, then $s(x \wedge y) = 0$, and hence $s(x) \wedge s(y) = 0$. Since ${}^*[0,1]_{\mathfrak{U}}$ is totally ordered, $s(x) = 0$, or $s(y) = 0$, i.e. $x \in \mathfrak{p}_s$, or $y \in \mathfrak{p}_s$, whence $\mathfrak{p}_s$ is prime filter.

In the case of states, the space of extremal states $\mathrm{ext}(\mathcal{S}(A))$, and the space $\mathcal{M}(A)$ of maximal filters of A are homeomorphic. This parallelism seems not to be recoverable when we move to hyperreal states, and hence when we consider the space $Spec(A)$ (endowed with spectral topology) and $\mathrm{ext}(\mathcal{HS}(A))$. Actually, $Spec(A)$ is a T_0 space, while on the other hand ${}^*[0,1]$ endowed with the S-topology is not T_0 (cf. [4, Theorem 1.4]).

Finally we applied hyperreal states to define a class of generators for the variety of SMV-algebras (cf. [7]). Recall that an SMV-algebra is a pair (A, σ) where A is an MV-algebra, and $\sigma : A \to A$ satisfies: (1) $\sigma(\top) = \top$; (2) $\sigma(\neg x) = \neg\sigma(x)$; (3) $\sigma(\sigma(x) \oplus \sigma(y))$; (4) $\sigma(x \oplus y) = \sigma(x) \oplus \sigma(y \ominus (x \odot y))$; where in (4), $a \odot b$ stands for $\neg(\neg a \oplus \neg b)$, and $a \ominus b$ stands for $\neg(\neg a \oplus b)$.

The class of SMV-algebras forms a variety that can be generated by hyperreal states. In fact consider the following construction: for every MV-algebra B, and every hyperreal state $s : B \to {}^*[0,1]_{\mathfrak{U}}$, define:

- The MV-algebra $A = {}^*[0,1]_{\mathfrak{U}} \otimes B$, where $\otimes$ denotes the *MV-tensorial product* (cf. [10]). We denote by $\alpha \otimes b$ (where $\alpha \in {}^*[0,1]_{\mathfrak{U}}$, and $b \in B$) the generic element of A.

- The operator $\sigma_s : A \to A$ such that, for every $\alpha \otimes b \in A$, $\sigma_s(\alpha \otimes a) = s(a) \cdot \alpha \otimes \top$.

Then (A, σ_s) is an SMV-algebra. It is not hard to see that and the variety of SMV-algebras can be generated by this class of structures.

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Intermediate Fuzzy Quantifiers and Their Properties

*Antonín Dvořák**   *Petra Murinová**   *Vilém Novák**

In this contribution, we will present new results on *intermediate fuzzy quantifiers*. We also show some of their semantic properties, and mention connections to fuzzy quantifiers determined by measures.

Intermediate fuzzy quantifiers were introduced by Novák in [4] as an interpretations of *intermediate quantifiers* [5] in the frame of fuzzy logic. Examples of these quantifiers are *most*, *large part of*, *a few*, etc. It is natural to study these quantifiers in the frame of multi-valued and fuzzy logics, because truth degrees of sentences involving them intuitively run continuously from falsity to truth, when cardinalities of sets of objects in their interpretations change.

Intermediate fuzzy quantifiers are developed in the frame of *fuzzy type theory* (FTT) [3]. It permits to define them as special formulas of FTT (not as new logical symbols). Hence, all proofs involving intermediate quantifiers are carried out in the basic formal system of FTT. To be able to define generalized quantifiers, it is necessary to introduce special formulas representing *measures* into FTT.

In [4], there were proved several tens of 105 syllogisms introduced in [5]. An example of such syllogism is:

Almost all Y are M.
All M are X.

Some X are Y.

Now, we succeeded to prove all of them. We present several interesting cases of these syllogisms and discuss their proofs.

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Further, we will investigate some properties of interpretations of these quantifiers. We are following definitions of semantic properties of fuzzy quantifiers in [2]. It can be easily shown that interpretations of intermediate fuzzy quantifiers are conservative and possess the property of extension. Finally, we discuss how recently introduced *fuzzy quantifiers determined by measures* (see [1], where case of quantifiers of one argument is studied, case of quantifiers of two arguments is in preparation) can be used as models of intermediate quantifiers.

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An Advertisement for Kleisli Categories

*Patrik Eklund**   *Robert Helgesson**

Overview

Monads have been known for more than fifty years [13, 16], and have been successfully used within algebra, logic and topology, by both mathematicians as well as computer scientists.

Our kind of favourite 'root monad' has been the term monad, and in this advertisement we tend to like Kleisli [17] categories more than Eilenberg-Moore [2] categories. Algebras [20, 21] are also nice, but here we need mostly substitutions, i.e. morphisms of the Kleisli categories, for our logical considerations. Roughly speaking, Eilenberg-Moore categories lean a bit more on 'topology and algebra', and are computationally interesting from semantic point of view, whereas Kleisli categories appear more as part of 'logic and algebra', and they are syntactic.

Over `Set`, the category of sets, composition of suitable monads with the term monad gives generalized views on substitutions [7]. Fuzzy sets of terms is a good example, and also provides a concept of non-classical non-determinism.

Clearly, we can then move over to trying out term functors over other categories, e.g. like Goguen's $\mathtt{Set}(L)$, and it works well [3, 10], so that we place uncertainty in a different way as compared to composing (over `Set`) with the many-valued powerset monad [7]. From there we can then basically try out any category, and that is now our present and future work.

All this we want to integrate into a general logics ([22, 14, 15]) machinery involving these composed monads. Some results we have [9], and here we need partially ordered monads, i.e. monads (F, η, μ), where $(\mathsf{F}X, \preceq)$ is always partially ordered with some order-preserving conditions for μ.

We have some distributive laws also for these partially ordered monads [9]. The history of partially ordered monads, as far as we know, comes from late

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90's, when partially ordered monads [12] were used for generalized notions of convergence spaces [18, 11, 4, 5, 6].

We have also established how the Kleisli morphisms of partially ordered monads form Kleene algebras [8], which opens up avenues towards investigation of entirely new types of programming language constructions [19].

Presently we show what happens when we do 'Kleisli over Kleisli', i.e. composing distributive laws. Beck [1] did something on that already in those days, but again, everything was always focused on algebras and Eilenberg-Moore categories, and not Kleisli categories. For 'Kleisli over Kleisli, and so on, composition' a kind-of associativity can be established, and this gives yet another technique for creating new interesting monads from old, in turn opening up more possibilities for substitution theories.

Syntax apart from semantics

The distinction between variable substitutions and variable assignments is more clear for computer scientists than for mathematicians. However, also in computer science they need to be kept apart.

Additionally, terms are usually defined informally e.g. as in the following:

> Let Ω be a set of operators, and X a set of variables. All variables are terms, constants are terms, and then inductively, if ω is an n-ary operator, and $t_1, \ldots, t_n$ are terms, then $\omega(t_1, \ldots, t_n)$ is a term.

Books usually say 'we keep it single sorted, since many-sortedness is just an exercise'. It is not that simple. In particular, when aiming at doing something interesting and innovative in fuzzy logic, you need to be even more formal so as not to hide important issues behind informal definitions and statements.

Arithmetics is a good example. Are we doing fuzzy arithmetics or arithmetics with fuzzy? The application decides. Fuzzy arithmetics involves uncertainty on operation level, i.e. we then consider the term monad over $\mathtt{Set}(L)$. Arithmetics with fuzzy is more ad hoc, and means using the composed monad $\mathbf{L} \bullet \mathbf{T}_\Sigma$ over $\mathtt{Set}$. With many-sortedness you need to move over to $\mathtt{Set}_S$, S being the set of sorts (or types), with objects $\{X_\mathtt{s}\}_{\mathtt{s}\in S}$, where $X_\mathtt{s}$, $\mathtt{s} \in S$, are objects in $\mathtt{Set}$. And so on. Considerations enter the scene with signature morphisms being included.

In a categorical approach a most important question is where to draw the line between what is managed by categorical tools and constructions, and what resides in the metalanguage. There is an additional distinction between GB and ZF but this is outside the scope here. Category usually builds upon ZF. Take a signature (S, Ω). Is S a set in ZF or handled as an object in $\mathtt{Set}$. What about Ω? Clearly, Ω is neither, since it resides in $\mathtt{Set}_S$.

From there you go to algebras, and some say it's just Eilenberg-Moore. Where do you precisely move over from syntax to semantics? And when and where you do, what happens with the operators? Do you need a signature morphism somewhere there in that giant leap? When moving over, can your underlying category change? Should it change. An Italian and a Greek reads New York Times. What they read is the same. What they understand in their models is different. Is there anybody out there claiming that their underlying categories are identical? Does it make any sense to say that you are allowed syntactically to do arithmetics with fuzzy and in your interpretation doing fuzzy arithmetics.

Work referred to in our reference list contains the equipment and tools to answer these questions, and, in our view, is ambitious enough to claim production of new innovations in fuzzy logic thinking.

What we say in this advertisement is essentially 'unfold, don't hide', and be careful about explaining why something is left hanging in the metalanguage. Pure constructions, and careful considerations of what resides in the geography of categorical constructions, provides the required *transparency of constructions* (alternative title of this abstract!) needed when you want to place uncertainty on your map, in your application, for your end-user.

Generalized general logic

As mentioned, we wish to integrate these notions into the framework of general logic whereby it is then possible to describe and reason about non-classical logics in a firmly grounded manner. Central to this is the extension of classical general logic in the style of ([22, 14]) to something akin to a *generalized* general logic.

We observe the difference between the classical setting and generalized setting by noting the implicit existence of the power set monad throughout general logic. Making the underlying monads explicit opens up possibilities for useful non-classical generalizations.

Traditionally we find entailment, denoted $\vdash$, described as a relation over $\mathsf{PSen}(\Sigma) \times \mathsf{Sen}(\Sigma)$ where P is the power set functor and Sen is the sentence functor taking signatures to the set of logical sentences or statements over that signature. In [9] we described the generalization of $\vdash$ to an L-valued relation $\vdash : \Phi\mathsf{Sen}(\Sigma) \times \Phi\mathsf{Sen}(\Sigma) \to L$ where Φ is a general functor that is also part of a partially ordered monad $\mathbf{\Phi}$ and L is a lattice. Consider now that any lattice L naturally give rise to a partially monad $\mathbf{L} = (\mathsf{L}, \preceq, \eta^{\mathbf{L}}, \mu^{\mathbf{L}})$ with $\mathsf{L}X = L^X$, i.e., the set of mappings from X to L and morphisms $f : X \to Y$ in $\mathtt{Set}$ mapped according to

$$\mathsf{L}f(A)(y) = \bigvee_{f(x)=y} A(x).$$

Here we adopt the convention that $\bigvee \emptyset = 0$. Further, $\preceq$ on $\mathsf{L}X$ is defined pointwise, $\eta_X : X \to \mathsf{L}X$ is given by

$$\eta_X(x)(x') = \begin{cases} 1 & \text{if } x = x' \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

and $\mu_X : \mathsf{L}(\mathsf{L}X) \to \mathsf{L}X$ by

$$\mu_X(\mathcal{M})(x) = \bigvee_{A \in \mathsf{L}X} A(x) \wedge \mathcal{M}(A). \tag{2}$$

Thus, this *fuzzy power set monad* is a prime example of a useful choice for $\mathbf{\Phi}$. Further, since $\Phi\mathsf{Sen}(\Sigma) \times \Phi\mathsf{Sen}(\Sigma) \to L$ is isomorphic to $\Phi\mathsf{Sen}(\Sigma) \to \mathsf{L}\Phi\mathsf{Sen}(\Sigma)$, we find that the generalized $\vdash$ relation can quite readily be seen as a morphism in the Kleisli category of our underlying truth monad.

Within this generalized framework we define equational and first order logic in a way readily available to non-classical extensions.

This logic approach provides an elegant approach to abstraction from logical operators, yet maintaining expressive power with respect to entailment and representation of non-classical sets of sentences and clauses. Substitution as morphisms in the Kleisli category over underlying monads and monad compositions is a key language construct as it includes both representation of generalized terms as well as shareable knowledge in a generalized general logics. Here we are then able to communicate knowledge between respective selected logics, enabled by mappings between logics and their respective proof calculi.

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On Dialogue Games for Multi-Valued Logics

*Christian G. Fermüller**   *Christoph Roschger**

Non-classical logics are often only presented syntactically by Hilbert style proof systems and semantically by corresponding algebraic structures. As an alternative approach to formal logics, dialogue games provide an independent characterization that turns out to be closely related to analytic Gentzen style proof systems. Our starting point is Robin Giles's game for Łukasiewicz logic $\mathbf{Ł}_\infty$. We investigate to which extent the game can be generalized in a manner that still yields a characterization of a truth functional many-valued logic.

Giles's game [5, 6] is a combination of a *dialogue game* and a *betting scheme* originally motivated for characterizing reasoning in physical theories. Arguments about logically complex statements are reduced to arguments about atomic ones governed by dialogue rules that are intended to capture the meaning of logical connectives. In the final state of the dialogue game, the players place bets on the results of dispersive experiments that decide about 'truth' and 'falsity' of occurrences of corresponding atomic statements.

The dialogue part of Giles's game is a two-player zero-sum game with perfect information. The players are called *y*ou and *m*e, with *m*e initially asserting a logically complex statement. The game can be considered an evaluation game, since the players devise their strategies with respect to a pay off function that is determined by given success probabilities of experiments associated with atomic assertions.

At any point in the game each player asserts a multi-set of propositions, called *tenet*. Accordingly a game state is denoted as $[\psi_1, \ldots, \psi_n \mid \phi_1, \ldots, \phi_m]$ where $[\psi_1, \ldots, \psi_n]$ is your tenet and $[\phi_1, \ldots, \phi_m]$ is mine, respectively. Initial game states take the form $[\mid \phi]$; i.e., I assert a single statement ϕ, while your tenet is empty. In each move of the game one of the players picks one of the statements asserted by her opponent and either challenges it or grants it

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explicitly. In both cases the picked statement is deleted from the game and therefore cannot be challenged again. The other player has to respond to the challenge in accordance with the following rules, that can be traced back to Lorenzen [8].

($R_\supset$) A player asserting $\phi \supset \psi$ agrees to assert ψ if his opponent will assert ϕ.

($R_\vee$) A player asserting $\phi \vee \psi$ undertakes to assert either ϕ or ψ at his own choice.

($R_\wedge$) A player asserting $\phi \wedge \psi$ undertakes to assert either ϕ or ψ at his opponent's choice.

Negation is considered equivalent to the implication of a statement $\perp$ that is always evaluated as 'false'. Thus we obtain:

($R_\neg$) A player asserting $\neg\phi$ agrees to pay 1€ to his opponent if he (the opponent) will assert ϕ.

As already indicated, Giles stipulated that at the final state of the game the players have to pay a fixed amount of money, say 1€, for each atomic statement in their tenet that is evaluated as 'false' according to an associated experiment. These experiments may show dispersion, i.e., they may yield different answers upon repetition. However a fixed *risk value* $\langle p \rangle$ specifies the probability that the experiment associated with the atomic statement p results in a negative answer. My risk, i.e., the expected amount of money that I have to pay you, for asserting an atomic tenet $[p_1, \ldots, p_n]$ therefore amounts to $\sum_{i=1}^{n} \langle p_i \rangle$. Consequently my total risk for an atomic game state is calculated as the difference between the risks of out tenets.

Giles proved the following:

Theorem 1. For every formula ϕ, every risk value assignment $\langle \cdot \rangle$, the following are equivalent:

- initially asserting ϕ, I have a strategy for the game to enforce a final elementary state, where my risk according to $\langle \cdot \rangle$ is not higher than x€, while you have a strategy to enforce a final elementary state, where my risk according to $\langle \cdot \rangle$ is not lower than x.
- $v(F) = x$, where v is the standard evaluation function for Łukasiwicz logic $\mathbf{Ł}_\infty$, extending the valuation given by $v(p) = 1 - \langle p \rangle$ for all propositional variables p.

It follows immediately from Theorem 1 that a formula ϕ is valid in $\mathbf{Ł}_\infty$ iff for every (properly restricted) risk value assignment I have a strategy to avoid expected loss in games that start with my initial assertion of ϕ.

As has been demonstrated in [4] an alternative rule for conjunction, that corresponds to the 'strong conjunction' interpreted by the Łukasiewicz t-norm $\otimes$, can be specified as follows.

($R_\&$) A player asserting $\phi \& \psi$ undertakes to assert either both, ϕ and ψ, or else to pay 1€ to his opponent.

A simple parameter one can change when systematically looking for variants of Giles's game is the presence of the *principle of limited liability*. Although implicitly included in Giles's original rules, we may simplify the dialogue rules and re-introduce the principle of limited liability in a more systematic manner than done by Giles. Especially the respective choices the players are allowed to make in the dialogue rules for $\supset$ and & do not have to be explicitly stated by the dialogue rule. Consider the following:

Attack principle of limited liability: Every player can, instead of attacking a compound formula ϕ asserted by the opponent player, declare that he will not attack ϕ at all.

Defense principle of limited liability: Every player can, instead of defending a compound formula ϕ asserted by him according to the dialogue rules, alternatively assert $\bot$.

These two principles basically ensure that both players can limit their expected loss from asserting one proposition to 1€. By dropping them, one can instantly turn Giles's game into a game adequate for Abelian logic **A**.

Note that Giles's elaborated story about dialogues about why and how to evaluate atomic formulas by betting on the results of associated dispersive experiments for which success probabilities are known, boils down to just some assignment of concrete payoff values to final states in a game. In particular, the reference to probabilities completely disappears: only the expected amount of money to be paid or received is relevant. From the game theoretic point of view, we only need a real number as payoff value for each final state.

The central question in our contribution is to what extent Giles's game can be generalized while still retaining the same close connection to some many-valued logic. This way truth tables for the connectives are defined by means of the game. Possible parameters for extending or adapting Giles's game are the evaluation scheme for atomic game states, the dialogue rules themselves, and whether the so-called *principle of limited liability* remains in force. Note that the variants of Giles's game described in [1, 2], covering also Gödel logic **G** and Product logic **P**, are not of concern here, as these do not share the same correspondence between truth values and risk assignments. There, only the (weaker) correspondence between validity and strategies to avoid expected loss under all risk assignments.

As a first step we abstract away from risk assignments and simply speak (inversely) of some payoff assigned to final, atomic states. We formulate three conditions on these payoff functions.

Definition (Symmetry). *A payoff function* $\langle\cdot\rangle$ *is* symmetric *if for all (atomic) tenets* Γ *the following holds:* $\langle\Gamma \mid \Delta\rangle = -\langle\Gamma \mid \Delta\rangle$.

Definition (Context-independence). *An payoff function* $\langle\cdot\rangle$ *is* context-independent *if for any tenets* Γ,Δ,Γ', Δ', Γ'', *and* Δ'' *the following holds: If* $\langle\Gamma' \mid \Delta'\rangle = \langle\Gamma'' \mid \Delta''\rangle$ *then* $\langle\Gamma, \Gamma' \mid \Delta', \Delta\rangle = \langle\Gamma, \Gamma'' \mid \Delta'', \Delta\rangle$.

A context-independent symmetric payoff function can be characterized by a binary associative, commutative function $\circ$ such that the payoff value of an atomic game state can be computed from the payoff values of the respective atomic propositions.

Finally, the following condition ensures that adding the same propositions to different atomic game states preserves the order of the respective payoffs.

Definition (Monotonicity). *An payoff function* $\langle\cdot\rangle$ *is* monotone *if for any tenets* Γ,Δ,Γ', Δ', Γ'', *and* Δ'' *the following holds:*
If $\langle\Gamma' \mid \Delta'\rangle \leq \langle\Gamma'' \mid \Delta''\rangle$ *then* $\langle\Gamma, \Gamma' \mid \Delta', \Delta\rangle \leq \langle\Gamma, \Gamma'' \mid \Delta'', \Delta\rangle$.

Definition (Discriminating). *A payoff function* $\langle\cdot \mid \cdot\rangle$ *is* discriminating *if* $\langle\cdot \mid \cdot\rangle$ *is symmetric, context-independent, and monotone.*

For the dialogue rules the most important requirement is that they are *decomposing.* This ensures that the game is of finite depth and, thus, determined.

Definition (Decomposition). *A dialogue rule is* decomposing *if an attack on the occurrence of a compound formula* $\diamond(\phi_1, \ldots, \phi_n)$ *results in states where it is replaced by occurrences of subformulas* ϕ_i *and truth constants.*

Moreover it is necessary to require the rules are symmetric with respect to the two players.

Definition (Duality). *Two rules are called* dual *if one results from the other by systematically switching the roles of the two players.*

Theorem 2. Let $\Game$ be a game with a discriminating payoff function $\langle \cdot \mid \cdot \rangle$ and decomposing dialogue rules respecting duality. Then one can extract from $\langle \cdot \mid \cdot \rangle$ and the rules a set of truth functions $\mathcal{F}_{\Game}$ over $\mathbb{R}$ such that the following two values are the same for every formula ϕ:

- the highest payoff guaranteed by my best strategy for a $\Game$-play starting in the game state $[\mid \phi]$,
- the truth value of ϕ according to $\mathcal{F}_{\Game}$ under the interpretation that assigns $\langle \mid p \rangle$ to p for all atomic formulas p.

Conversely, we offer a characterization of the family of many-valued logics that can be described by such games. This sheds new light on the expressivity of Giles's style evaluation games.

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A Duality for Quasi Ordered Structures

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Recently, several authors extended Priestley duality for distributive lattices [9] to other classes of algebras, such as, e.g. distributive lattices with operators [7], MV-algebras [8], MTL and $IMTL$ algebras [1]. In [4] necessary and sufficient conditions for a normally presented variety to be naturally dualizable, in the sense of [5], i.e. with respect to a discrete topology, have been provided. Under this perspective, also bdq-lattices are naturally dualizable. Nonetheless, quasi lattices, introduced in [3], constitute a generalization of lattice ordered structures to preordered ones, i.e. structures in which the ordering relation $\leq$ is reflexive and transitive, but it may fail to be anti-symmetric. Consequently, a sensible question arises: is there any "natural" candidate which stands to Priestley spaces as bounded distributive quasi lattices stand to bounded distributive lattices?

In this work we present two alternative form of dualization for *bdq*-lattices: by using the notion of *preordered Priestley spaces* and by *covering spaces*.

A *distributive bounded q-lattice*, introduced by Ivan Chajda in [3], (bdq-lattice for short) is an algebra $\langle A, \vee, \wedge, 0, 1\rangle$ of type $\langle 2, 2, 0, 0\rangle$ satisfying the following conditions:

1. $\langle A, \vee\rangle$ and $\langle A, \wedge\rangle$ are commutative semigroups;
2. $x \vee (x \wedge y) = x \vee x$; $\qquad x \wedge (x \vee y) = x \wedge x$;
3. $x \vee (y \vee y) = x \vee y$; $\qquad x \wedge (y \wedge y) = x \wedge y$;
4. $x \wedge x = x \vee x$;
5. $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$;
6. $x \wedge 0 = 0$; $\qquad x \vee 1 = 1$.

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In this work we first introduce the notion of preordered Priestley space and we present its main properties. Preordered Priestley spaces are, in our opinion, of a certain interest in that they share with Priestley spaces several desirable features; moreover, they interpret, with respect to bdq-lattices, the same rôle Priestley spaces play with respect to bounded distributive lattices. Upon recalling that a subset Y of a preordered set $\langle X, \preceq \rangle$ is said to be *increasing* if and only if, for any $x \in X$ and $y \in Y$, if $y \preceq x$, then $x \in Y$, and also that Y is said to be *almost increasing* if and only if $Y/_{\equiv} = \{[x]_{\equiv} \in X/_{\equiv} : x \in Y\}$ is increasing in $X/_{\equiv}$, where, $\forall x, y \in X$, $x \equiv y$ iff $x \preceq y$ and $y \preceq x$, we introduce the notion of preordered Priestley space:

A *preordered Priestley space* (*pp-space*) $\chi = \langle X, \tau, \preceq \rangle$ is a preordered (i.e. $\langle X, \preceq \rangle$ is a preordered set) topological space which is compact and satisfies the *weak separation axiom*: if $x \not\preceq y$ then there is a *clopen almost increasing set* U such that $x \in U$ and $y \notin U$.

In the first part of this work we show that the category whose objects are bdq-lattices, and whose arrows are bdq-lattice homomorphisms, is equivalent to the category whose objects are pp-spaces, and whose arrows are pp-space morphisms.

In the second part, we propose an alternative form of dualizing the category of bdq-lattices. To this aim we will recur to the notion of covering space.

A *covering map* [6] $p : A \to I$ is a continuous map between topological spaces such that each $x \in I$ has an open neighborhood U with $x \in U \subseteq I$, for which $p^{-1}(U)$ is a disjoint union of open set $(U_j)_j$, each of which is mapped homeomorphically onto U by p. Thus a covering map is a local homeomorphism. In this case the triple $\langle A, p, I \rangle$ is called a *covering space* of I. The special open neighborhoods U of x given in the definition are called *evenly-covered neighborhoods*. The evenly-covered neighborhoods form an open cover of the space A. In this framework we introduce the *Priestley covering map* given by covering maps over Priestley spaces with a distiguished cross-section and a special property in the evenly-covered neighborhoods. We show that the category whose object are Priestley covering maps and whose arrows are the usual étale space-arrows that preserve a distiguished evenly-covered neighborhoods is equivalent to the category of bdq-lattices.

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Local Finiteness in t-Norm-Based Structures

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An algebraic structure is *locally finite* iff each of its finite subset generates a finite subalgebra only.

In this paper we look at this property of local finiteness for t-norm based structures. By $T_{\mathsf{L}}, T_{\mathsf{P}}, T_{\mathsf{G}}$ we denote the basic t-norms, i.e. the Łukasiewicz, the product, and the Gödel t-norm, respectively. Furthermore $I_{\mathsf{L}}, I_{\mathsf{P}}, I_{\mathsf{G}}$ shall be their residuation operations, and $N_{\mathsf{L}}, N_{\mathsf{P}}, N_{\mathsf{G}}$ the corresponding standard negation functions defined via $N_\alpha(x) = I_\alpha(x, 0)$ in all these cases (and yielding $N_{\mathsf{P}} = N_{\mathsf{G}}$).

Generally, given a continuous t-norm T and its residuation operation I_T, we denote by N_T the corresponding standard negation function given as $N_T(x) = I_T(x, 0)$.

Proposition 1. *The t-norm-monoid* $([0,1], T_{\mathsf{G}}, 1)$ *is locally finite, and so is its extended version* $([0,1], T_{\mathsf{G}}, N_{\mathsf{G}}, 1)$.

Proof: Obvious.

Proposition 2. *The t-norm-monoid* $([0,1], T_{\mathsf{P}}, 1)$ *is not locally finite, and so is its extended version* $([0,1], T_{\mathsf{P}}, N_{\mathsf{P}}, 1)$.

Proof: Any $a \in (0,1)$ generates an infinite submonoid of $([0,1], T_{\mathsf{P}}, 1)$.

Proposition 3. *The t-norm-monoid* $([0,1], T_{\mathsf{L}}, 1)$ *is locally finite.*

Proof: Instead of T_{L} we can consider the conorm $+_{\mathsf{L}}$ with $x +_{\mathsf{L}} y = \min\{x+y, 1\}$. Inside $([0,1], +_{\mathsf{L}}, 0)$ each finite $G \subseteq (0,1]$ generates only a finite number of elements: 1 together with all the finitely many sums $k_1 a_1 + \cdots + k_n a_n$, $k_1, \ldots, k_n \in \mathbb{N}$, of $+_{\mathsf{L}}$-multiples of $a_1, \ldots, a_n \in G$.

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Proposition 4. *The (extended) t-norm-monoid* $([0,1], T_{\mathsf{L}}, N_{\mathsf{L}}, 1)$ *is not locally finite.*

Proof:[1] Any $a \in (0,1) \setminus \mathbb{Q}$ generates an infinite substructure.

The problem here really comes from the irrational numbers.

Proposition 5. *The (extended) t-norm-monoid* $([0,1] \cap \mathbb{Q}, T_{\mathsf{L}}, N_{\mathsf{L}}, 1)$ *is locally finite.*

Proof: Any finite set $G \subseteq [0,1] \cap \mathbb{Q}$ is a subset of a suitable finite truth degree set of a finitely-valued Łukasiewicz system.

Therefore we introduce an additional notion.

Definition 1. *A t-norm based algebraic structure* $\mathfrak{A}$ *over the unit interval is rationally locally finite iff each finite set* $G \subseteq [0,1] \cap \mathbb{Q}$ *generates only a finite substructure of* $\mathfrak{A}$.

Theorem 6. *A t-norm-monoid* $([0,1], T, 1)$ *with continuous t-norm* T *is (rationally) locally finite if and only if* T *does only have (rationally) locally finite summands in its representation as ordinal sum of archimedean summands.*

Corollary 7. *A t-norm-monoid* $([0,1], T, 1)$ *with continuous t-norm* T *is locally finite if and only if* T *does not have a product-isomorphic summand in its representation as ordinal sum of archimedean summands.*

Corollary 8. *An extended t-norm-monoid* $([0,1], T, N_T, 1)$ *with continuous t-norm* T *and their standard negation* N_T *is rationally locally finite if and only if* T *does not have a product-isomorphic summand in its representation as ordinal sum of archimedean summands.*

Corollary 9. *An extended t-norm-monoid* $([0,1], T, N_T, 1)$ *with a continuous t-norm* T *and their standard negation* N_T *is locally finite if and only if it is the Gödel monoid, i.e. iff* $T = T_{\mathsf{G}}$.

The final version will also discuss the inclusion of the lattice structure of $[0,1]$ into these considerations.

[1] The idea of this proof I owe to Jürgen Stückrad (Leipzig).

Free and Projective Bimodal Symmetric Gödel Algebras

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The variety $\mathbf{MG^2}$ of bimodal symmetric Gödel algebras, which represent the algebraic counterparts of bimodal symmetric Gödel logic MG^2, is investigated. Description of free algebras and characterization of projective bimodal symmetric MG^2-algebras is given.

1 Introduction

A "symmetric" formulation of intuitionistic propositional calculus, suggested by various authors (G. Moisil, A. Kuznetsov, C. Rauszer), presupposes that any connective $\&, \vee, \rightharpoonup, \top, \bot$ has its dual $\vee, \&, \rightharpoondown, \bot, \top$, and the duality principle of classical logic is restored. The notion of double-Browerian algebras was introduced by J. McKinsey and A. Tarski in [7], based on the idea considered by T. Skolem in 1919. In [2] double-Browerian algebras were named Skolem algebras.

Heyting-Brouwer logic (alias symmetric Intuitionistic logic Int^2) was introduced by C. Rauszer as a Hilbert calculus with algebraic semantics [8]. Notice, that the variety of Skolem (Heyting-Brouwerian) algebras are algebraic models for symmetric Intuitionistic logic Int^2. Recall that Gödel logic G is an extension of intuitionistic logic Int by the linearity axiom

$$(p \rightharpoonup q) \vee (q \rightharpoonup p).$$

Denote by G^2 the extension of symmetric Intuitionistic logic Int^2 by Gödel (the linearity) axiom and dual Gödel axiom.

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The well known procedure for embedding the intuitionistic propositional calculus into Gödel-Löb modal system GL (alias, the provability logic) can be also extended on symmetric intuitionistic logic Int^2.

We recall that the proof-intuitionistic logic KM (=Kuznetsov-Muravitsky [6]) is the intuitionistic logic Int enriched by $\Box$ as $Prov$ modality satisfying the following conditions:

$$p \rightharpoonup \Box p,\ \Box p \rightharpoonup (q \vee (q \rightharpoonup p)),\ (\Box p \rightharpoonup p) \rightharpoonup p.$$

We refer also to K. Segerberg, who has formulated in his paper [9], bimodal (temporal) logical system K^2C4T and proved that the logical system mentioned has the finite model property.

In this paper we investigate the proof-symmetric logic G^2 enriched by two modalities $\Box$ (considered as $Prov$ modality) and $\Diamond$; we denote this logic by MG^2. We call this logic a bimodal symmetric Gödel logic. Semantically the logic MG^2 is defined in the following way: MG^2 is the set of all formulas which are valid in all finite Kripke models (X, R), where the binary relation R is transitive and irreflexive, while reflexive closure R^ρ of R is a totally ordered.

This paper is devoted to the description of finitely generated free algebras in the variety of algebras corresponding to the bimodal symmetric Gödel logic, which is equivalent to the description of non-equivalent formulas (with fixed number of variables) in this logic, and to the characterization of finitely generated projective algebras, which play an important role in the unification problem for the bimodal symmetric Gödel logic.

2 Preliminaries

An algebra $(T, \vee, \wedge, \rightharpoonup, \rightharpoondown, 0, 1)$ is a Skolem algebra [2] (or Heyting-Browerian algebra), if $(T, \vee, \wedge, 0, 1)$ is a bounded distributive lattice, $\rightharpoonup$ is an implication (relatively pseudo-complement), $\rightharpoondown$ is coimplication (relatively pseudo-difference) on T.

An algebra $(T, \vee, \wedge, \rightharpoonup, \rightharpoondown, 0, 1)$ is said to be G^2-*algebra*, if (i) $(T, \vee, \wedge, \rightharpoonup, 0, 1)$ is G-algebra, corresponding to Gödel logic; (ii) $(T, \vee, \wedge, \rightharpoondown, 0, 1)$ is dual G-algebra (alias Browerian algebra with linearity condition: $(p \rightharpoondown q) \wedge (q \rightharpoondown p) = 0$). G^2-algebras, which are algebraic models of the logical system G^2, represent a proper subclass of Skolem algebras.

MG^2-*algebra* is an algebra $(T, \vee, \wedge, \rightharpoonup, \rightharpoondown, \Box, \Diamond, 0, 1)$, if $(T, \vee, \wedge, \rightharpoonup, \rightharpoondown, 0, 1)$ is G^2-algebra and the operators $\Box, \Diamond$ satisfy the following conditions:

$$p \leq \Box p,\ \Box p \leq q \vee (q \rightharpoonup p),\ \Box p \rightharpoonup p = p,$$
$$(\Box p \rightharpoonup \Box q) \vee (\Box q \rightharpoonup \Box p) = 1;$$
$$\Diamond p \leq p,\ p \rightharpoondown \Diamond p = \Diamond p,\ \Diamond(p \vee q) = \Diamond p \vee \Diamond q,$$
$$(q \rightharpoondown p) \wedge (p \rightharpoondown q) = 0.$$

Let us denote the variety (and also the category) of all MG^2-algebras by $\mathbf{MG}^2$.

Now we are ready to define the logic MG^2 in algebraic terminology: MG^2 is the set of all formulas valid in all finite totally ordered MG^2-algebras. This definition of the logic MG^2 is equivalent to the Kripke semantic definition given in the introduction. Let us introduce some abbreviation: $\neg p = p \rightharpoonup 0$, $\ulcorner p = 1 \rightharpoondown p$.

Theorem 1. *The logic MG^2 has finite model property.*

A subset $F \subset T$ is said to be a Skolem filter [3], if F is a filter (i.e. $1 \in F$, if $x \in F$ and $x \leq y$, then $y \in F$, if $x, y \in F$, then $x \wedge y \in F$) and if $x \in F$, then $\neg\ulcorner x \in F$.

The results obtained in [3] can be adopted for MG^2-algebras.

Proposition 2. *Let T be an MG^2-algebra. The lattice of all congruences of the algebra T is isomorphic to the lattice of all Skolem filters of the algebra T.*

A system (X, R), where X is a non-empty set and R transitive relation, is said to be *Kripke model*. We shall say that a subset $Y \subset X$ is an *upper cone* (or cone) if $x \in Y$ and xRy imply $y \in Y$. The concept of a *lower cone* is defined dually. A subset $Y \subset X$ is called *a bicone* if it is an upper cone and a lower cone at the same time.

We say that (X, τ, R) is a perfect Kripke model (or descriptive frame, in another terminology) if

1) (X, τ) is a topological space, which is a Stone space (i.e. Hausdorf, zero-dimensional and compact space),

2) $R^{-1}(x) = \{y : yRx\}$ is closed, for each $x \in X$,

3) the smallest closed set containing a cone is itself a cone,

4) the smallest cone containing a closed set is closed.

Hereinafter instead of (X, τ, R) we will write (X, R). Let (X, R) and (X', R') be perfect Kripke models; a mapping $f : X \to X'$ is said to be strongly isotone (or p-morphism) if

$$f(y)R'x \Leftrightarrow (\exists y')(yRy' \& f(y') = x)$$

for any $x \in X',\ y \in X$.

A perfect Kripke model (X, R) is called *symmetric* if R is order relation, $(X, \widetilde{R})$ is a perfect Kripke model as well, where $x\widetilde{R}y \Leftrightarrow yRx$. The category of symmetric perfect Kripke models (X, R), where R is an order relation on X, is dually equivalent to the category of Skolem algebras (Heyting-Browerian algebras) [2].

3 Free and Projective MG^2-algebras

A Kripke model (X, R) is called *strongly symmetric* if (X, R_ρ) is a symmetric Kripke model, (X, R_ρ) is a disjoint union of chains, where R_ρ is the reflexive closure of R, and, in addition, for every clopen A of X and every element $x \in A$ there is an element $y \in A - R^{-1}(A)$ such that either xRy or $x \in A - R^{-1}(A)$. Notice, that if strongly symmetric Kripke model is finite, then R is irreflexive.

Proposition 3. *The category* $\mathbf{MG^2}$ *of* MG^2*-algebras and algebraic homomorphism is dually equivalent to the category* $\mathbf{SK}$ *of strongly symmetric Kripke models and strongly isotone maps.*

Proposition 4. *Let* T *be an* MG^2*-algebra and* (X, R) *corresponding to* T *strongly symmetric Kripke model. Then the lattice of all congruences of the algebra* T *is anti-isomorphic to the lattice (by the inclusion relation* $\subseteq$*) of all closed bicones of* (X, R)*.*

Let $\mathbf{K}$ be any variety of algebras. Then $F_{\mathbf{K}}(m)$ denotes the m-generated free algebra in the variety $\mathbf{K}$. An algebra A is said to be *a retract* of the algebra B, if there are homomorphisms $\varepsilon : A \to B$ and $h \colon B \to A$ such that $h\varepsilon = Id_A$, where Id_A denotes the identity map over A. An algebra $A \in \mathbf{K}$ is called *projective*, if for any $B, C \in \mathbf{K}$, any epimorphism (that is an onto homomorphism) $\gamma : B \to C$ and any homomorphism $\beta \colon A \to C$, there exists a homomorphism $\alpha \colon A \to B$ such that $\gamma\alpha = \beta$. Notice that in varieties, projective algebras are characterized as retracts of free algebras. A subalgebra A of $F_{\mathbf{K}}(m)$ is said to be *projective subalgebra* if there exists an endomorphism $h : F_{\mathbf{K}}(m) \to F_{\mathbf{K}}(m)$ such that $h(F_{\mathbf{K}}(m)) = A$ and $h(x) = x$ for every $x \in A$.

Now we describe the one-generated free MG^2-algebra. Such a description can be easily generalized to the m-generated case ($m > 1$). Let (C_n^m, R_n^m) $(0 \le m \le n > 0)$ be a strongly symmetric Kripke model, where C_n^m is an n-element set $\{c_1^m, ..., c_n^m\}$ and R_n^m is an irreflexive and transitive relation such that $c_1^m R_n^m c_2^m$ ... $c_{n-1}^m R_n^m c_n^m$. Let $X_n = \coprod_{m=0}^n C_n^m$ be a disjoint union of C_n^m, $R_n = \bigcup_{m=0}^n R_n^m$ and $(X, R) = \bigcup_{n=1}^\infty (X_n, R_n)$. Let g_n^m $(0 \leq m \leq n > 0)$ be an m-element upper cone of C_n^m and $g_n = \{g_n^0, ..., g_n^n\}$. Then $G = \bigcup_{n=1}^\infty g_n \subset X$. A part of X is depicted in the Fig. 1, where the generator is represented by cycles or ovals.

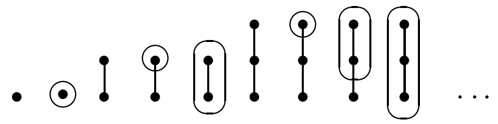

Fig. 1

Let $(T, \cup, \cap, \rightharpoonup, \neg\!\!\rightarrow, \Box, \Diamond, \emptyset, X)$ be an algebra generated by G by the following operations: the union $\cup$, the intersection $\cap$, $A \rightharpoonup B = -R^{-1} - (-A \cup B)$, $A \neg\!\!\rightarrow B = R(A \cap -B)$, $\Box(A) = -R^{-1} - (A)$, $\Diamond(A) = R(A)$ for any upper cones of A and B.

Observe, that if A is an upper cone of a strongly symmetric Kripke model, then the action of the operator $\Box$ on A increases the upper cone, and the action of $\Diamond$ on A decreases the upper cone (because of irreflexivity of R).

Lemma 5. *The MG^2-algebra*

$$T_n^m = \mathfrak{S}(C_n^m) = (Con(C_n^m), \cup, \cap, \rightharpoonup, \neg\!\!\rightarrow, \Box, \Diamond, \emptyset, C_n^m)$$

is generated by any element of T_n^m, where $Con(C_n^m)$ is the set of all upper cones of (C_n^m, R_n^m), $\cup$ is the union, $\cap$ is the intersection, $A \rightharpoonup B = -(R_n^m)^{-1} - (-A \cup B)$, $A \neg\!\!\rightarrow B = R_n^m(A \cap -B)$, $\Box A = -(R_n^m)^{-1} - (A)$, $\Diamond A = R_n^m(A)$.

Theorem 6. *The algebra $(T, \cup, \cap, \rightharpoonup, \neg\!\!\rightarrow, \Box, \Diamond, \emptyset, X)$ is a one-generated free MG^2-algebra with free generator G in the variety $\boldsymbol{MG}^2$.*

Theorem 7. *Let A be m-generated subalgebra of the m-generated free MG^2-algebra $F_{\mathbf{MG^2}}(m)$ and $a_1, ..., a_m$ the generators of A. Let A_i be the subalgebra of A generated by a_i for $i = 1, ..., m$. If $a_i = \Diamond^{n_i} g_i$ or $a_i = \Box^{n_i} g_i$ for some $n_i \in \omega$, then the algebra A is projective, where $g_1, ..., g_m$ are the free generators of $F_{\mathbf{MG^2}}(m)$.*

Notice, that any finite MG^2-algebra is not projective, since it is not a subalgebra of a free MG^2-algebra.

Say that join irreducible element a of MG^2-algebra A has a *height* n if $\Box^n \Diamond^n a = a$. According to this definition C_n^m is a join irreducible element of height n of the algebra T.

Theorem 8. *If A is an m-generated projective MG^2-algebra, then for every positive integer n there exists a join irreducible element $a \in A$ of height n.*

Problems. 1) *Let A be an m-generated MG^2-algebra. Is it true, that if for every positive integer n there exists a join irreducible element $a \in A$ of height n, then A is projective?*

2) *Does there exist finitely axiomatized subvariety of* $\mathbf{MG^2}$ *which is not generated by its finite members?*

3) *Is every subvariety of* $\mathbf{MG^2}$ *finitely axiomatizable?*

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Some Theories over Łukasiewicz Logic

Petr Hájek[*]

Łukasiewicz infinite-valued predicate logic Ł∀ is one of most important mathematical fuzzy logics; it will be quickly described. Then we survey some results (by Restall and Hájek) on axiomatic arithmetics of natural numbers over Ł∀, properties of a formal truth predicate, essential incompleteness, essential undecidability and ω-inconsistency.

The main part will concern Cantor-Łukasiewicz set theory $C\text{Ł}_0$ (with full comprehension - each formula determines a set of all objects satisfying the formula). Its consistency (over Łukasiewicz logic) was proved by White. It has two equality predicates: extensional $=_e$ and Leibniz equality $=$. It is proved that there are many pairs of sets x, y such that $x =_e y \,\&\, x \neq y$ is true. In particular, x may be the set ω of natural numbers, defined together with ternary predicates for addition and multiplication. The main (Hájek's) result says that the Cantor-Łukasiewicz set theory is essentially undecidable. The proof is difficult since it is not supposed that the set ω is crisp (non-fuzzy). Finally, we present Yatabe's result showing in which sense $C\text{Ł}_0$ is ω-inconsistent.

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On Logics with Truth Constants for Delimiting Idempotents

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The results below come from [5], a work in progress. We investigate the propositional logic of BL-algebras given by continuous t-norms (*standard algebras*) as described in [3], in a language expanded with truth constants for the idempotent elements delimiting the Ł-, G-, and Π-components. Note that the logics of continuous t-norms without constants were completely axiomatized in [2]. For a given standard algebra, we try to present a suitable axiomatization of its tautologies in the expanded language under a given semantics. A particular case of this general setting was already discussed in [6], where only one delimiting constant is considered. We mention the papers [4] on logic of hedges, and [1] on general expansions with truth constants, as related material.

It follows from Mostert-Shields representation theorem that with each continuous t-norm $*$, one can distinguish intervals on which $*$ is isomorphic to the Łukasiewicz t-norm or to the product t-norm and (maximal) intervals of idempotent elements. Each interval of the three above types is delimited by two idempotent elements, its *endpoints*.

For a given standard algebra $[0, 1]_*$ let $\mathrm{EP}(*)$ be the (countable) set of endpoints of its Ł-, G-, and Π-intervals. It follows from Mostert-Shields that if two standard algebras $[0, 1]_{*_1}$ and $[0, 1]_{*_2}$ have order-isomorphic sets of endpoints and for $x, y \in \mathrm{EP}(*_1)$ we have $[x, y]$ is an Ł-component, (G-component, Π-component) in $[0, 1]_{*_1}$ iff $[x, y]$ is an Ł-component (G-component, Π-component respectively) in $[0, 1]_{*_2}$, then $[0, 1]_{*_1}$ and $[0, 1]_{*_2}$ are isomorphic.

Definition 1. *(i) Fix* $*$ *and let* EP *be its set of endpoints. Assume* $a : \mathbb{N} \longrightarrow \mathrm{EP}$ *is a given enumeration of* EP*, i. e., a maps (some initial segment of)* $\mathbb{N}$ *bijectively onto* EP*. Denote* $\mathbb{N}_0 = \mathrm{Dom}(a)$*. So* $a_i = a(i)$

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is the i-th endpoint in the enumeration of EP$(*)$*. Assume for convenience* $a_0 = 0$.

(ii) Furthermore, let $+ : \mathbb{N}_0 \longrightarrow \mathbb{N}_0$ *be the function assigning to each* $i \in \mathbb{N}_0$ *an index* j *s. t.* $a_j = \min\{x : x \in \mathrm{EP}\,\mathrm{and}\, a_i < x\}$, $+(i) = i$ *if no such* j *exists. We write* i^+ *for* $+(i)$.

Given $*$, introduce a set of truth constants $\mathcal{C}_* = \{c_i\}_{i=0}^{\mathbb{N}_0 - 1}$, and stipulate $e(c_i) = a_i$ for any evaluation e in $[0,1]_*$. For each $i \in \mathbb{N}_0$ we define $c_i^+ = c_{(i^+)}$.

The semantics for the propositional BL-language expanded with the set $\mathcal{C}_*$ is given by a continuous t-norm *and* the mapping a enumerating the endpoints of $*$; different enumerations (in the same algebra) result in different sets of tautologies in the expanded language.

For each $*$, we define the propositional logic $BL_{EP(*)}$.

Definition 2. *Let* $*$ *be a continuous t-norm. The axioms of the logic* $BL_{EP(*)}$ *are the axioms of BL plus the following formulas:*

$$(EP_1^i) \qquad c_i \,\&\, c_i \equiv c_i \text{ for each } i \in \mathbb{N}_0$$

$$(EP_2^{i,j}) \qquad c_i \to c_j \text{ for each } i, j \in \mathbb{N}_0 \text{ s.t. } a_i < a_j$$

$$(EP_3^{i,j}) \qquad (c_j \to c_i) \to c_i \text{ for each } i, j \in \mathbb{N}_0 \text{ s.t. } a_i < a_j$$

The deduction rule is modus ponens.

It is important to notice that this logic does not aim at a complete description of $[0,1]_*$; it describes the ordering of the components, but not their nature.

General algebraic semantics is defined naturally: for $*$ and EP$(*)$ fixed, a $BL_{EP(*)}$-algebra is a structure for the language of BL-algebras expanded with a set $\mathcal{S}_*$ of constants that makes valid all the axioms of $BL_{EP(*)}$, evaluating $e(c_i) = s_i, i \in \mathbb{N}_0, s_i \in \mathcal{A}$ for all evaluations e.

$BL_{EP(*)}$-algebras are defined by a set of propositional formulas and therefore form a variety in the given language. By a *standard* $BL_{EP(*)}$-algebra we mean those standard algebras that are members of the variety generated by the axioms of $BL_{EP(*)}$.

Let $*$ be a continuous t-norm, EP the set of its endpoints, $\mathbf{A}$ a $BL_{EP(*)}$-chain and $s_i = e(c_i)$ in $\mathbf{A}$. Assume $a_i, a_j, a_k \in EP$. Then it holds (cf. also [1], Lemma 20) that

(i) if $a_i < a_j$ in $[0,1]_*$, then $s_i \leq s_j$ in $\mathbf{A}$;
(ii) if $s_i, s_j < 1$ in $\mathbf{A}$ and $a_i < a_j$ in $[0,1]_*$, then $s_i < s_j$ in $\mathbf{A}$.

The following completeness theorem is a matter of course:

Theorem 3. (Completeness) *Let $*$ be a continuous t-norm and $EP(*)$ the set of its endpoints. Let ϕ be a formula in the language of $BL_{EP(*)}$. Then the following are equivalent:*
(i) $\vdash_{BL_{EP}} \phi$
(ii) ϕ holds in any $BL_{EP(*)}$-algebra A
(iii) ϕ holds in any $BL_{EP(*)}$-chain A.

Moreover, under the above definition of standard algebras, the following can be proved (cf. [5]):

Theorem 4. (Standard completeness) *Let $*$ be a continuous t-norm and $EP(*)$ be the set of its endpoints. Let ϕ be a formula in the language of $BL_{EP(*)}$. Then $BL_{EP(*)} \vdash \phi$ iff ϕ holds in all standard $BL_{EP(*)}$-algebras.*

For each particular continuous t-norm, the goal of our endeavour is to find a complete axiomatics for the BL_{EP}-algebra given by it. For a given $*$, and for each $i \in \mathbb{N}_0$, we define a translation function, operating on formulas of the language of BL. The result of the translation of a formula φ will be denoted $\varphi^{[c_i,c_i{}^+]}$. The translation function is defined by induction on the formula structure as follows:

$$\begin{aligned}
\overline{0}^{[c_i,c_i^+]} &= c_i \\
\overline{1}^{[c_i,c_i^+]} &= c_i^+ \\
p^{[c_i,c_i^+]} &= (p \vee c_i) \wedge c_i^+ \\
(\varphi \& \psi)^{[c_i,c_i^+]} &= \varphi^{[c_i,c_i^+]} \& \psi^{[c_i,c_i^+]} \\
(\varphi \to \psi)^{[c_i,c_i^+]} &= (\varphi^{[c_i,c_i^+]} \to \psi^{[c_i,c_i^+]}) \wedge c_i^+
\end{aligned}$$

Lemma 5. *Let $*$ be a continuous t-norm, EP($*$) the set of its endpoints, and A the BL_{EP}-algebra given by $*$ on $[0,1]$. Let $i \in \mathbb{N}_0$ be such that $*$ on $[c_i, c_i^+]$ is isomorphic to the Łukasiewicz t-norm (the Gödel t-norm, the product t-norm respectively). Then $\varphi \equiv \psi$ is a tautology of $[0,1]_Ł$ ($[0,1]_{\mathrm{G}}$, $[0,1]_\Pi$ respectively) iff $\varphi^{[c_i,c_i^+]} \equiv \psi^{[c_i,c_i^+]}$ is a tautology of A.*

In particular, if φ is a tautology of $[0,1]_Ł$ ($[0,1]_{\mathrm{G}}$, $[0,1]_\Pi$ respectively), and the interval $[a_i, a_i^+]$ in $*$ is an Ł-component (G-component, Π-component respectively), then $\varphi^{[c_i,c_i^+]} \equiv c_i^+$ is a tautology of the $BL_{EP(*)}$-algebra given by $*$.

Let (Ł) denote the additional axiom $\neg\neg\varphi \to \varphi$ of Łukasiewicz logic, (G) denote the axiom $\varphi \to \varphi\&\varphi$ of Gödel logic, and (Π) denote the axiom $(\neg\varphi) \vee ((\varphi \to (\varphi \,\&\, \psi)) \to \psi)$ of product logic. For $i \in N$, denote

Ł^i the formula Ł$^{[c_i,c_i^+]} \equiv c_i^+$

G^i the formula $\mathrm{G}^{[c_i,c_i^+]} \equiv c_i^+$

Π^i the formula $\Pi^{[c_i,c_i^+]} \equiv c_i^+$

We refine the calculus BL_{EP} with a specification of the isomorphism type of each of the components of $*$.

Definition 6. *Let $*$ be a continuous t-norm, $EP(*)$ the set of its endpoints. The logic $BL_{COMP(*)}$ has as axioms the axioms of $BL_{EP(*)}$ plus the following formulas, for all $i \in \mathbb{N}_0$:*

$(COMP^i_Ł)$ Ł^i whenever $[a_i, a_i^+]$ in $[0,1]_*$ is a copy of $[0,1]_Ł$

$(COMP^i_G)$ G^i whenever $[a_i, a_i^+]$ in $[0,1]_*$ is a copy of $[0,1]_G$

$(COMP^i_\Pi)$ Π^i whenever $[a_i, a_i^+]$ in $[0,1]_*$is a copy of $[0,1]_\Pi$

The deduction rule is modus ponens.

As in the case of the logic BL_{EP}, one can state a general completeness theorem w. r. t. (linearly ordered) BL_{COMP}-algebras. It remains open whether the logic BL_{COMP} is complete with respect to the single standard BL_{COMP}-algebra given by $*$. This is true for algebras with only finitely many endpoints, but for some other algebras it seems more axioms are necessary.

We further analyze the computational complexity of the set of propositional 1-tautologies of each of the BL_{EP}-algebras given by $*$.

If $*$ is a finite ordinal sum, we show that the set of propositional 1-tautologies of $[0,1]_*$ in the language enriched with the constants $\mathcal{C}$ is in coNP (in fact, it is coNP-complete).

Next we address infinite sums. Although there exist infinite sums whose sets of tautologies (in the language of BL_{EP}-algebras) are in coNP, it is also true that some others are undecidable. There are (classes of) standard algebras which are infinite sums with a less favourable ordering of components and whose sets of 1-tautologies in the enriched language are non-arithmetical. That is unsurprising, taking into regard that the tautologies of some of the infinite ordinal sums in the enriched language allow for coding of infinite sequences of 0's and 1's.

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Evaluating Many Valued Modus Ponens

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1. Introduction

The aim of this paper is to have a sound and complete deduction in knowledge systems where uncertainty, vagueness and preference is modeled by many valued logic with arbitrary connectives (possibly obtained by an inductive procedure, see e.g. [4]).

In many systems domain (background) knowledge is modeled using IF-THEN rules (Prolog/Datalog rule based systems). From the very beginning we face a problem. In two-valued logic

$$B \longrightarrow H \equiv \neg B \vee H$$

is a tautology. This need not be true in many valued logic. As far as our main concern is to make modeling as much as possible realistic to real world data, we do not make any restriction here. Instead we study both possibilities separately and compare them.

$$\frac{(\boldsymbol{B}, \boldsymbol{b}), (\boldsymbol{B} \rightarrow \boldsymbol{H}, \boldsymbol{r})}{\boldsymbol{H}, \boldsymbol{f}_{\rightarrow}(\boldsymbol{b}, \boldsymbol{r})}, \qquad \frac{(\boldsymbol{B}, \boldsymbol{b}), (\neg \boldsymbol{B} \vee \boldsymbol{H}, \boldsymbol{r})}{\boldsymbol{H}, \boldsymbol{g}_{\vee}(\boldsymbol{b}, \boldsymbol{r})}.$$

We give some formula for evaluation of $\boldsymbol{f}_{\rightarrow}$ for evaluation of modus ponens with implicative rules and of $\boldsymbol{g}_{\vee}$ for evaluation of modus ponens with clausal rules.

We build on works [6, 10], in [10] there is estimate of full resolution and in [6] there is estimate of modus ponens for implicative rules.

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We assume our language consists of set of propositional variables and connectives: conjunction $\wedge$, disjunction $\vee$ and the negation $\neg$ (our language does not contain implications). Note that connectives in MV-logic with truth values range $[\mathbf{0}, \mathbf{1}]$ are monotone extensions of the classical connectives. Commonly used conjunctors (disjunctors) in MV-logic are the triangular norms (conorms). In this contribution the truth functions for conjunctions and disjunctions are t-seminorms $\boldsymbol{C}$ and t-semiconorms $\boldsymbol{D}$ which are given by next definition.

Definition 1.

(i) A t-seminorm $\boldsymbol{C}$ *is a conjunctor that satisfied the boundary condition*

$$\boldsymbol{C}(\mathbf{1}, \boldsymbol{x}) = \boldsymbol{C}(\boldsymbol{x}, \mathbf{1}) = \boldsymbol{x} \textit{ for all } \boldsymbol{x} \in [\mathbf{0}, \mathbf{1}].$$

(ii) A t-semiconorm $\boldsymbol{D}$ *is a disjunctor that satisfied the boundary condition*

$$\boldsymbol{D}(\mathbf{0}, \boldsymbol{x}) = \boldsymbol{D}(\boldsymbol{x}, \mathbf{0}) = \boldsymbol{x} \textit{ for all } \boldsymbol{x} \in [\mathbf{0}, \mathbf{1}].$$

2 . The aggregation deficits and full resolution truth function

In Pavelka's language of evaluated expresions, we would like from $(\boldsymbol{C} \vee \boldsymbol{A}, \boldsymbol{x})$ and $(\boldsymbol{B} \vee \neg \boldsymbol{A}, \boldsymbol{y})$ infer $(\boldsymbol{C} \vee \boldsymbol{B}, \boldsymbol{f}_{\vee}(\boldsymbol{x}, \boldsymbol{y}))$ where $\boldsymbol{f}_{\vee}(\boldsymbol{x}, \boldsymbol{y})$ should be the best promise we can give based on $\boldsymbol{D}$ the truth function of disjunction $\vee$ and $\boldsymbol{x}$ and $\boldsymbol{y}$.

In [10] there was introduced a new operator, let us call it *aggregation deficit* $\boldsymbol{R_D}$, which is based on a disjunctor $\boldsymbol{D}$. We recall its definition and important theorems, their proofs can be found in [10].

Definition 2. *The aggregation deficit is defined by*

$$\boldsymbol{R_D}(\boldsymbol{x}, \boldsymbol{y}) = \inf\{\boldsymbol{z} \in [0, 1]; \boldsymbol{D}(\boldsymbol{z}, \boldsymbol{x}) \geq \boldsymbol{y}\}.$$

Theorem 3. *If* $\boldsymbol{D}(\boldsymbol{c}, \boldsymbol{a}) \geq \boldsymbol{x}$ *then* $\boldsymbol{c} \geq \boldsymbol{R_D}(\boldsymbol{a}, \boldsymbol{x})$.

If moreover D *is right continuous then the opposite implication holds.*

Example 4. *For the basic t-conorms* $\boldsymbol{S_M}, \boldsymbol{S_P}$ *and* $\boldsymbol{S_L}$ *we obtain the following aggregation deficits:*

$$R_{S_M}(x,y) = \begin{cases} 0 & \text{if } x \geq y, \\ y & \text{otherwise,} \end{cases} \qquad R_{S_P}(x,y) = \begin{cases} 0 & \text{if } x \geq y, \\ \frac{y-x}{1-x} & \text{otherwise,} \end{cases}$$

$$R_{S_L}(x,y) = \begin{cases} 0 & \text{if } x \geq y, \\ y-x & \text{otherwise.} \end{cases}$$

Remark 5. *Note, that $R_D(x,y) \leq y$ for $(x,y) \in [0,1]^2$. If $x \geq y$, then $R_D(x,y) = 0$. It means, that for any aggregation deficit R_D it holds that $R_D \leq R_{S_M}$. More, if the partial mappings of disjunctor D are infimum-morfism ($\inf_{a\in M} D(x,a) = D(x, \inf_{a\in M} a)$, where M is subset of interval $[0,1]$) then $x \geq y$ if and only if $R_D(x,y) = 0$. It follows from boundary condition and monotonicity of t-semiconorm D. Consider an aggregation deficit R_D, then the partial mapping $R_D(.,1)$ is negator on $[0,1]$. The aggregation deficit R_S of t-conorm S coincides with residual coimplicator J_S, which was introduced by Bernard De Baets in [1] for different purpose.*

For formulation of a result on sound and complete full resolution, we investigated the *resolution truth function* $f : [0,1]^2 \to [0,1]$, which is defined by

$$f_{R_D}(x,y) = \inf_{a\in[0,1]} \{D(R_D(a,x), R_D(1-a,y))\}.$$

Example 6. *For the aggregation deficits R_{S_M}, R_{S_P} and R_{S_L} which are corresponded with the basic t-conorms we obtain the following functions:*

$$f_{R_{S_M}}(x,y) = \begin{cases} 0 & \text{if } x+y \leq 1, \\ \min(x,y) & \text{otherwise,} \end{cases}$$

$$f_{R_{S_P}}(x,y) = \begin{cases} 0 & \text{if } x+y \leq 1, \\ \frac{x+y-1}{\max(x,y)} & \text{otherwise,} \end{cases}$$

$$f_{R_{S_L}}(x,y) = \begin{cases} 0 & \text{if } x+y \leq 1, \\ x+y-1 & \text{otherwise.} \end{cases}$$

Remark 7. Note that arbitrary resolution truth function $\boldsymbol{f_{R_D}(x,y)}$ *satisfies*

$$\textit{if } \boldsymbol{x+y\leq 1} \textit{ then } \boldsymbol{f_{R_D}(x,y)=0.}$$

Theorem 8. *Assume the truth evaluation of propositions is a model of* $\boldsymbol{(C\vee A,x)}$ *and* $\boldsymbol{(B\vee\neg A,y)}$*. Then*

$$\boldsymbol{TV(C\vee B)\geq f_{R_D}(x,y).}$$

3. Modus ponens for clause based rules

For implicative rules, there is in [6] an estimation of modus ponens

$$\frac{\boldsymbol{(B,b),(B\to H,r)}}{\boldsymbol{H,f_{\to}(b,r)}}.$$

Let $\boldsymbol{I}$ be truth function of implication $\boldsymbol{\to}$, then truth function $\boldsymbol{f_{\to}}$ is residual conjunctor of implicator $\boldsymbol{I}$ (note mnemonic body-head-rule notation of variables)

$$\boldsymbol{f_{\to}(b,r)=C_I(b,r)=\inf\{h:I(b,h)\geq r\}.}$$

To be consistent with body-head-rule notation of [6], we will use it also here for clausal rules. In Pavelka's language of evaluated expresions, we would like from $\boldsymbol{(B,b)}$ and $\boldsymbol{(\neg B\vee H,r)}$ infer $\boldsymbol{(H,g_{\vee}(b,r))}$ where $\boldsymbol{g_{\vee}(b,r)}$ should be the best promise we can give based on $\boldsymbol{D}$ the truth function of disjunction $\vee$ and $\boldsymbol{b}$ and $\boldsymbol{r}$.

Example 9. *The following are the logical operators of material implication which are corresponding to basic t-norms Gödel* $\boldsymbol{T_M}$*, product* $\boldsymbol{T_P}$*, and Łukasiewicz* $\boldsymbol{T_L}$*.*

$$\boldsymbol{I_{T_M}(b,h)=\max(1-b,h),\quad I_{T_P}(b,h)=1-b+b\cdot h,}$$

$$\boldsymbol{I_{T_L}(b,h)=\min(1-b+h,1).}$$

First idea to mimic implicative rules, is to take residua to material implications. The residual conjunctors of previous implicators are:

$$\boldsymbol{C_{I_{T_M}}(b,r)}=\begin{cases}\boldsymbol{0} & \textit{if } \boldsymbol{b+r\leq 1},\\ \boldsymbol{r} & \textit{otherwise},\end{cases}\qquad \boldsymbol{C_{I_{T_P}}(b,r)}=\begin{cases}\boldsymbol{0} & \textit{if } \boldsymbol{b+r\leq 1},\\ \boldsymbol{\frac{b+r-1}{b}} & \textit{otherwise},\end{cases}$$

$$\boldsymbol{C_{I_{T_L}}(b,r)=\max(0,b+r-1).}$$

We can see, that all residua to material implication of classical connectives are zero in triangle $b + r \leq 1$. Indeed, this is true in general for all conjunctors with left additive generator.

Theorem 10. *Let $f : [0,1] \to [0,\infty]$ be a left-continuous additive generator of a conjunctor C. Let $f(0) = \infty$ and the function f be right-continuous in the point 0 or $f(0) < \infty$ and there exists $x \in [0,1]$ such that $f(x)+f(1-x) \neq f(0)$ and function f be continuous in the point x. Then $I_C(x,y) = 1 - C(x,1-y)$ is implicator and the conjunctor $C_{I_C} : [0,1]^2 \to [0,1]$ can be expressed by*

$$C_{I_C}(x,y) = l(g(x) + h(y)),$$

where

$$l(x) = 1 - f^{(-1)}(x),\ g(x) = -f(x),\ h(x) = f(1-x),$$

and

$$C_{I_C}(x,y) = 0 \text{ holds whenever } x + y \leq 1.$$

Another possibility, is to calculate the lower bound on truth value of H using aggregation deficit.

Example 11. *To have a sound clause based modus ponens, we make following observation. If for all $b, r \in [0,1]$*

$$(B,b) \text{ and } (\neg B \vee H, r) \text{ should imply } (H, g_\vee(b,r))$$

then using Theorem 2.2

$$r \leq D(1-b,h) \iff h \geq R_D(1-b,r).$$

Hence the best estimation for h is

$$g_\vee(b,r) = \inf_{b' \geq b} R_D(1-b',r).$$

Remark 12. *Usualy $g_\vee(b,r) = R_D(1-b,r)$. Note also that $g_\vee(b,r) = 0$ if $r + b \leq 1$.*

Theorem 13. *Let $g_{\vee_S}$ be truth function based on R_S and C_{I_T} be a truth function based on I_T, where T and S are dual triangular norm and conorm, as in Example 3.1. If T is T_M, T_L or T_P, then*

$$C_{I_T}(b,r) = g_{\vee_S}(b,r)$$

for all $b, r \in [0,1]$.

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Minimal Varieties of Representable Commutative Residuated Lattices

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Let RL be the variety of residuated lattices. The aim of this talk is to answer several open questions on the cardinality of minimal nontrivial subvarieties of RL. Given a variety of algebras V, we denote its subvariety lattice by $\mathbf{\Lambda}(\mathsf{V})$. In [3, Problem 8.6] the authors posed a question whether there are uncountably many atoms in $\mathbf{\Lambda}(\mathsf{RL})$ that satisfy $x \cdot y = y \cdot x$ or $x^2 = x^3$. This question was answered in [1] by giving continuum many idempotent representable atoms. Concerning the commutative atoms, [1] gives only a partial answer by showing that there are at least countably many commutative representable atoms leaving as an open question whether there are uncountable many of them or not. The same question appears also in [2] together with related problems on FL-algebras; see [2, Problems 17–19, pp. 437]. We solve this problem by constructing continuum many 4-potent commutative representable atoms in $\mathbf{\Lambda}(\mathsf{RL})$. The related problems on FL-algebras can be solved by easy modifications.

Theorem 1.

1. *There are* $2^{\aleph_0}$ *representable commutative 4-potent atoms in* $\mathbf{\Lambda}(\mathsf{RL})$.
2. *There are* $2^{\aleph_0}$ *representable 4-potent atoms in* $\mathbf{\Lambda}(\mathsf{FL_{ei}})$.
3. *There are* $2^{\aleph_0}$ *representable 4-potent atoms in* $\mathbf{\Lambda}(\mathsf{FL_{eo}})$.

In the above theorem all the atoms are 4-potent and non-integral. Thus there is a natural question how many integral (resp. 3-potent) representable commutative atoms we have. First, we can show that there are only two integral representable commutative atoms in the subvariety lattice $\mathbf{\Lambda}(\mathsf{RL})$.

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Let $\mathbf{L}_n = \langle L_n, \wedge, \vee, \cdot, \rightarrow, 0\rangle$ denote the $(n+1)$-valued MV-chain, where $L_n = \{-n, -n+1, \ldots, -1, 0\}$, $x \cdot y = -n \vee (x+y)$, and $x \rightarrow y = 0 \wedge (y-x)$. Further, let $\mathbf{Z}$ be the totally ordered additive group of integers viewed as a residuated chain. It is well known that its negative cone $\mathbf{Z}^-$ is residuated as well.

Theorem 2. *There are exactly two integral representable commutative atoms in* $\mathbf{\Lambda}(\mathsf{RL})$*, namely varieties generated by* $\mathbf{L}_1$ *and* $\mathbf{Z}^-$*.*

If we replace integrality with 3-potency, we can show that there are exactly five representable commutative atoms. We start by defining strictly simple finite algebras generating those atoms. We define an integral commutative residuated chain $\mathbf{U}_3 = \langle U_3, \wedge, \vee, \circ, \Rightarrow, 0\rangle$ which behaves almost like $\mathbf{L}_3 = \langle L_3, \wedge, \vee, \cdot, \rightarrow, 0\rangle$. Let $U_3 = L_3 \cup \{-2^*\}$. The order is given by $-3 < -2 < -2^* < -1 < 0$. The multiplication is defined $x \circ y = x \cdot y$ for $x, y \in L_3$, $-2^* \circ x = -3 = x \circ -2^*$ for $x \neq 0$, and $-2^* \circ 0 = -2^* = 0 \circ -2^*$. The residuum $\Rightarrow$ is fully determined by $\circ$ and the order. The operations $\circ$ and $\Rightarrow$ are described in Figure 1.

$\circ$	-3	-2	-2^*	-1	0
-3	-3	-3	-3	-3	-3
-2	-3	-3	-3	-3	-2
-2^*	-3	-3	-3	-3	-2^*
-1	-3	-3	-3	-2	-1
0	-3	-2	-2^*	-1	0

$\Rightarrow$	-3	-2	-2^*	-1	0
-3	0	0	0	0	0
-2	-1	0	0	0	0
-2^*	-1	-1	0	0	0
-1	-2^*	-1	-1	0	0
0	-3	-2	-2^*	-1	0

Figure 1: The multiplication and residuum in $\mathbf{U}_3$.

Having the integral commutative residuated chains $\mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3, \mathbf{U}_3$, we need to produce non-integral chains from them. Let $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \rightarrow, 1\rangle$ be an integral commutative residuated chain with a coatom $a = \max(A \setminus \{1\})$. We will extend the 1-free reduct of $\mathbf{A}$ by adding a new neutral element e in order to obtain $\mathbf{A}' = \langle A', \wedge, \vee, \circ, \rightarrow', e\rangle$, where $A' = A \cup \{e\}$. The lattice order $\wedge, \vee$ (denoted the same way as in $\mathbf{A}$) is the extension of the original order letting $a \leq e \leq 1$. Let $x \in A \cup \{e\}$ and $y \in A \setminus \{1\}$. The operations are extended as follows:

$$e \circ x = x = x \circ e\,, \quad e \rightarrow' x = x\,, \quad y \rightarrow' e = 1\,, \quad 1 \rightarrow' e = a\,.$$

Then one can show that $\mathbf{A}'$ is a commutative residuated chain which is not integral since the top element 1 is strictly greater than the monoidal identity e.

Theorem 3. *There are exactly five* 3*-potent representable commutative atoms in* $\mathbf{\Lambda}(\mathsf{RL})$*. Namely, varieties generated by* $\mathbf{L}_1, \mathbf{L}'_1, \mathbf{L}'_2, \mathbf{L}'_3, \mathbf{U}'_3$.

The construction of atoms from Theorem 1 can be modified in order to produce a large class of 1-generated algebras. In fact, we can prove that each finite integral commutative residuated chain embeds into one of them. Thus we obtain a new generating class for the variety of integral representable commutative residuated lattices.

Theorem 4. *The variety of representable integral commutative residuated lattices is generated as a quasi-variety by* 1*-generated finite members.*

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Dempster-Shafer Degrees of Belief in Łukasiewicz Logic

Tomáš Kroupa[*]

Dempster-Shafer theory of evidence [7] is an alternative framework for representing partial knowledge and reasoning under uncertainty in the situations when classical probabilistic models do not faithfully capture a problem in question. The uncertainty is described by a pair consisting of a *belief function* (lower probability) and a *plausibility function* (upper probability). The two functions can be also expressed by a so-called *mass assignment,* which is a certain finitely additive probability on the power set of all elements of the original Boolean algebra of events.

A main aim of this contribution is to study belief functions in the more general setting than Boolean algebras of events. This effort is in the line with a growing interest in the generalization of classical probability towards "many-valued" events, such as those resulting from formulas in Łukasiewicz infinite-valued logic—see [1] and [5]. The general case of lower and upper probabilities is investigated in [2]. The approach pursued here is different from that of [3], where a belief degree of a formula is the truth degree of the corresponding modality. Namely, we will define belief functions as particular real functionals on Lindenbaum algebra of Łukasiewicz logic. Generalizing the integral representation theorem for states [4, 6], it can be proven that every belief function is just a Choquet integral over the unit hypercube or, equivalently, over the set of all possible worlds. Moreover, we will provide a natural MV-algebraic generalization of the notion of mass assignment. The generalized mass assignment will be defined as a state on the MV-algebra of continuous functions over the space of all closed subsets of the unit hypercube, which is endowed with the topology given by Hausdorff metric. The introduced model enables interpreting belief degrees in Pavelka extension of

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Łukasiewicz logic: every "elementary"degree of belief associated with a given formula is a truth degree of this formula with respect to some theory.

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Constructing Operational Logics on Non-Associative Residuated Structures

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In the von Neumann interpretation of quantum mechanics, the states are unitary vectors of a Hilbert space while unitary transformations, together with observations, form a part of the operator algebra on the space. Certain subspaces of the Hilbert space can be interpreted as observables of the system and those, which commute with others, form a Boolean substructure representing the classical information respected by the quantum system. In terms of lattice theory, there is an orthomodular lattice L of observables and its centre Z. The lattice analog of the operator algebra is then a quantale Q acting on the orthomodular lattice. It makes L a left Q-module and right Z-module. Moreover, the embedding $Z \subseteq L$ is *open* (i.e. it has both adjoints and satisfies the Frobenius reciprocity condition) and thus admits a well-defined inner product $L \times L \to Z$. We can find more examples of such a situation among residuated lattices, e.g. a *quantum frame* (an intuitionistic generalization of orthomodular lattice). I wish to explain how one of the three parts Q, L and Z can be constructed when we know the other two, especially Q from L and Z. The construction provides an associative structure of "operational logic" acting on L which need not admit suitable multiplication; in the motivating example of orthomodular lattice the classical lattice operations are not distributive while the Sasaki projection (considered as a binary multiplication) is not associative.

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MV-Algebras with Constant Elements

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In this work we deal with algebraic counterparts of Lukasiewicz logic enriched by finite number of truth constants. A propositional many-valued logical system which turned out to be equivalent to the expansion of Lukasiewicz Logic L by adding into the language a truth-constant r for each real r (0, 1), together with a number of additional axioms was proposed by Pavelka in [1].

Many authors have been studied many-valued logics enriched by truth constants with respect to their relationship to other parts of mathematics, as well as to various structures (see, for example, [2, 3, 4]).

We investigate the varieties of algebras, corresponding to of Lukasiewicz logic enriched by finite number of truth constants. Specifically we show that these varieties contain non-trivial minimal subvariety generated by finite linearly ordered algebra which is functionally equivalent to Post algebra.

1. Definition

Recall that a universal algebra $(A, \oplus, \otimes, ^*, 0, 1)$ is called an MV-algebra, if it satisfies the following identities: for any $x, y \in A$

A1. $x \oplus y = y \oplus x$

A2. $x \oplus (y \oplus z) = (x \oplus y) \oplus z$

A3. $x \oplus x^* = 1$

A4. $x \oplus 1 = 1$

A5. $x \oplus 0 = x$

A6. $x \otimes y = (x^* \oplus y^*)^*$

A7. $x = (x^*)^*$

A8. $0^* = 1$

A9. $(x^* \oplus y)^* \oplus y = (x \oplus y^*) \oplus x$

A10. $x \oplus x \oplus x = x \oplus x$

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In order to read and write the axioms in a compact form, we make definitions: i) $0x = 0$, $(i+1)x = ix \oplus x$ ii) $x^0 = 1$, $x^{i+1} = x^i \otimes x$

The MV-algebra is called the MV$_m$-algebra, if it additionally satisfies the following identities:

A11. $(m-1)x \oplus x = (m-1)x$

A12. If $m > 3$, $[(jx) \otimes (x^* \oplus [(j-1)x]^*)]^{m-1} = 0$, where $1 < j < m-1$ and j does not divide $m-1$

Now in the MV-algebra with fix n-2 additional constants. By the MVS$_n$-algebra we consider the algebra $(A, \oplus, \otimes, ^*, 0 = C_0, C_1, \ldots, C_{n-2}, 1 = C_{n-1})$ where the $(A, \oplus, \otimes, ^*, 0, 1)$-reduct is an MV-algebra and additionally the nullary operators $C_i (i = 1, ..., n-1)$ satisfy the following properties:

C1. $iC_1 = C_i$, $i = (2, ..., n-1)$ and C2. $C_1 = C^*_{n-2}$

Thus we obtain the algebra with n fixed elements. For each $i, j = 0, \ldots, n-1$ we have $C_i \oplus C_j = iC_1 \oplus jC_1 = (i+j)C_1 = kC_1$ where $k = min(n-1, i+j)$. Also one can prove that $C^*_i = C_{n-1-i}$. Hence these fixed elements form the subalgebra of the MV-algebra.

The same way we define the MV$_m$S$_n$ algebras for such m-s, that $n-1$ divides $m-1$: These are the MV$_m$-algebras with constants $C_1, ..., C_{n-2}$ and additional axioms C1-C2 for them. From here on when we write MV$_m$S$_n$, we consider that $n-1$ divides $m-1$.

Algebra $S_(n) = (\{0, \frac{1}{n-1}, \frac{2}{n-1}, ..., \frac{n-2}{n-1}, 1\}, \oplus, \otimes, ^*, 0, C_1, ..., C_{n-2}, 1)$, where $C_i(x) = \frac{i}{n-1}$ is a MVS$_n$-chain.

The variety of the MVS$_n$-algebras and MV$_m$S$_n$-algebras denote by $\mathbf{MVS_n}$ and $\mathbf{MV_mS_n}$ respectively.

2. Characterization

Theorem. The only subdirectly irreducible algebras in the variety $\mathbf{MV_mS_n}$ are the algebras $S_(k)$, where $n-1$ divides $k-1$ and $k-1$ divides $m-1$.

Corollary. Every MV$_m$S$_n$-algebra A is isomorphic to the subdirect product of the algebras $S_(k) = (\{0, \frac{1}{k-1}, \frac{2}{k-1}, ..., \frac{k-2}{k-1}, 1\}, \oplus, \otimes, ^*, 0, C_1, ..., C_{n-2}, 1)$ where $n-1$ divides $k-1$ and $k-1$ divides $m-1$.

Now let us define the following recursive sequence: $p(n, k) = n^k$, $p(i, k) = i^k - \sum_{n \le j < i}^{j-1 \in div(i-1)} p(j, k)$ for every $i > n$.

Theorem. Algebra $F_{\mathrm{MV}_m\mathrm{S}_n}(k) = \prod_{j\geq n}^{j-1\in div(m-1)} S_j^{p(j,k)}$ is k-generated free $\mathrm{MV}_m\mathrm{S}_n$-algebra over the variety $\mathbf{MV_mS_n}$.

Consider the set recursive set j_i, $i = 1, 2, ...$: constructed the following way: $j_1 = n$ and for every i, $(j_i - 1)$ divides $(j_{i+1} - 1)$. Let $J = \{j_i | i = 1, 2, ...\}$.

Theorem. The subalgebra $F_{\mathrm{MVS}_n}(k)$ of the direct product $\prod_{j_i\in J} F_{\mathrm{MV}_{j_i}S_n}(k)$ generated by $s_m = (g_m^{(j_1)}, g_m^{(j_2)}, ...) \in \prod_{j_i\in J} F_{\mathrm{MV}_{j_i}S_n}(k)$ $(m = 1, ..., k)$ is a free MVS_n-algebra with the free generators $s_1, ..., s_k$, where $g_1^{(j_i)}, ..., g_k^{(j_i)}$ are the free generators of the k-generated free algebra $F_{\mathrm{MV}_{j_i}S_n}(k)$.

Theorem. Algebra A is projective in the variety $\mathbf{MV_mS_n}$ if it is isomorphic to the algebra $S_n \times A'$ where A' is some $\mathrm{MV}_m\mathrm{S}_n$-algebra.

In the algebra S_n we can construct the cyclic operator by means of the $\mathrm{MV}_m\mathrm{S}_n$-algebra operations: $f(x) = ((n-1)x)^* \vee (x \otimes C_{n-2})$.

Theorem. An algebra S_n is functionally equivalent to the n-valued Post algebra.

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Non-Safe Structures in Fuzzy Logics and Game Semantics

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The well known problem of non-safe structures in the semantics of first order fuzzy logics is a straightforward consequence of the interpretation of a quantified formulas: the value of $\forall x\varphi(x)$ is the infimum and the value of $\exists x\varphi(x)$ is the supremum of the set of values assigned to the formula $\varphi(x)$ in a particular structure M. The models, in which all the required infima and suprema exist are called safe and all semantic statements, most importantly completeness theorems, have to be restricted to safe models.

Game theoretical semantics for fuzzy logics proposed in [1] was shown to deal with all models, not just the safe ones. The aim of this paper is to define a notion of game theoretical truth and to prove a completeness theorem for game-theoretical semantics without the restriction to safe models. We confine ourselves to the case of Lukasiewicz logic, but it is possible to show, that the result can be generalised to Godel and Product logic as well.

1 First-order Łukasiewicz logic

Our starting point is the Łukasiewicz fuzzy logic as presented e.g. in [2]. The language of Łukasiewicz logic contains classical connectives plus the strong disjunction $\oplus$ and strong conjunction $\otimes$. The standard set of truth values is the unit interval, but theere are completeness results with respect to more general structures - linearly ordered MV algebras (MV-chains).

For any MV-chain $\mathbf{L}$, an $\mathbf{L}$-structure for a predicate language $\mathbf{\Gamma}$ is $\mathbf{M} = (M, (P_{\mathbf{M}})_{P\in\mathbf{\Gamma}}, (f_{\mathbf{M}})_{f\in\mathbf{\Gamma}})$ where $M \neq \emptyset$ is the domain of the model, for each predicate P of arity n, $P_{\mathbf{M}}$ is an n-ary $\mathbf{L}$-fuzzy relation on M (a mapping $M^n \to \mathbf{L}$), and for each function f, $f_{\mathbf{M}}$ is a mapping $M^n \to M$. Then we define for each formula φ (of the given language), the *truth value*

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$||\varphi||^{\mathbf{L}}_{\mathbf{M},v}$ of φ in the MV-chain $\mathbf{L}$ with respect to $\mathbf{M}$ and an $\mathbf{M}$-evaluation v of free variables of φ in M in the usual (Tarskian) way.

Definition 1. *Let* $\mathbf{\Gamma}$ *be a predicate language,* $\mathbf{L}$ *an* MV-*algebra,* $\mathbf{M}$ *an* $\mathbf{L}$-*structure for* $\mathbf{\Gamma}$*,* v *an* $\mathbf{M}$-*evaluation. The* value of a term *is defined as:* $||x||_{\mathbf{M},v} = v(x)$ *and* $||f(t_1,\dots,d_n)||_{\mathbf{M},v} = f_{\mathbf{M}}(||t_1||_{\mathbf{M},v},\dots,||t_n||_{\mathbf{M},v})$. *A* truth value of the formula φ *in* $\mathbf{M}$ *for an evaluation* v *is defined*[1]*:*

$$\begin{aligned} ||P(t_1,\ t_2,\ \dots,\ t_n)||^{\mathbf{L}}_{\mathbf{M},v} &= P_{\mathbf{M}}(||t_1||_{\mathbf{M},v},\ ||t_2||_{\mathbf{M},v},\ \dots,\ ||t_n||_{\mathbf{M},v})\,, \\ ||\varphi \oplus \psi||^{\mathbf{L}}_{\mathbf{M},v} &= ||\varphi||^{\mathbf{L}}_{\mathbf{M},v} \oplus ||\psi||^{\mathbf{L}}_{\mathbf{M},v}\,, \\ ||\neg\varphi||^{\mathbf{L}}_{\mathbf{M},v} &= \neg||\varphi||^{\mathbf{L}}_{\mathbf{M},v}\,, \\ ||0||_{\mathbf{M},v} &= 0\,, \\ ||(\forall x)\varphi||_{\mathbf{M},v} &= \inf\{||\varphi||^{\mathbf{L}}_{\mathbf{M},v'} \mid v' \equiv_x v\}\,. \\ ||(\exists x)\varphi||_{\mathbf{M},v} &= \sup\{||\varphi||^{\mathbf{L}}_{\mathbf{M},v'} \mid v' \equiv_x v\}\,. \end{aligned}$$

If infimum (supremum) does not exist, we take its value as undefined.

As we can see, in general the truth assignment is a *partial* function. To overcome this difficulty we define two classes of models:

Definition 2 (Safe structures)**.** *Let* $\mathbf{\Gamma}$ *be a predicate language,* $\mathbf{L}$ *an* MV-*chain,* $\mathbf{M}$ *an* $\mathbf{L}$-*structure for* $\mathbf{\Gamma}$*. We say that* $\mathbf{M}$ *is:*

- *a* safe $\mathbf{L}$-*structure, if* $||\varphi||^{\mathbf{L}}_{\mathbf{M},v}$ *is defined for each* φ *and* v*.*
- *a* witnessed $\mathbf{L}$-*structure, if* $||\varphi||^{\mathbf{L}}_{\mathbf{M},v}$ *is defined for each* φ *and* v *if we replace* sup *and* inf *in Definition 1 by* max *and* min*.*

We use the symbol $\models_{\mathbf{L}}$ for the semantical consequence over given MV-algebra $\mathbf{L}$ ($T \models_{\mathbf{L}} \varphi$ iff for each $\mathbf{L}$-model e of T we have $e(\varphi) = 1$). It is well known, that Łukasiewicz logic is complete with respect to the *safe* $\mathbf{L}$-structures over all MV-chains $\mathbf{L}$. For the proof see again [2].

Theorem 3 (Completeness Theorem)**.** *Let* $\mathbf{\Gamma}$ *be a predicate language and* φ *a formula. Then the following are equivalent:*

- $\vdash \varphi$*.*
- $(\mathbf{M}, \mathbf{L}) \models \varphi$ *for each* MV-*chain* $\mathbf{L}$ *and each* safe $\mathbf{L}$-*structure* $\mathbf{M}$*.*
- $(\mathbf{M}, \mathbf{L}) \models \varphi$ *for each* MV-*chain* $\mathbf{L}$ *and each* witnessed $\mathbf{L}$-*structure* $\mathbf{M}$*.*

[1] We shall use the same symbols for both connectives and corresponding operations of the MV-algebra. By $v \equiv_x v'$ we mean that $v(y) = v'(y)$ for each object variable y different from x.

2 Evaluation games

Evaluation game for fuzzy logics is a generalization of the classical evaluation game in the sense of Hintikka-Sandu (e.g. [3]). It is a zero-sum game of two players, traditionally called Eloise and Abelard, who disagree about truth of a formula φ in a model $\mathbf{M}$ with respect to an $\mathbf{M}$-evaluation v. To decide the matter they play an evaluation game, the rules of which are given by the structure of the formula in question. The formula φ is (game-theoretically) true in $(\mathbf{M}, v)$ iff there exits a winning strategy for the initial verifier in the corresponding evaluation game.

Evaluation games for fuzzy logics include one more parameter—the degree of truth of the formula in question. (The argument is not just about a formula being true, but 'how much' it is true.) Eloise as the initial Verifier wants to show that $||\varphi||^{\mathbf{L}}_{\mathbf{M},v} \geq r$ in a model $\mathbf{M}$ with an $\mathbf{M}$-evaluation v, Abelard as the initial Falsifier wants to deny that. Let $\mathbf{L}$ be an MV-chain, $\mathbf{M}$ be an $\mathbf{L}$-structure, φ a formula, v an $\mathbf{M}$-valuation, and $r \in L$. The fuzzy evaluation game $G^{\mathbf{L}}_{M}(\varphi, v, r)$ has the following moves:

Propositional moves

($\oplus$) $(\psi_1 \oplus \psi_2, v, r)$: $\mathcal{V}$ chooses $r' \leq r$, $\mathcal{F}$ chooses whether to play (ψ_1, v, r') or $(\psi_2, v, r - r')$.

($\vee$) $(\psi_1 \vee \psi_2, v, r)$: $\mathcal{V}$ chooses whether to play (ψ_1, v, r) or (ψ_2, v, r).

($\otimes$) $(\psi_1 \otimes \psi_2, v, r)$: $\mathcal{V}$ chooses $r' \leq 1 - r$, $\mathcal{F}$ chooses whether to play $(\psi_1, v, r + r')$ or $(\psi_2, v, r + (1 - r - r'))$.

($\wedge$) $(\psi_1 \wedge \psi_2, v, r)$: $\mathcal{F}$ chooses whether to play (ψ_1, v, r) or (ψ_2, v, r).

($\neg$) $(\neg\psi, v, r)$: $\mathcal{F}$ chooses $r', r \geq r' > 0$, *role switch*, game continues as $(\psi, v, (1 - r) + r')$

(*at*) (ψ, v, r), where ψ is an atomic formula: the end of the game, if $||\psi||_{\mathbf{M},v} \geq r$ (the current) $\mathcal{V}$ wins, otherwise $\mathcal{F}$ wins.

(0) $(\varphi, v, 0)$: the end of the game, the current $\mathcal{V}$ wins.

Quantifier moves

The existential move for witnessed models consists of Verifier's choice of an element from the domain of the model witnessing the claim $||(\exists x)\psi||_v \geq r$, equivalently $\sup(||\psi||_{v[x]}) \geq r$

($\exists'$) $((\exists x)\psi, v, r)$: $\mathcal{V}$ chooses $a \in M$, the game continues as $(\psi, v[x : a], r)$.

If the supremum is proper, i.e., $||\psi||_{v[x:a']} < \sup(||\psi||_{v[x]})$ for all $a' \in M$, Verifier cannot in principle provide a witness, so we weaken the classical rule: she has to provide a witness for any r' strictly smaller than r. We let Falsifier to decrease Verifier's stake (it is Falsifier's interest to decrease it as little as possible) and then Verifier finds an element in the domain to meet the weakened condition.

($\exists$) $((\exists x)\psi, v, r)$: $\mathcal{F}$ chooses $r' < r$ and $\mathcal{V}$ chooses $a \in M$, the game continues as $(\psi, v[x : a], r')$.

Let us note, that this rule is applicable even in the case, when the supremum does not exist. It is also 'fair' to Falsifier as it keeps his winning condition untouched. If it is the case, that $\sup(||\psi||_{v[x]}) < r$, Verifier can always find r' between the supremum and r to justify this claim (no matter if the supremum exists or not or is witnessed or not).

The position $((\forall x)\psi, v, r)$ corresponds to Verifier's claim that $\inf(||\psi||_{v[x]}) \geq r$. $\mathcal{F}$ is to move and he has to provide a counterexample, i.e., to find an a' such that $(||\psi||_{v[x:a']} < r)$. In this case the (non)existence of the witnessing element does not influence Falsifier's choice:

($\forall$) $((\forall x)\psi, v, r)$: $\mathcal{F}$ chooses $a \in M$, game continues as $(\psi, v[x : a], r)$.

This finishes the specification of the fuzzy evaluation game.

Correspondence theorem

Fuzzy evaluation games are zero-sum games of a finite depth, so by Zermelo's theorem they are determined—either Eloise or Abelard has a winning strategy for every $(\varphi, \mathbf{M}, v, r)$. Moreover, the game-theroretical value (existence of winning strategies for Eloise for a certain r) coincides in the case of safe structures with the standard Tarskian value.

Theorem 4. *Let* $\mathbf{L}$ *be an* MV*-chain,* $\mathbf{M}$ *be a safe* $\mathbf{L}$*-structure,* φ *a formula,* v *an* $\mathbf{M}$*-valuation, and* $r \in L$*. Then Eloise has a winning strategy for the* $(\mathbf{M}, \mathbf{L})$*-Game* (φ, v, r) *iff* $||\varphi||^{\mathbf{L}}_{\mathbf{M},v} \geq r$*.*

3 Game-theoretical truth

From the game-theoretical point of view it makes good sense to speak about powers of players in a particular game, We shall identify the power of Eloise (Abelard) in a fuzzy evaluation games corresponding to a formula φ with the set of values for which she (he) has a winning strategy.

Definition 5 (Powers of players)**.** *Let* **L** *be an* MV*-chain,* **M** *be an* **L***-structure,* φ *a formula, and* v *an* **M***-valuation. We define:*

$\mathcal{E}(\mathbf{M}, \mathbf{L}, v, \varphi) =_{\mathrm{df}} \{r \mid$ *Eloise has a winning strategy for the game* $G^{\mathbf{L}}_{\mathbf{M}}(\varphi, v, r)\}$.

$\mathcal{A}(\mathbf{M}, \mathbf{L}, v, \varphi) =_{\mathrm{df}} \{r \mid$ *for any* $r' \in \mathbf{L}$ *if* $r' > r$*, then Abelard has a winning strategy for the game* $G^{\mathbf{L}}_{\mathbf{M}}(\varphi, v, r')\}$.

If **L**, **M** or v are clear from the context we will omit them.

The definition of $\mathcal{A}$ seems to be less straightforward than necessary; we need it to obtain a duality of the operations over the powers of players defined below. The powers of the players have following properties:

Lemma 6 (Properties of the powers)**.** *Let* **L** *be an* MV*-chain,* **M** *be an* **L***-structure,* φ *a formula, and* v *an* **M***-valuation. Then:*

(i) $0_L \in \mathcal{E}(\varphi)$ *(ii)* $\mathcal{E}(\varphi)$ *is a lower set;*

(iii) $1_L \in \mathcal{A}(\varphi)$; *(iv)* $\mathcal{A}(\varphi)$ *is an upper set;*

(v) $\mathcal{A}(\varphi) \cup \mathcal{E}(\varphi) = L$; *(vi)* $||\mathcal{A}(\varphi) \cap \mathcal{E}(\varphi)|| \leq 1$;

(vii) For a safe **M***:* $\mathcal{E}(\mathbf{M}, \mathbf{L}, v, \varphi) \cap \mathcal{A}(\mathbf{M}, \mathbf{L}, v, \varphi) = \{||\varphi||^{\mathbf{L}}_{\mathbf{M},v}\}$.

Let us note that (iii) does not mean Abelard has a winning strategy for the value 1_L; it follows from the fact, that there is no game $G(\varphi, v, r')\}$ for $r' > r = 1$. This lemma gives us also a characterisation of safe structures (for a given **L**) in the terms of evaluation games: a structure **M** is safe iff the powers of players have a nonepmty intersection for any formula φ and valuation v.

It is quite natural to identify (full) truth of a formula φ with the situation when Eloise has a winning strategy for the value 1_L, in this case $\mathcal{E} = \mathbf{L}$, so Eloise has winning strategy for any value from **L**.

Definition 7 (G-truth)**.** *Let* **L** *be an* MV*-chain,* **M** *be an* arbitrary **L***-structure and* v *an* **M***-valuation. Then we say that*

- φ *is true in* $(\mathbf{M}, \mathbf{L})$ *with respect to* v *in the game-theoretical semantics (G-true) and write* $(\mathbf{M}, \mathbf{L})_v \models_G \varphi$ *iff* $\mathcal{E}(\mathbf{M}, \mathbf{L}, v, \varphi) = \mathbf{L}$.
- φ *is G-true in the* $\mathbf{L}$*-structure* $\mathbf{M}$*:* $(\mathbf{M}, \mathbf{L}) \models_G \varphi$ *iff* $\mathcal{E}(\mathbf{M}, \mathbf{L}, v, \varphi) = \mathbf{L}$ *for any* v.
- φ *is a G-tautology* $\models_G \varphi$ *iff* φ *is G-true in all* $\mathbf{L}$*-structures* $\mathbf{M}$.

We can show, that in the case of safe structures the notion of G-truth (G-tautology) corresponds to the standard (tarskian) one.

4 Completeness

With the definition of G-truth we can proceed to the main result of the article.

Theorem 8 (Completeness for non-safe structures)**.** *Let* $\mathbf{\Gamma}$ *be a predicate language and* φ *a formula. Then the following are equivalent:*

1. $\vdash \varphi$.
2. $(\mathbf{M}, \mathbf{L}) \models_G \varphi$ *for each* MV*-chain* $\mathbf{L}$ *and each* $\mathbf{L}$*-structure* $\mathbf{M}$.

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A Theory of Modal Natural Dualities with Applications to Many-Valued Modal Logics

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By proposing the notion of $\mathbb{ISP}_{\mathbb{M}}$, we extend the theory of natural dualities (see [4]) so that it encompasses Jónsson-Tarski's topological duality (see [6]) and Kupke-Kurz-Venema's coalgebraic duality (see [9]) for the class of all modal algebras. As applications of our duality theory, we can obtain new coalgebraic dualities for algebras of many-valued modal logics including Łukasiewicz n-valued modal logic (see [1, 13]) and a version of Fitting's many-valued modal logic (see [5, 11]).

The theory of natural dualities by Davey et al. is a powerful general theory of Stone-Priestley-type dualities based on the machinery of universal algebra. It basically considers duality theory for $\mathbb{ISP}(M)$ where M is a finite algebra. It is useful for obtaining new dualities and actually encompasses many known dualities, including Stone duality for Boolean algebras (see [7]), Priestley duality for distributive lattices (see [4]), and Cignoli duality for MV_n-algebras, i.e., algebras of Łukasiewicz n-valued logic (see [2, 3]), to name but a few (for more instances, see [4]).

However, as far as the author knows, it has not encompassed Jónsson-Tarski's topological duality or Kupke-Kurz-Venema's coalgebraic duality for the class of all modal algebras. We consider that this is mainly because the class of all modal algebras cannot be expressed as $\mathbb{ISP}(M)$ for a finite algebra M, in contrast to the fact that any of the class of Boolean algebras, the class of distributive lattices and the class of MV_n-algebras can be expressed as $\mathbb{ISP}(M)$ for a suitable finite algebra M.

In this talk, we propose the new notion of $\mathbb{ISP}_{\mathbb{M}}$ in order to extend the theory of natural dualities so that it encompasses Jónsson-Tarski's topological duality and Kupke-Kurz-Venema's coalgebraic duality for the class of all

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modal algebras. It is crucial that the class of all modal algebras coincides with $\mathbb{ISP}_{\mathbb{M}}(\mathbf{2})$ for the two-element Boolean algebra $\mathbf{2}$ (i.e., the class of all modal algebras can be generated from a single algebra in this way). Moreover, we have the following facts: for $\mathbf{n} = \{0, 1/(n-1), 2/(n-1), \dots, 1\}$ with the usual operations of MV-algebras, $\mathbb{ISP}_{\mathbb{M}}(\mathbf{n})$ coincides with the class of all algebras of Łukasiewicz n-valued modal logic; a similar thing holds also for algebras of a version of Fitting's many-valued modal logic. Thus, the notion of $\mathbb{ISP}_{\mathbb{M}}$ seems to be natural and useful for our goal.

Our main results are topological and coalgebraic dualities for $\mathbb{ISP}_{\mathbb{M}}(L)$ where L is a quasi-primal algebra with a bounded lattice reduct (for the details of these, see [12]). Our results encompass both Jónsson-Tarski's and Kupke-Kurz-Venema's dualities as the case $L = \mathbf{2}$. They also encompass topological dualities in [11, 13] for algebras of many-valued modal logics. Our dualities are obtained based on semi-primal duality theorem (see [4, 8]) in the theory of natural dualities and may be considered as modalized versions of semi-primal duality theorem. As applications of our results, we obtain new coalgebraic dualities for algebras of Łukasiewicz n-valued modal logic and for algebras of a version of Fitting's many-valued modal logic. It also follows from our dualities that the category of relevant coalgebras has a final coalgebra and cofree coalgebras.

We emphasize that the notion of $\mathbb{ISP}_{\mathbb{M}}$ makes it possible to incorporate both Jónsson-Tarski's topological duality and Kupke-Kurz-Venema's coalgebraic duality into the theory of natural dualities.

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Modal MTL-Algebras

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The variety of MTL-algebras is the equivalent algebraic semantics for monoidal t-norm logic, or MTL, as defined in [3]. We investigate extensions of MTL-algebras by an additional unary 'modal' operation. Extensions of various non-classical logical systems by additional modal operators are increasingly being studied. On substructural logics, existing examples of modalities include the linear logic exponential ! and the Baaz Delta Δ. In [1] and [6], operators on MTL related to ! are considered. MTL is also closely related to fuzzy set theory, in which many natural modalities can be found. In particular, the notion of a 'hedge' is widely used, which is essentially a modality on the logic of fuzzy sets with meaning such as 'very' or 'more-or-less'. For example, the hedge 'very' A on a fuzzy set A with membership function $\mu_A(x)$ is often taken to be the fuzzy set with membership function $\mu_{\text{very } A}(x) = (\mu_A(x))^2$. Another interesting modality on fuzzy sets (or any logic which has as primary model an algebra on $[0, 1]$) is the operation $\sim x = 1 - x$. This operation usually differs from the negation $\neg x = x \to 0$.

We recall that a (commutative, bounded, integral) *residuated lattice* is a lattice-ordered algebra with bounds 0, 1 and binary operations $\circ, \to$ such that $\circ$ is associative, commutative, has identity 1 and the residuation property holds: $x \circ z \leq y$ iff $z \leq x \to y$. An *MTL-algebra* is a residuated lattice that satisfies the prelinearity axiom: $(x \to y) \vee (y \to x) = 1$. An *MTL-chain* is just a linearly ordered MTL-algebra.

By a *modal MTL-algebra* (*modal MTL-chain*) we mean an MTL-algebra (MTL-chain) with an additional unary operation, called a modality and usually denoted by f. Initially, the only assumption we make on the modality is that it be order-preserving, or order-reversing as in the case of $\sim$. We then consider additional identities that arise naturally. On an MTL-chain, the order-preserving assumption on a modality f is equivalent to either of the

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identities: $f(x \wedge y) = f(x) \wedge f(y)$ or $f(x \vee y) = f(x) \vee f(y)$. To borrow terms from the theory of modal algebras, f is both 'box-like' and 'diamond-like'. In the context of MTL-chains, it makes sense to instead call an operator f 'box-like' if it satisfies the identity $f(x) \circ f(y) \leq f(x \circ y)$, and 'diamond-like' if it satisfies $f(x \oplus y) \leq f(x) \oplus f(y)$, where $x \oplus y = \neg((\neg x) \circ (\neg y))$. In the case of involutive MTL-chains the following correspondences hold: $f(x)$ is box-like iff $\neg f(\neg x)$ is diamond-like, and $f(x)$ is diamond-like iff $\neg f(\neg x)$ is box-like.

Our research is motivated by considerations on modal algebras. An important problem in modal algebras is that of canonicity; that is, the problem of preservation of identities under completion. A classical result here is that the class of Sahlqvist identities, a syntactically defined class, is preserved by the canonical completion. We investigate the problem of preservation of identities under completions of modal MTL-chains. We prove, for example, a Sahlqvist-like result by giving a syntactic description of a class of identities preserved by the MacNeille completion. This class is closely related to the Sahlqvist identities for modal algebras but makes use of the notions of box-like and diamond-like in the MTL-chain context. Our algebraic approach is motivated by the algebraic approach for modal algebras in [5] and [4], and extends the methods used in [8].

We investigate 'filtrations' on modal MTL-chains as methods for proving finite model properties. In modal algebras the study of filtrations makes strong use of the theory of Kripke frames, however a purely algebraic description of filtrations on modal algebras is also possible [2]. We extend this algebraic approach to MTL-chains although it is mainly possible in the n-potent case. In the non-n-potent case, a finite embedding construction as used in [8] may be used to prove the finite model property. In both cases, the stronger 'finite embeddability property' is obtained, which implies decidability of the universal theory of finitely axiomatized classes. We give a syntactic description of classes of identities that are preserved by these constructions. That is, we describe some classes for which the finite embeddability property holds.

We also consider some structural properties of modal MTL-algebras. We describe the filters (equivalently, ideals) of such algebras and give a finite basis of ideal terms. Such structural properties are important for the classification of varieties of modal MTL-algebras and questions related to the generation of such varieties by the MTL-chains in the variety.

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Priestley Duality and Nelson Lattices

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In this talk, we consider the Priestley duality for algebraic models of paraconsistent Nelson's logic with strong negation, more exactly, for its version $\mathbf{N4}^{\perp}$ suggested in [6] and containing the additional constant $\perp$ used to define the intuitionistic negation $\neg\varphi := \varphi \to \perp$. It was shown in [6] that the class of $\mathbf{N4}^{\perp}$-extensions has more regular structure then the class of extensions of the logic **N4** having only strong negation in its language. Further, algebraic models of **N4** are unbounded in general case, where as the models of $\mathbf{N4}^{\perp}$ are bounded lattices with additional operations, which makes them more convenient for constructing the Priestley duality.

For **N3**-lattices, i.e., for models of explosive Nelson's logic **N3** [4], the elegant representation theory of Priestley type was developed independently by R. Cignoli [1] and A. Sendlewski [10] (they used the term Nelson algebras). Both Cignoli and Sendlewski essentially used the original algebraic representation of **N3**-lattices obtained by A. Monteiro [3] and based on the so called interpolation property.

The algebraic semantics for paraconsistent Nelson's logic **N4** in terms of **N4**-lattices was only recently developed [5]. These results were transferred to the logic $\mathbf{N4}^{\perp}$ and $\mathbf{N4}^{\perp}$-lattices in [6] (see also [7]). This allows to pose a question on Priestley duality for algebraic models of paraconsistent Nelson's logic. It looks interesting especially due to the fact that unlike **N3**-lattices, the class of $\mathbf{N4}^{\perp}$-lattices is not contained in the well studied class of Kleene algebras. Another difficulty is connected with the fact that the weak implication $\to$ of **N3**-lattices can be represented via the relative pseudocomplement operation $\supset$ as follows: $a \to b = a \supset (\sim a \vee b)$. So, defining the structure of **N3**-lattice on clopen increasing sets of the dual space we can use the formula for $\supset$ known from the duality theory for Heyting algebras. This way is impossible for $\mathbf{N4}^{\perp}$-lattices, where the equality $a \to b = a \supset (\sim a \vee b)$ fails. The

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closest analog, which was found, looks as $a \to b = \sim \neg a \supset (\neg a \vee b)$, where $\neg a = a \to \bot$. It involves the weak implication and is not useful. Due to this reason we have to find a rather complicated formula for the weak implication. However, the resulting duality theory for $\mathbf{N4}^{\bot}$-lattices agrees very well with the duality theory developed by R. Cignioli [1] and A. Sendlewski [10].

We continue with the algebraic semantics for the logic $\mathbf{N4}^{\bot}$.

Definition 1. *An algebra* $\mathcal{A} = \langle A, \vee, \wedge, \to, \sim, \bot, 1\rangle$ *is said to be an* $\mathbf{N4}^{\bot}$-lattice *if the following hold.*

1. *The* $\to$*-free reduct* $\langle A, \vee, \wedge, \sim, \bot, 1\rangle$ *is a De Morgan algebra, i.e.,* $\langle A, \vee, \wedge, \bot, 1\rangle$ *is a bounded distributive lattice and the following identities hold:* $\sim (p \vee q) = \sim p \wedge \sim q$ *and* $\sim\sim p = p$.
2. *The relation* $\preceq$*, where* $a \preceq b$ *denotes* $(a \to b) \to (a \to b) = a \to b$*, is a preordering on* $\mathcal{A}$.
3. *The relation* $\approx$*, where* $a \approx b$ *if and only if* $a \preceq b$ *and* $b \preceq a$*, is a congruence relation with respect to* $\vee, \wedge, \to$ *and the quotient-algebra* $\mathcal{A}_{\bowtie} := \langle A, \vee, \wedge, \to, \bot, 1\rangle / \approx$ *is a Heyting algebra.*
4. *For any* $a, b \in A$, $\sim (a \to b) \approx a \wedge \sim b$.
5. *For any* $a, b \in A$, $a \leq b$ *if and only if* $a \preceq b$ *and* $\sim b \preceq \sim a$*, where* $\leq$ *is a lattice ordering on* $\mathcal{A}$.

An $\mathbf{N3}$-*lattice* $\mathcal{A}$ is an $\mathbf{N4}^{\bot}$-lattice modelling $\mathbf{N3}$, i.e., $\sim a \wedge a \approx \bot$ for all $a \in \mathcal{A}$.

Unlike Heyting algebras the notions of lattice filter and implicative filter are not equivalent on Nelson lattices. Following H. Rasiowa [9] we define two classes of implicative filters on $\mathbf{N4}^{\bot}$-lattices. A non-empty subset F of an $\mathbf{N4}^{\bot}$-lattice $\mathcal{A}$ is said to be a *special filter of the first kind* (*sffk*) on $\mathcal{A}$ if 1. $a \wedge b \in F$ for any $a, b \in F$, 2. $a \preceq b$ implies $b \in F$ for any $a \in F$ and $b \in \mathcal{A}$. And F is a *special filter of the second kind* (*sfsk*) on $\mathcal{A}$ if 1. $a \wedge b \in F$ for any $a, b \in F$, 2. $\sim b \preceq \sim a$ implies $b \in F$ for any $a \in F$ and $b \in \mathcal{A}$.

It turns out that every prime lattice filter on an $\mathbf{N4}^{\bot}$-lattice is either *sffk*, or *sfsk*.

Definition 2. *Let* $\mathcal{X} = (X, X^1, \leq, \tau, g)$ *be a tuple, where* X *is a set,* $X^1 \subseteq X$*,* $\leq$ *a partial order on* X*,* $g : X \to X$*, and* τ *is a topology on* X*. Put*

$$X^2 := g(X^1),\ X^+ := \{x \in X \mid x \leq g(x)\},\ X^- := \{x \in X \mid g(x) \leq x\}.$$

The structure $\mathcal{X}$ is said to be an $N4$-space *if the following conditions are satisfied:*

1. $(X, \leq, \tau, g)$ *is a De Morgan space, i.e.,* $(X, \leq, \tau)$ *is a Priestley space [2] and* g *is an order reversing homeomorphism such that* $g^2 = id_X$*;*
2. X^1 *is closed in* τ*,* $X = X^1 \cup X^2$*, and* $X^1 \cap X^2 = X^+ \cap X^-$*;*
3. $(X^1, \leq\restriction_{X^1}, \tau\restriction_{X^1})$ *is a Heyting space;*
4. *for any* $x \in X^1$ *and* $y \in X^2$*, if* $x \leq y$*, then* $x \in X^+$*,* $y \in X^-$*, and there exists* $z \in X$ *such that* $x, g(y) \leq z \leq g(x), y$*;*
5. *for any* $x \in X^2$ *and* $y \in X^1$*, if* $x \leq y$*, then* $x \in X^+$*,* $y \in X^-$*, and* $x \leq g(y)$*.*

Elements of the subspace X^1 (X^2) correspond to *sffk* (*sfsk*) on $\mathbf{N4}^{\perp}$-lattices. Now we define mappings dual to $\mathbf{N4}^{\perp}$-lattice homomorphisms.

Definition 3. *Let* $\mathcal{X} = (X, X^1, \leq, \tau, g)$ *and* $\mathcal{Y} = (Y, Y^1, \leq', \tau', g')$ *be* $N4$*-spaces. The mapping* $f : X \to Y$ *is an* $N4$-function *if the following conditions hold: 1)* $f : (X \leq, \tau, g) \to (Y, \leq', \tau', g')$ *is a De Morgan function, i.e., an order preserving continuous mapping such that* $fg = g'f$*; 2)* $f(X^1) \subseteq Y^1$*; 3)* $f\restriction_{X^1}$ *is a Heyting function, i.e., for any* U *open in* τ'*,*

$$f^{-1}((U \cap Y^1]) \cap X^1 = (f^{-1}(U \cap Y^1)] \cap X^1.$$

Now we denote by $\mathcal{N}_4$ the category of $\mathbf{N4}^{\perp}$-lattices and their homomorphisms and by $\mathcal{N}_4^*$ the category of $N4$-spaces and $N4$-functions. The main result is the following:

Theorem 4. *The categories* $\mathcal{N}_4$ *and* $\mathcal{N}_4^*$ *are dually equivalent.*

In conclusion we note that $N4$-spaces corresponding to $\mathbf{N3}$-lattices are exactly those satisfying the equality $X^1 = X^+$, and, consequently, $X^2 = X^-$. In this case condition 4 of Definition 2 turns into Monteiro's interpolation property and condition 5 becomes trivial. So, the equality $X^1 = X^+$ turns $N4$-space into Nelson space in the sense of [1].

The full version of the announced results will be presented in [8].

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Web-Geometric Approach to Continuous Triangular Subnorms

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Web geometry [1] is a branch of differential geometry providing several geometric concepts and tools which are known to characterize algebraic properties of *loops* in a surprisingly transparent geometric way. These are known as *closure conditions*; in particular, the associativity of loops is characterized by the *Reidemeister closure condition* [7]. This result is of a great interest since the algebraic property of associativity (unlike, e.g., the monotonicity, commutativity, or existence of a neutral element) usually seems to resist all intuitive geometric interpretations.

In previous results of the authors it has been shown that the concept of the Reidemeister closure condition can be used not only for loops but also for triangular norms [4] and for totally ordered monoids [5]. This allows a visual characterization of their associativity in a way similar to the case of the loops. Both these structures are important in the monoidal triangular norm based logic (MTL) [2], a prototypical logic of truth degrees. The semantical part of MTL is represented by MTL-algebras which are known to be subdirect products of MTL-chains [2]; the conjunction-interperting binary operation of an MTL-chain turns its underlying set into a totally ordered monoid. MTL is, however, complete even with respect to left-continuous triangular norms [3] which establishes their importance.

However, a convenient characterization of associativity of general triangular norms and totally ordered monoids is still an open problem. This contribution intends to focus on a specific subset of general left-continuous triangular norms: those which are constructed by ordinal sums of continuous

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triangular subnorms. The fact that every triangular subnorm can be easily extended to a triangular norm [6] will allow us to modify and apply web geometry in a similar way as in the previous results.

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Varieties of Interlaced Bilattices

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We present some algebraic results on several varieties of algebras having a bilattice reduct; we point out an interesting correspondence between these varieties and well-known varieties of lattices.

Bilattices are algebraic structures introduced by Ginsberg [7] as a uniform framework for inference in Artificial Intelligence, but since then they have found applications in many fields. In the 1990s, Arieli and Avron [1] developed the first bilattice-based logical system in the traditional sense, defined from logical matrices that they called *logical bilattices*. In a previous work [3] we studied, from the perspective of Abstract Algebraic Logic, the implicationless fragment of Arieli and Avron's logic, which they called *the basic logic of logical bilattices* (we call it $\mathcal{LB}$). We characterized the reduced models and reduced generalized models of $\mathcal{LB}$; in particular, we showed that the class of algebraic reducts of the reduced generalized models of this logic is the variety of distributive bilattices. In a later work [4], we completed this study by considering Arieli and Avron's logic in the full language, obtained by adding two (interdefinable) implication connectives. We proved that this logic (which we call $\mathcal{LB}_{\supset}$) is algebraizable and defined its equivalent algebraic semantics through an equational presentation. We called the algebras in this variety *implicative bilattices*.

Our aim here is to present some algebraic results concerning the new classes of algebras that emerged during our investigation of Arieli and Avron's logic, relating them to analogous results on some classes of bilattices already known in the literature.

Let us recall the main definitions appearing in our framework. A *pre-bilattice* is an algebra $\mathbf{B} = \langle B, \wedge, \vee, \otimes, \oplus \rangle$ such that both $\langle B, \leq_t, \wedge, \vee \rangle$ and $\langle B, \leq_k, \otimes, \oplus \rangle$ are lattices. A *bilattice* is an algebra $\mathbf{B} = \langle B, \wedge, \vee, \otimes, \oplus, \neg \rangle$ such that $\langle B, \wedge, \vee, \otimes, \oplus \rangle$ is a pre-bilattice and the *negation* $\neg$ is a unary operation satisfying that for every $a, b \in B$,

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(neg1) if $a \leq_t b$, then $\neg b \leq_t \neg a$

(neg2) if $a \leq_k b$, then $\neg a \leq_k \neg b$

(neg3) $a = \neg\neg a$.

Pre-bilattices and bilattices are both varieties. Particularly interesting subvarieties of the class of pre-bilattices are the variety of interlaced pre-bilattices and the variety of distributive pre-bilattices. A pre-bilattice (bilattice) is *interlaced* when all the operations $\{\wedge, \vee, \otimes, \oplus\}$ are monotone w.r.t. both orders; it is called *distributive* if all twelve possible distributive laws concerning $\{\wedge, \vee, \otimes, \oplus\}$ hold. Distributive (pre-)bilattices are a proper subvariety of interlaced (pre-)bilattices.

We also consider some algebras in expansions of the standard bilattice language, for example *bilattices with conflation* [5], obtained by adding a kind of dual of the bilattice negation (an involutive unary operator that is monotone w.r.t. $\leq_t$ and antimonotone w.r.t. $\leq_k$), and *implicative bilattices*.

An *implicative bilattice* is an algebra $\mathbf{B} = \langle B, \wedge, \vee, \otimes, \oplus, \supset, \neg \rangle$ such that the reduct $\langle B, \wedge, \vee, \otimes, \oplus, \neg \rangle$ is a bilattice and the following equations are satisfied:

(IB1) $(x \supset x) \supset y \approx y$

(IB2) $x \supset (y \supset z) \approx (x \wedge y) \supset z \approx (x \otimes y) \supset z$

(IB3) $((x \supset y) \supset x) \supset x \approx x \supset x$

(IB4) $(x \vee y) \supset z \approx (x \supset z) \wedge (y \supset z) \approx (x \oplus y) \supset z$

(IB5) $x \wedge ((x \supset y) \supset (x \otimes y)) \approx x$

(IB6) $\neg(x \supset y) \supset z \approx (x \wedge \neg y) \supset z$.

The class of implicative bilattices is a discriminator variety generated by a single finite algebra; let us also note that the bilattice reduct of any implicative bilattice is distributive.

One of the key results for our approach to the study of bilattices is a representation theorem stating that any interlaced pre-bilattice is isomorphic to a certain product (similar to a direct product) of two lattices. An analogous theorem holds for bilattices: in this case we have that any interlaced bilattice is isomorphic to a product of two copies of the same lattice.

interlaced pre-bilattices	product of the category of lattices with itself
distributive pre-bilattices	product of the category of distributive lattices with itself
interlaced bilattices	lattices
distributive bilattices	distributive lattices
commutative interlaced bilattices with conflation	lattices with involution
commutative distributive bilattices with conflation	De Morgan lattices
Kleene bilattices with conflation	Kleene lattices
classical bilattices with conflation	Boolean lattices
implicative bilattices	classical implicative lattices

Table 1: Some categorical equivalences

These results were proved by Avron [2] for bounded interlaced (pre-) bilattices, then generalized in [3] to the unbounded case. In [8] they are formulated in categorical terms, as follows: the category of bounded interlaced pre-bilattices is equivalent to the product of the category of bounded lattices (whose objects are pairs of bounded lattices) with itself, and the category of bounded interlaced bilattices is equivalent to the category of bounded lattices. We show that these results can be straightforwardly extended not only to the unbounded case, but also to several other classes of algebras having an interlaced bilattice reduct.

For each class of interlaced (pre-)bilattices listed in Table 1 we provide an equational presentation, prove a representation theorem analogue to the one known for interlaced bilattices, and characterize the subdirectly irreducible members in terms of those of its lattice counterpart. We use these results to show that every category on the left column of Table 1 is equivalent to the category on the right.

Focusing on the last row of the table, we sketch a way to prove similar correspondence results by weakening the axioms that define implicative bilattices, thus obtaining categories of bilattices corresponding to relatively pseudo-complemented lattices and all its subvarieties (Heyting algebras, Gödel algebras and so on).

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Bisymmetric Gödel Algebras with Special Modal Operators

*Mircea Sularia**

A *Heyting algebra* [1] is an algebra $\mathbf{H} = (H, \vee, \wedge, \to, 0, 1)$ of type (2,2,2,0,0) such that $(H, \vee, \wedge, 0, 1)$ is a bounded lattice and the following condition holds, for all $x, y, z \in H$:

$$(H) \qquad z \leq x \to y \text{ if and only if } z \wedge x \leq y.$$

A *Gödel algebra* [4, 5] (an *L-algebra* [1, 6] or a *Linear Heyting algebra* [8]) is a Heyting algebra $\mathbf{G} = (G, \vee, \wedge, \to, 0, 1)$ satisfying the Dummett prelinearity condition, for all $x, y \in G$:

$$(D) \qquad (x \to y) \vee (y \to x) = 1.$$

A *symmetric Gödel algebra* [7, 9] is a system $\mathbf{G}^* = (G, \vee, \wedge, \to, ^*, 0, 1)$ such that its reduct $\mathbf{G} = (G, \vee, \wedge, \to, 0, 1)$ is a Gödel algebra and $^* : G \to G$ is a De Morgan negation of the lattice reduct $(G, \vee, \wedge, 0, 1)$ i.e. for all $x, y \in G$:

$(M1)$ $(x^*)^* = x$,
$(M2)$ $(x \vee y)^* = x^* \wedge y^*$,
$(M3)$ $(x \wedge y)^* = x^* \vee y^*$.

The condition $(M1)$ is called *the law of double negation* and the conditions $(M2)$ and $(M3)$ are called *the De Morgan laws.*

The new structure of *bisymmetric Gödel algebra* (*S_2G-algebra*) is introduced by a special symmetric Gödel algebra equipped with an involutive automorphism. An *S_2G-algebra* is an algebra $\mathbf{A} = (A, \vee, \wedge, \to, ^*, \sigma, 0, 1)$, where A is a nonempty set together with three binary operations $\vee, \wedge, \to$, two operators $^*, \sigma$ and two constants $0, 1$ such that the following conditions hold, for all $x \in A$:

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(1) The reduct $G^*(\mathbf{A}) = (A, \vee, \wedge, \rightarrow, ^*, 0, 1)$ is a symmetric Gödel algebra.
(2) $(x \rightarrow 0)^* = (x \rightarrow 0) \rightarrow 0$.
(3) The operator $\sigma : A \rightarrow A$ is an involutive automorphism of $G^*(\mathbf{A})$ (symmetry in the sense of Moisil [12]).

The condition (2) is introduced in [2, 3] as an axiom for the structure of *Heyting Wajsberg algebra* and it seems to be of interest for mathematical fuzzy logic [4, 5] because of the fact that any symmetric Gödel algebra satisfying the identity (2) is isomorphic to a subdirect product of symmetric Gödel chains. In order to obtain an algebraic semantics for a suitable formal logic of two criteria decision making the structure of *modal bisymmetric Gödel algebra* (*μS_2G-algebra*) is introduced by a bisymmetric Gödel algebra together with a pair of special modal operators. A *μS_2G-algebra* is a triple $(\mathbf{A}, \lambda, \rho)$, where $\mathbf{A} = (A, \vee, \wedge, \rightarrow, ^*, \sigma, 0, 1)$ is an S_2G-algebra and λ, ρ are operators such that the following conditions hold:

(4) The function $\lambda : A \rightarrow A$ is an *interior operator* of the lattice reduct $L(\mathbf{A}) = (A, \vee, \wedge, 0, 1)$ and for all $x, y \in A$, $\lambda(\lambda(x) \rightarrow \lambda(y)) = \lambda(x) \rightarrow \lambda(y)$.
(5) The function $\rho : A \rightarrow A$ is a *closure operator* of $L(\mathbf{A})$ and for all $x, y \in A$, $\rho(\rho(x) - \rho(y)) = \rho(x) - \rho(y)$, where $u - v = (v^* \rightarrow u^*)^*$ if $u, v \in A$.
(6) The system $\left(A, \vee, \wedge, \dot{\rightarrow}, \dot{-}, ^d, 0, 1\right)$ is an *involutive Brouwerian D-algebra* [10, 11], where for all $x, y \in A$,

$$x \dot{\rightarrow} y = \lambda(x) \rightarrow \lambda(y), x \dot{-} y = \rho(x) - \rho(y) \text{ and } x^d = \sigma(x^*).$$

Basic properties of μS_2G-algebras are presented. The structure of μS_2G-algebra is proved to be the algebraic semantics of some special system of *modal symmetric Gödel sentential logic for two criteria decison making*. The structure of *multisymmetric Gödel algebra* including S_2G-algebra is introduced as a starting point to develop a formal logic for multicriteria decision making in connection with mathematical fuzzy logic.

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Paraconsistent Fuzzy Logic—A Review

*Esko Turunen**

We take a review of our papers [12] and [13] where points in common with paraconsistent logic, Pavelka style fuzzy logic and GUHA data mining logic are studied. By [9], the contemporary logical orthodoxy has it that, from contradictory premises, anything can be inferred. To be more precise, let $\models$ be a relation of logical consequence, defined either semantically or proof-theoretically. Call $\models$ *explosive* if it validates $\{A, \neg A\} \models B$ for every A and B (*ex contradictione quodlibet*). The contemporary orthodoxy, i.e., classical logic, is explosive, but also some non-classical logics such as intuitionist logic and most other standard logics are explosive. The major motivation behind paraconsistent logic is to challenge this orthodoxy. A logical consequence relation, $\models$, is said to be *paraconsistent* if it is not explosive. Thus, if $\models$ is paraconsistent, then even if we are in certain circumstances where the available information is inconsistent, the inference relation does not explode into triviality. Thus, paraconsistent logic accommodates inconsistency in a sensible manner that treats inconsistent information as informative.

In Belnap's paraconsistent logic [1], four possible values associated with atomic formulas α are interpreted as `told only True`, `told only False`, `both told True and told False` and `neither told True nor told False` (or just `true`, `false`, `contradictory` and `unknown`): if there is evidence for α and no evidence against α, then α obtains the value `true` and if there is no evidence for α and evidence against α, then α obtains the value `false`. A value `contradictory` corresponds to a situation where there is simultaneously evidence for α and against α and, finally, α is labeled by value `unknown` if there is no evidence for α nor evidence against α. More formally, the values are associated with ordered couples $T = \langle 1, 0\rangle$, $F = \langle 0, 1\rangle$, $K = \langle 1, 1\rangle$ and $U = \langle 0, 0\rangle$, respectively.

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In [10] Tsoukiás introduced a first order extension of Belnap's logic named DDT. In [8] and [6], a continuous valued extension of DDT logic was studied. The authors came to the conclusion that the graded values are to be computed via

$$t(\alpha) = \min\{a, 1-b\}, \tag{1.1}$$
$$k(\alpha) = \max\{a+b-1, 0\}, \tag{1.2}$$
$$u(\alpha) = \max\{1-a-b, 0\}, \tag{1.3}$$
$$f(\alpha) = \min\{1-a, b\}. \tag{1.4}$$

where an ordered couple $\langle a, b\rangle$, called *evidence couple*, is given. The intuitive meaning of a and b is the degree of evidence for a statement α and against α, respectively. Moreover, the set of 2×2 *evidence matrices* of a form

$$\begin{bmatrix} f(\alpha) & k(\alpha) \\ u(\alpha) & t(\alpha) \end{bmatrix}$$

is denoted by $\mathcal{M}$. The values $f(\alpha), k(\alpha), u(\alpha)$ and $t(\alpha)$ are values on the real unit interval $[0, 1]$ such that $f(\alpha) + k(\alpha) + u(\alpha) + t(\alpha) = 1$. Their intuitive meaning is *f(α) = falsehood, k(α) = contradictory, u(α) = unknown* and *t(α) = truth* of the statement α.

In [8] it is shown how such a fuzzy version of Belnap's logic can be applied in preference modeling. However, in [8] the authors listed some open problems, e.g. (i) a missing complete truth calculus for logics conceived as fuzzy extensions of four valued paraconsistent logics (ii) a more thorough investigation of valued sets and valued relations (when the valuation domain is $\mathcal{M}$).

In [13] we demonstarted that, instead of a Boolean structure that was proposed in [8], the valuation domain $\mathcal{M}$ should be equipped with a more general algebraic structure called *injective MV-algebra*. The standard Łukasiewicz structure is an example of an injective MV-algebra. This associates fuzzy extensions of four valued paraconsistent logics with Pavelka style fuzzy sentential logic [7]. As a consequence a complete truth calculus is obtained. Indeed, our basic observation is that the algebraic operations in (1.1)–(1.4) are expressible only by the Łukasiewicz t-norm and the corresponding residuum, i.e. in the standard Łukasiewicz structure. This fact was implicitely shown in the analysis done in [8] and [6]. Thus, if we would start with some other t-norm conjunction and an involutive negation then the reasonable conditions a continuous valued extension of paraconsistent logic should obey would cease to hold.

Pavelka's ideas were generalized in [11] by introducing a Pavelka style fuzzy sentential logic with truth values in an injective MV-algebra, thus generalizing $[0,1]$-valued logic. Indeed, in [11] it is proved that Pavelka style fuzzy sentential logic is a complete logic in a sense that if the truth value set L forms an injective MV-algebra $\mathbf{L}$, then the set of a-tautologies and the set of a-provable formulae coincide for all $a \in L$.

We therefore considered in [13] the problem that, given a set of evidence values in an injective MV-algebra, is it possible to transfer an injective MV-structure to the set $\mathcal{M}$, too. The answer turned out to be affirmative, consequently, the corresponding paraconsistent sentential logic is Pavelka style fuzzy logic with new semantics. Thus, a rich semantics and syntax is now available. For example, Łukasiewicz tautologies as well as Intuitionistic tautologies can be expressed in the framework of this logic. This follows by the fact that we have two sorts of logical connectives conjunction, disjunction, implication and negation interpreted either by the monoidal operations $\bigodot, \bigoplus, \longrightarrow, ^*$ or by the lattice operations $\wedge, \vee, \Rightarrow, ^\star$, respectively (however, neither * nor * is a lattice complementation). Besides, there are many other logical connectives available.

GUHA - General Unary Hypotheses Automaton - introduced in [2] (see also [3]) is a method of automatic generation of hypotheses based on empirical data, thus a method of data mining. The GUHA method, based on well-defined first order monadic logic containing generalized quantifiers on finite models, is a kind of automated exploratory data analysis: it generates systematically hypotheses supported by the data. A GUHA *procedure*, i.e. a computer software, generates statements on association between complex Boolean attributes denoted by ϕ, ψ. These attributes are constructed from the predicates corresponding to columns of the data matrix.

Since mathematical fuzzy logic and the GUHA method are both extension of classical Boolean logic and are related to vagueness and partial truth, it is not a surprising news that there are several approaches to connect mathematical fuzzy logic to the GUHA method. We mention Holeňa's paper [4] and Novák et al. who show in [5] that, by evaluating real-valued data by linguistic expressions and then using the GUHA method, one obtains data mining outcomes that are easily understandable as they are close to human way of thinking. Such a target we, too, had when we wrote [12], there we demostrated that the GUHA method has a natural interpretation in paraconsistent mathematical fuzzy logic.

More precisely, assume we have a data file composed of k columns and m rows. A *four-fold contingency table* $\langle a, b, c, d\rangle$ related to the attributes ϕ, ψ is composed from numbers of objects in the data satisfying four different binary combinations of these attributes:

	ψ	$\neg\psi$
ϕ	a	b
$\neg\phi$	c	d

where
- a is the number of objects satisfying both ϕ and ψ,
- b is the number of objects satisfying ϕ but not ψ,
- c is the number of objects not satisfying ϕ but satisfying ψ,
- d is the number of objects not satisfying ϕ nor ψ,
- $m = a + b + c + d$.

Various relations between ϕ and ψ can be measured in the data by different *four-fold table quantifiers*, which here are understood as functions with values in the real unit interval $[0, 1]$. Among the most well-known is the following: a statement Φ connecting two attributes ϕ and ψ by *basic double implicational quantifier* is defined to be true in a given data if

$$a \geq n \text{ and } \frac{a}{a+b+c} \geq p,$$

where $n \in \mathbb{N}$ and $p \in [0, 1]$ are parameters given by user. We proved

THEOREM 1 *Given a data, all statements Φ such that the truth value of Φ is at least k times greater than the falsehood of Φ in the sense of paraconsistent logic, can be found by using basic double implicational quantifier in a GUHA procedure.*

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Toward a Forcing Model Construction of H

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The study of a logical theory of circularity is important not only in logic but also in computer science. For, one of the key concepts, recursion, has a circular nature since we should calculate the value of $4 + 2$ in order to calculate the value of $4 + 3$. However, it is well-known that the full form of circularity implies a contradiction, e.g. Russell paradox: the comprehension principle, it guarantees the existence of term $\{x : \varphi(x)\}$ for any formula $\varphi(x)$, implies a contradiction (an infinite loop $R \in R \to R \notin R \to R \in R \cdots$) in classical logic. Therefore we have to restrict the form of recursion to have a consistent theory if we keep classical logic.

It is well-known that the comprehension principle does not imply a contradiction in many non-classical logics by paying a cost of weakening logical rules. These theories allow a very strong form of circularity, namely a *general form of the recursive definition*: for any formula $\varphi(x, \cdots, y)$, we can define a term θ such that $x \in \theta \leftrightarrow \varphi(x, \cdots, \theta)$ for any x in a set theory with the comprehension principle within Grisin Logic (classical logic minus the contraction rule) [2]. This allows us to define a set ω of natural numbers and to make any recursive function numerically representable. Therefore we can develop arithmetic to some extent. Such form of circularity is not only interesting as itself, but also worth studying, for it is an ideal generalization of a recursion in classical recursion theory.

Generally speaking, it is very difficult to construct a model of a theory which allows a strong form of circularity is really difficult (remember, it takes time to construct a domain model of λ-calculus). **H** is a typical case: **H** is a set theory with the comprehension principle within Łukasiewicz infinite-valued predicate logic $\forall$. **H** is known as one of the strongest theory in which the comprehension principle does not imply a contradiction, therefore **H**

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shows the limit of the consistency of the general form of recursive definition. In 1957 Skolem [8] conjectured that the comprehension principle does not imply a contradiction within ∀. Skolem, Chang [4] and Fenstad [5] proved it when the principle restricted to formulae with quantifier in a few special forms. However they could not prove the whole consistency of H because they failed to construct a model of full **H**. Only known model of **H** is an infinite set of sentences whose truth value are 1 constructed by a proof theoretic way [9].

The aim of this paper is to try to introduce a method of constructing a natural model of the set theory **H**. Since ∀ is Π^0_2-complete, treating ∀ via models is a more antural way by one's nature. Therefore, treating **H** via models should be a more natural way too. We review a new construction method, forcing in **ZFC**: we discuss how it can be applied to construct a model of $\mathbf{H}^{\mathbf{qf}}$, and a possibility to extend the result to construct a model of full **H**.

1. Preliminaries

In this paper, we work in **H** which is a set theory with the comprehension principle in ∀. We note that ∀ is Łukasiewicz predicate logic **Ł**∀ with standard semantics [6].

Definition 1 (**H**)**.** *Let* **H** *be the set theory within* ∀

- *which has a binary predicate* ∈*, and terms of the form* $\{x : \varphi(x)\}$*,*
- *whose only axiom scheme is the comprehension principle: for any* φ *not containing* u *freely,*

$$(\forall u)[u \in \{x : \varphi(x, \cdots)\} \equiv \varphi(u, \cdots)]$$

Definition 2 (the quantifier-free fragment of **H**)**.** $\mathbf{H}^{\mathbf{qf}}$ *is the set theory within* ∀*, which consists of a restricted form of the comprehension principle,* $(\exists x)(\forall y)y \in x \equiv \varphi(y, \vec{z})$ *for any set parameters* $\vec{z}$*, for all open (quantifier-free) formulae* φ*.*

Since nothing can be defined in pure $\mathbf{H}^{\mathbf{qf}}$, we add a syntactical identity relation ≡ over terms to $\mathbf{H}^{\mathbf{qf}}$ as Skolem did for sake of simplicity.

In 1957 Skolem [8] conjectured that the comprehension principle does not imply a contradiction within ∀. In fact, it is easy to construct a model of $\mathbf{H}^{\mathbf{qf}}$ by using Tychonoff-Schauder's fixed point theorem [3] because any truth function of $\mathbf{H}^{\mathbf{qf}}$ is continuous. Similarly, Chang proved the consistency of $\mathbf{H}(\Sigma_2)$, the fragment of **H** with the comprehension principle for all parameter-free formulae by using the compactness argument [4]. Actually, these methods cannot be extended to construct a model of **H**:

- the truth value of $\exists x\varphi(x)$ is the supremum of that of $\varphi(a)$ for any term a, therefore truth function might not be continuous in **H**: the fixed point construction cannot be applied for such case,
- it is difficult to evaluate formulae with parameters in compactness argument since any model is with finite domain.

In 1979 White proved the consistency of H in a proof theoretic way [9]. However, it does not provide the intuitive model of the set theory. The only known model of **H** is a maximally consistent set of formulae constructed as an infinite path of a proof search tree of a natural deduction: that is neither intuitive nor *natural* (in a natural model, for example, the truth value of $\emptyset \in \omega$ and $\omega \in \emptyset$ are different) in the sense of Fenstad. White wrote that "the chief open problem [...] is to find a natural interpretation for it, an interpretation which justifies the formal system in the way in which the cumulative type structure justifies the axioms of **ZF**" [9].

2. A forcing model construction of $\mathbf{H}^{\mathbf{qf}}$

A forcing is a partial order (which is a sort of a Kripke frame [1]). In the context of set theory, working in some model of **ZFC** (which is called *ground model*), forcing is used to provide some new target objects (e.g. generic real numbers [7], a model of H, etc.) which are usually not contained in the ground model. Each condition of forcing has a partial information of the target object: it approximates a some part (e.g. a finite part) of the target object in the ground model.

2.2 Forcing $\mathbb{P}$

First, we define basic notations. Let **Form** be the set of Godel codes of formulae, and **Term** be the set of Goedel codes of terms (which are of the form $\{x : \varphi(x)\}$).

Definition 3 (forcing notion $\mathbb{P}$)**.** *Let $\mathbb{P}$ be a set of p such that p is a* **finite** *subset of* **Term** $\times$ **Term** $\times [0,1]$ *(p consists of $\langle\langle s,t\rangle, r\rangle$).*

In other words, p is an approximation of a natural Tarskian model of $\mathbf{H}^{\mathbf{qf}}$ by finite domain: "$s \in t$ is of the truth value r in the target model". We write $s\epsilon_p t = r$ if $\langle\langle s, t\rangle, r\rangle \in p$ as an abbreviation. The syntactical equality is defined as usual ($s \equiv t$ is truth value 1 if s is syntactically equivalent to t: otherwise it is 0).

2.2 Consistency of conditions

Some conditions of $\mathbb{P}$ might contain *potentially* inconsistent information though it seems not to contain a contradiction demonstratively.

Example 4. *p is contradictory if, for some term t and some formula $\varphi_0(s) \wedge \varphi_1(s)$,*

(i) **call-by-name:** $s_0 = \{x : \varphi_0(x) \wedge \varphi_1(x)\}$ *and* $t\epsilon_p s_0 = 0.7$*,*

(ii) **call-by-value:** $s_1 = \{x : \varphi_0(x)\}$ *and* $t\epsilon_p s_1 = 0$*,*

$\varphi_0(t) \wedge \varphi_1(t)$ is of the value 0.7 from (i), but the truth value of $\varphi_0(t) \wedge \varphi_1(t)$ should be 0 from (ii).

We have at least two ways to evaluate truth values of any formulae. Therefore this violates the consistency of the model if two values are inconsistent. To exclude such inconsistency, first we introduce two sorts of truth value calculation, **cbv** value and **cbn** value, and next we define the inconsistency (and potential inconsistency) in terms of these values.

Definition 5.
- **cbn**-*value is defined by* **name**, *or by* **top-down** *evaluation, i.e.* $\mathbf{cbn}(p, \ulcorner\varphi(s)\urcorner) = r$ *iff* $t = \{x : \varphi(x)\}$ *and* $s\epsilon_p t = r$*,*
- **cbv**-*value is calculated by* **bottom-up** *evaluation from atomic case, i.e.* $\mathbf{cbv}(p, \ulcorner\varphi \to \psi\urcorner) = \min\{1 - \mathbf{cbv}(p, \ulcorner\varphi\urcorner) + \mathbf{cbv}(p, \ulcorner\psi\urcorner), 1\}$*, etc.*

Definition 6.
- *p decides φ ($p||\varphi$) iff p defines both* **cbn**, **cbv** *values of φ.*
- *p is directly inconsistent iff, for some φ, $p||\varphi$ and* $\mathbf{cbv}(p, \varphi) \neq \mathbf{cbn}(p, \varphi)$ *holds.*

Definition 7. *p is potentially inconsistent iff for any $q \supseteq p$ there exists $q' \supseteq q$ such that q' is directly inconsistent.*

2.3 Forcing $\mathbb{P}^{cons}$

Definition 8.
- *$\mathbb{P}^{\mathbf{cons}} \subseteq \mathbb{P}$ such that $\mathbb{P}^{\mathbf{cons}}$ consists of not-potentially inconsistent conditions,*
- *$\leq$ is a partial order on $\mathbb{P}^{\mathbf{cons}}$ such that*
  - *$p \leq q$ iff $p \supseteq q$,*
  - *$1_{\mathbb{P}^{\mathbf{cons}}}$ is a $\leq$-largest element ($1_{\mathbb{P}^{\mathbf{cons}}} = \emptyset$).*

$\mathbb{P}^{\mathbf{cons}}$ is not an empty but an infinite set by Brower's fixed point theorem.

Let $\mathbf{G}$ be a generic filter of $\mathbb{P}^{\mathbf{cons}}$, and $m_{\mathbf{G}} \in \mathbf{M}[\mathbf{G}]$ be a function $Form \to [0,1]$ defined by

$$m_{\mathbf{G}}(\ulcorner\varphi\urcorner) = \mathbf{cbv}(p, \ulcorner\varphi\urcorner) \text{ for some } p \in \mathbf{G}$$

Then $\langle A, m_{\mathbf{G}}\rangle$ is a natural model of $\mathbf{H}^{\mathbf{qf}}$.

3 Toward a forcing construction of a model of full-H

The forcing construction is not of service if it is only applied to construct a model of $\mathbf{H}^{\mathbf{qf}}$. However it is worth being considered because it has a possibility which can be extended to construct a model of full $\mathbf{H}$. The key idea is an approximation of non-continuous truth functions (by taking a sup, $\exists x\varphi(x)$) by continuous functions (say, $\varepsilon_n x\varphi(x)$).

Here, as an example, a plausible extension is introduced: the following forcing might provide a model of a fragment of Σ_1-formulae with parameters of $\mathbf{H}$.

Definition 9.
- *let $\mathbb{K}_n$ be a set of p such that p is a finite subset of $Term \times Term \times [0,1]$,*
- *$\mathbf{cbv}, \mathbf{cbn}$-values for open formulae are defined as definition 5 except*
  - *$\mathbf{cbv}(p, \exists x\varphi(x)) = \max\{\mathbf{cbv}(p, \varphi(s)) : p||\varphi(s)\}$,*
- *$p \in \mathbb{K}_n$ is potentially n-inconsistent if*
  - *$\mathbf{cbn}(p, \exists x\varphi(x)) < \mathbf{cbv}(p, \exists x\varphi(x))$,*
  - *or $\mathbf{cbn}(p, \exists x\varphi(x)) - \mathbf{cbv}(p, \exists x\varphi(x)) > \frac{1}{2^n}$.*

- *let* $\mathbb{K}_n^{cons}$ *be a set of* p *such that* p *is a not* n*-potentially inconsistent conditions of* $\mathbb{K}$ *where*

$\mathbb{K}_n^{cons}$ approximates the truth value of $\exists x\varphi(x)$ whose range of error is less than $\frac{1}{2^n}$.

On first glance, it is enough to do forcing by using the product forcing $\Pi_{n\in\omega}\mathbb{K}_n^{cons}$ or the finite support iterated forcing $\langle\mathbb{K}_n^{cons} : n \in \omega\rangle$. However, it is difficult to show that these forcings are neither empty nor non-trivial because it is difficult to define an adequate ordering. In the talk, we discuss technical problems of this construction, and prospects of forcing construction of model of full **H**.

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