A counterexample to Wegner's conjecture on good covers

Martin Tancer*

September 2, 2010

Abstract

In 1975 Wegner conjectured that the nerve of every finite good cover in $\mathbb{R}^d$ is d-collapsible. We disprove this conjecture.

A good cover is a collection of open sets in $\mathbb{R}^d$ such that the intersection of every subcollection is either empty or homeomorphic to an open d-ball. A simplicial complex is d-collapsible if it can be reduced to an empty complex by repeatedly removing a face of dimension at most $d-1$ which is contained in a unique maximal face.

1 Introduction

In 1975 Wegner [Weg75] introduced d-collapsible simplicial complexes. His definition comes from studying intersection patterns of convex sets. He proved that simplicial complexes coming from finite collections of convex sets (as their nerves) are d-collapsible. He also conjectured that his result has a topological extension when collections of convex sets are replaced by good covers. The purpose of this article is to disprove this conjecture.

We assume that the reader is familiar with simplicial complexes, otherwise we refer to introductory chapters of books like [Hat01, Mat03, Mun84].

Good covers. A *d-cell* is an open subset of $\mathbb{R}^d$ homeomorphic to a ball. A *good cover* in $\mathbb{R}^d$ is a collection $\mathcal{G}$ of d-cells in $\mathbb{R}^d$ such that for every subcollection $\mathcal{G}' \subseteq \mathcal{G}$ the intersection of the sets in $\mathcal{G}'$ is either a d-cell or

---

empty.[1] We remark that a collection of open convex sets in $\mathbb{R}^d$ always form a good cover.

d**-representable complexes.** A *nerve* of a collection $\mathcal{F}$ of sets in $\mathbb{R}^d$ is a simplicial complex whose vertices are the sets in $\mathcal{F}$ and whose faces are collections of vertices with a nonempty intersection. A finite simplicial complex is *convexly d-representable* if it is isomorphic to a collection of convex sets in $\mathbb{R}^d$; it is *topologically d-representable* if it is isomorphic to a good cover in $\mathbb{R}^d$. Standard notion appearing in the literature is *d-representable* instead of convexly d-representable. However, we add 'convexly' in order to clearly distinguish convex sets and good covers.

d**-collapsible complexes.** Let $\mathbf{K}$ be a simplicial complex. Assume that σ is a face of $\mathbf{K}$ of dimension at most $d-1$ such that there is only one maximal face of $\mathbf{K}$ containing σ. Then we say that σ is d-collapsible and that the complex

$$\mathbf{K}' := \mathbf{K} \setminus \{\eta \in \mathbf{K} : \sigma \subseteq \eta\}$$

arises from $\mathbf{K}$ by an *elementary d-collapse*. A simplicial complex $\mathbf{K}$ is *d-collapsible* if it can be reduced to an empty complex by a sequence of elementary d-collapses.

Wegner [Weg75] proved that every convexly d-representable simplicial complex is d-collapsible. He also conjectured that his result has the following topological extension.

Conjecture 1.1 (Wegner, 1975). *Every topologically d-representable simplicial complex is d-collapsible.*

We disprove this conjecture for every $d \geq 2$. We start with constructing a simplicial complex Ł which is topologically 2-representable but not 2-collapsible. In higher dimensions we obtain a counterexample by using suspensions of the complex Ł.

Theorem 1.2. *For every $d \geq 2$ there is a simplicial complex which is topologically d-representable but not d-collapsible.*

Additional background. Conjecture 1.1 is a natural question in the context of Helly-type theorems. For example the Helly theorem [Hel23] can be

---

[1]Different notions of a good cover appear in the literature. Sometimes the cells are assumed being compact instead of open. The intersection of $\mathcal{G}'$ can be also assumed contractible instead of being a d-cell. For our purposes there is not a big difference while constructing a counterexample. It can be easily modified to fulfill the above mentioned conditions.

formulated in such a way that a convexly d-representable simplicial complex containing all d-faces has to be already a full simplex. Once we know that convexly d-representable simplicial complexes are d-collapsible, it is a simple consideration to prove the Helly theorem[2]. In this case, the Helly theorem has a topological extension for topologically d-representable simplicial complexes [Hel30]. Many Helly type theorems have a similar topological extension; see, e.g., the introduction of [MT09] for more detailed list. However, the conclusion for d-collapsible complexes may be stronger than for topologically d-representable complexes, e.g., in [KM05]. Our counterexample thus shows that results for d-collapsible complexes cannot be generalized all at once for topologically d-representable complexes.

2 Planar case

We start this section with describing the complex Ł. Let $A_1, A_2, A_3, B_1, B_2, B_3,$ $C_1, C_2, C_3, D, X_1, X_2, X_3, Y_1, Y_2, Y_3, Z_1, Z_2, Z_3$ be the (open) sets from the Figure 2. We also set $\mathcal{A} := \{A_1, A_2, A_3\}$, $\mathcal{B} := \{B_1, B_2, B_3\}$, $\mathcal{C} := \{C_1, C_2, C_3\}$, $\mathcal{D} := \{D\}$, $\mathcal{X} := \{X_1, X_2, X_3\}$, $\mathcal{Y} := \{Y_1, Y_2, Y_3\}$, and $\mathcal{Z} := \{Z_1, Z_2, Z_3\}$. Let $\mathcal{L}$ be the collection of all these sets, i.e., $\mathcal{L} := \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{X} \cup \mathcal{Y} \cup \mathcal{Z}$. Finally, Ł is the nerve of $\mathcal{L}$.

We will show that Ł is topologically 2-representable but not 2-collapsible.

2.1 Topological representability

It is sufficient to show that $\mathcal{L}$ is a good cover. This property can be hand-checked; however, we offer an alternative approach.

First we realize that all sets of $\mathcal{L} \setminus \mathcal{Z}$ are convex. Thus $\mathcal{L} \setminus \mathcal{Z}$ is a good cover. It remains to check that adding sets of $\mathcal{Z}$ does not violate this property.

Let $Z \in \mathcal{Z}$ and let $\mathcal{L}^Z := \{L \cap Z : L \in \mathcal{L}\}$. We are done as soon as we show that $\mathcal{L}^{Z_1}, \mathcal{L}^{Z_2}$, and $\mathcal{L}^{Z_3}$ are good covers.

Because of the symmetry we show it only for $\mathcal{L}^{Z_1}$. The sets of $\mathcal{L}^{Z_1}$ can be transformed into convex sets by a homeomorphism of $\mathbb{R}^2$. See Figure 2.1. Thus they form a good cover.

2.2 Non-collapsibility by case analysis

Here we prove that Ł is not 2-collapsible by case analysis. We get a bit stronger results that will help us for higher dimensions. Disadvantage of this proof is that it does not give an explanation how is the complex constructed.

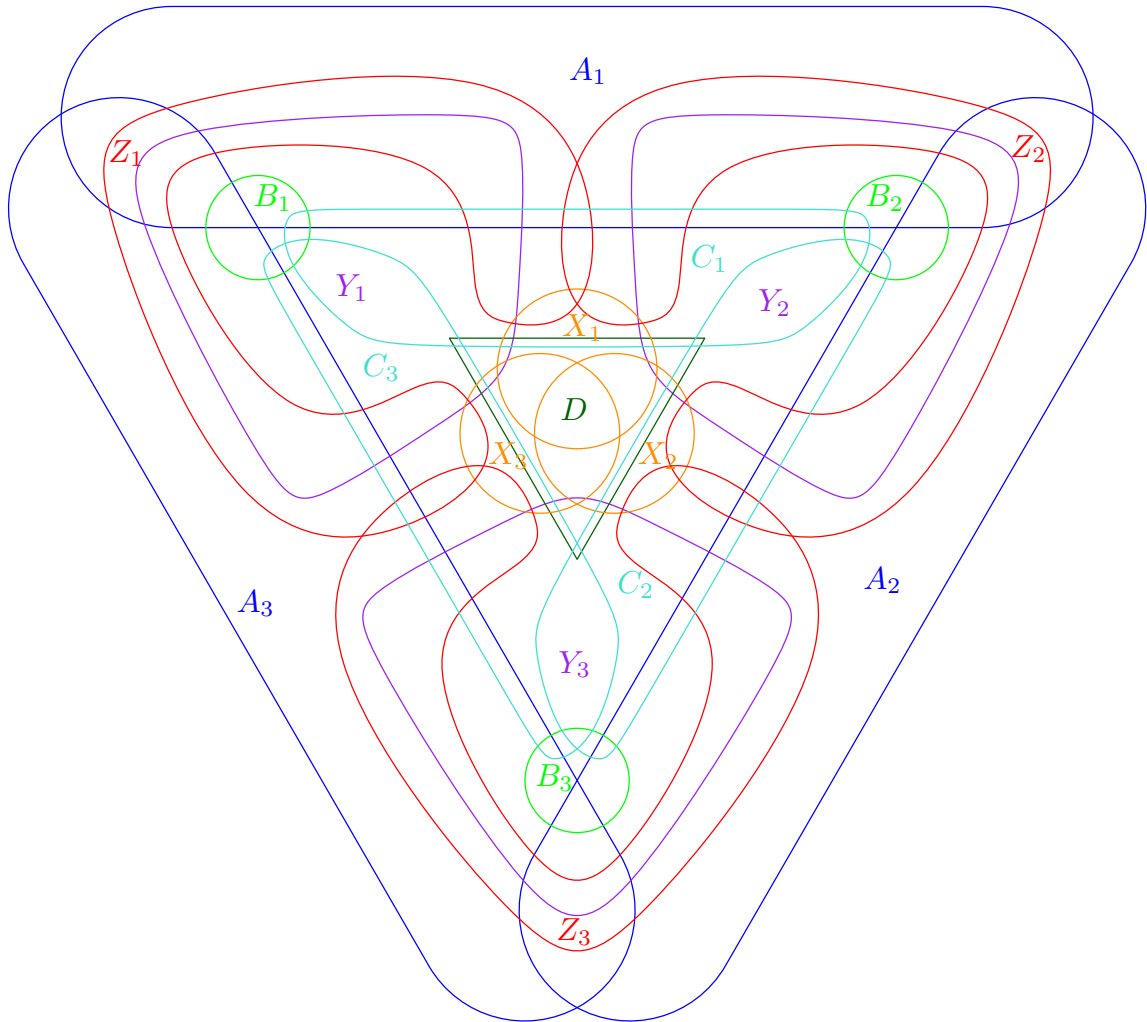[2]There is also a simple geometric proof of the Helly theorem, of course.

Figure 1: The sets $A_1, \ldots, Z_3$. We rather supply more detailed description of the sets if the picture is print only in black and white: The sets A_* are the ovals on the boundary; B_* are the small discs close to the boundary; C_* are the bread-shaped sets; D is the triangle in the center; X_* are the circles close to the center; Y_* are the bell-shaped sets; and Z_* are the boomerang-shaped sets
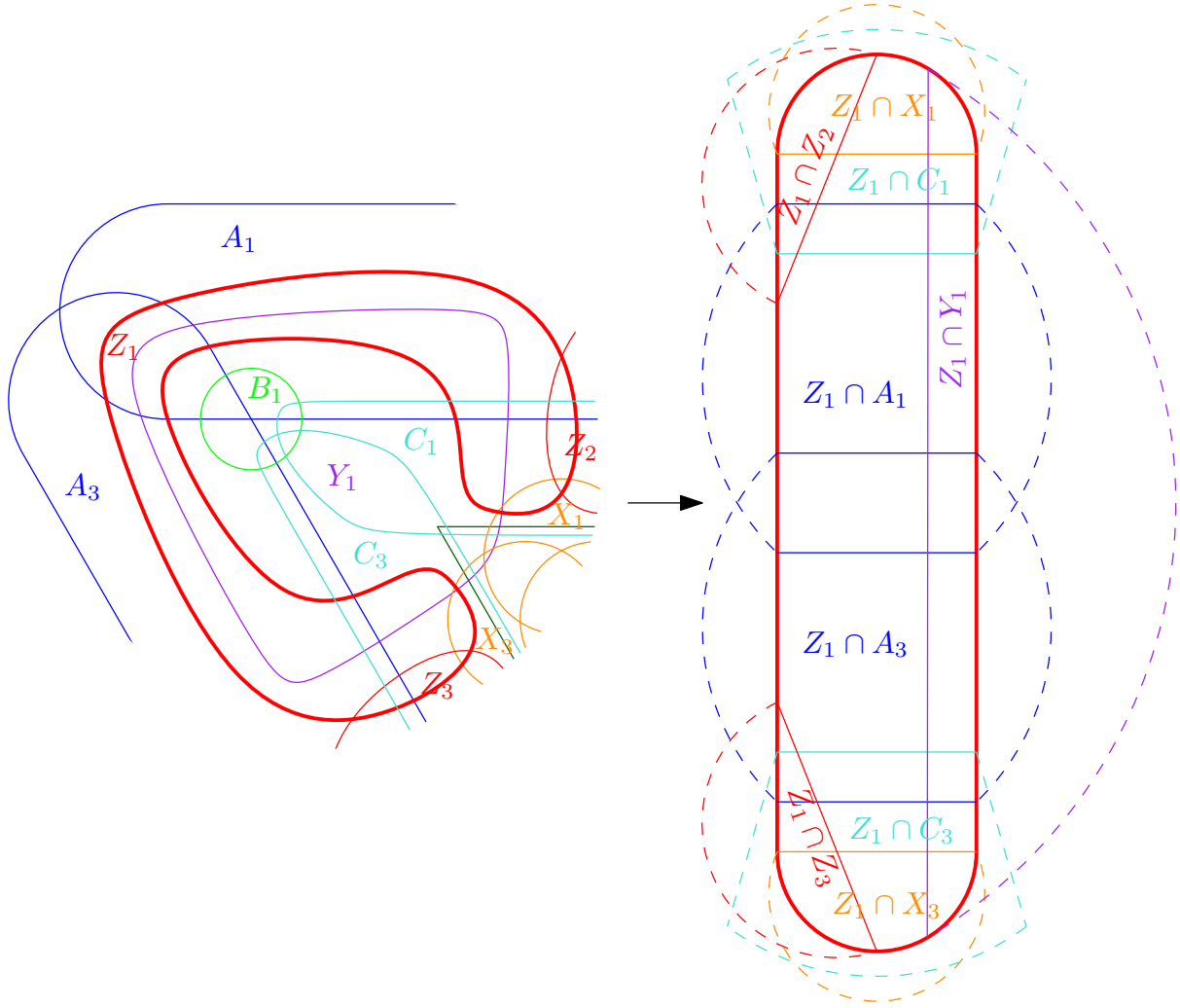
Figure 2: A transformation of $\mathcal{L}^{Z_1}$. Whatever is outside of Z_1 can be ignored.

Therefore we supply an additional heuristic explanation in the next subsection, although it would need a bit more effort to turn that explanation into a proof.

For a simplicial complex $\mathbf{K}$ we set

$$\gamma_0(\mathbf{K}) := \min\{d : \mathbf{K} \text{ has a } d\text{-collapsible face}\}.$$

The fact that Ł is not 2-collapsible is implied by the following proposition.

Proposition 2.1. $\gamma_0(\mathbb{L}) = 3$.

In order to prove the proposition we need a simple lemma.

Lemma 2.2. *Let $\mathbf{K}$ be a simplicial complex and σ be a 1-face (edge) of it. Assume that u and v are vertices of $\mathbf{K}$ not belonging to σ such that $\sigma \cup \{u\} \in \mathbf{K}$, $\sigma \cup \{v\} \in \mathbf{K}$, but $\sigma \cup \{u, v\} \notin \mathbf{K}$. Then σ is not a 2-collapsible face of $\mathbf{K}$.*

5

Proof. If τ is a unique maximal face of $\mathbf{K}$ containing σ then $u, v \in \tau$ due to the conditions of the lemma. However, $\sigma \cup \{u, v\} \notin \mathbf{K}$. $\square$

Proof of Proposition 2.1. In the spirit of Lemma 2.2 for every 1-face $\sigma \in \mathrm{L}$ we find a couple of vertices $u, v \in \mathrm{L}$ such that $\sigma \cup \{u\}, \sigma \cup \{v\} \in \mathrm{L}$, but $\sigma \cup \{u, v\} \notin \mathrm{L}$. It is sufficient to check 1-faces since if a 0-face (vertex) w is 1-collapsible then any 1-face containing w is 1-collapsible as well. Moreover, it is sufficient to check only some 1-faces because of the symmetries of the complex. The rest of the proof is given by the following table.

σ	u, v	σ	u, v	σ	u, v
$\{A_1, A_2\}$	B_2, Z_2	$\{A_1, B_1\}$	C_1, A_3	$\{A_1, C_1\}$	B_1, B_2
$\{A_1, Y_1\}$	B_1, Z_1	$\{A_1, Z_1\}$	C_1, A_3	$\{B_1, C_1\}$	A_1, C_3
$\{B_1, Y_1\}$	C_1, A_3	$\{C_1, C_2\}$	B_2, D	$\{C_1, D\}$	C_2, C_3
$\{C_1, X_1\}$	Y_1, Y_2	$\{C_1, Y_1\}$	B_1, Z_1	$\{C_1, Z_1\}$	Y_1, Z_2
$\{D, X_1\}$	Y_1, Y_2	$\{D, Y_1\}$	C_1, X_3	$\{X_1, X_2\}$	Y_2, X_3
$\{X_1, Y_1\}$	D, Z_1	$\{X_1, Z_1\}$	Y_1, Z_2	$\{Y_1, Z_1\}$	C_1, A_3
$\{Z_1, Z_2\}$	A_1, X_1				

$\square$

2.3 Sketch of non-collapsibility

The purpose of this subsection is to give a rough idea why the complex L should not be 2-collapsible. This description could be useful, for instance, for generalizations. However, the reader can easily skip this part. The author still prefer to include this discussion in order to explain how the complex is built up.

Let us split the collection $\mathcal{L}$ into two parts $\mathcal{L}^+ := \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$ and $\mathcal{L}^- := \mathcal{X} \cup \mathcal{Y} \cup \mathcal{Z}$. The nerve of $\mathcal{L}^+$, resp. $\mathcal{L}^-$, is denoted by L^+, resp. L^-. Both L^+ and L^- are triangulations of a disc with only three boundary edges $\{A_1, A_2\}$, $\{A_1, A_3\}$, and $\{A_2, A_3\}$; resp. $\{Z_1, Z_2\}$, $\{Z_1, Z_3\}$, and $\{Z_2, Z_3\}$; see Figure 2.3. Only these boundary faces are 2-collapsible faces of L^+, resp. L^-.

By suitable overlapping of $\mathcal{L}^+$ and $\mathcal{L}^-$ (i.e., obtaining $\mathcal{L}$) we get that also the above mentioned boundary faces are not 2-collapsible anymore (in whole L). For instance $Z_1 \cap Z_2$ intersects A_1 (in addition to X_1 already in $\mathcal{L}^-$); however, A_1 and X_1 are disjoint. Thus $\{Z_1, Z_2\}$ is not a 2-collapsible face of $\mathcal{L}$.

It remains to check that merging $\mathcal{L}^+$ and $\mathcal{L}^-$ does not introduce any new problems. It is, in fact, checked in a detail in the previous section. We just mention that there is no problem with 1-faces which already appear in L^+
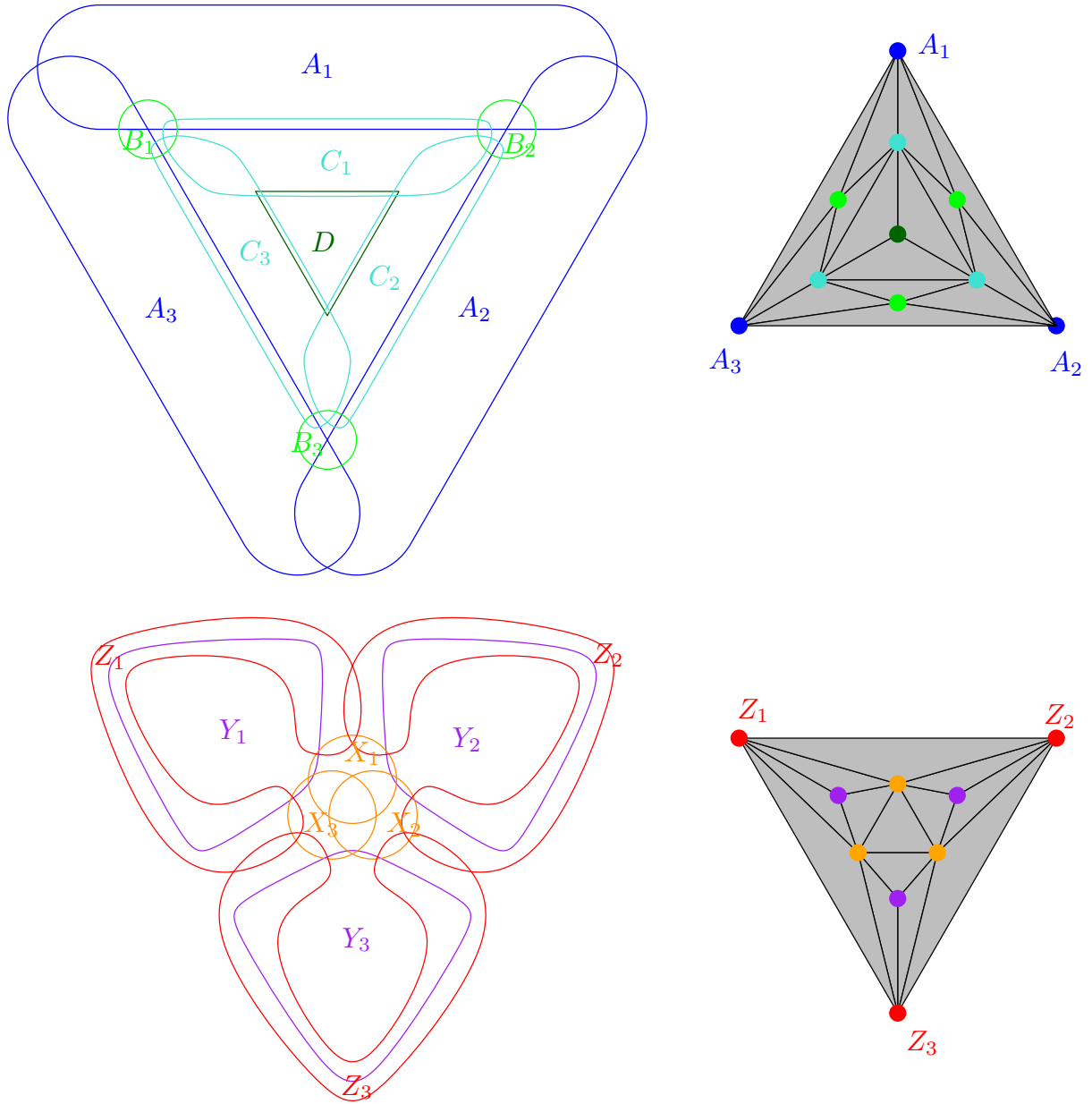
Figure 3: The complexes $Ł^+$ and $Ł^-$.

or L^-. However; new 1-faces are introduced when one vertex comes from L^+ and the second one from L^-. For another triangulations these newly introduced faces can be 2-collapsible.[3]

3 Higher dimensions

Joins of simplicial complexes will help us to generalize the counterexample to higher dimensions. Let $\mathbf{K}$ and $\mathbf{K}'$ be simplicial complexes with the vertex sets $V(\mathbf{K})$ and $V(\mathbf{K}')$. Their *join* is a simplicial complex $\mathbf{K} \star \mathbf{K}'$ whose vertex set is the disjoint union $V(\mathbf{K}) \sqcup V(\mathbf{K}')$;[4] and whose set of faces is $\{\alpha \sqcup \beta : \alpha \in \mathbf{K}, \beta \in \mathbf{K}'\}$.

We need the following two lemmas.

Lemma 3.1 ([MT09, Lemma 4.2])**.** *For every two simplicial complexes* $\mathbf{K}$, $\mathbf{K}'$ *we have* $\gamma_0(\mathbf{K} \star \mathbf{K}') = \gamma_0(\mathbf{K}) + \gamma_0(\mathbf{K}')$.

Lemma 3.2. *Let* $\mathbf{K}$ *be a convexly/topologically d-representable complex and* $\mathbf{K}'$ *be a convexly/topologically d'-representable complex. Then* $\mathbf{K} \star \mathbf{K}'$ *is a convexly/topologically $(d + d')$-representable complex.*

Proof. Let $\mathcal{F}$ be a collection of convex sets/good cover in $\mathbb{R}^d$ such that $\mathbf{K}$ is isomorphic to the nerve of $\mathcal{F}$. Similarly $\mathcal{F}'$ is a suitable collection in $\mathbb{R}^{d'}$ such that $\mathbf{K}'$ is isomorphic to the nerve of $\mathcal{F}'$.

Let us set

$$\mathcal{F} \star \mathcal{F}' := \{F \times \mathbb{R}^{d'} : F \in \mathcal{F}\} \cup \{\mathbb{R}^d \times F' : F' \in \mathcal{F}'\}.$$

Then it is easy to check that $\mathbf{K} \star \mathbf{K}'$ is isomorphic to the nerve of $\mathcal{F} \star \mathcal{F}'$. Moreover $\mathcal{F} \star \mathcal{F}'$ is a collection of convex sets/good cover in $\mathbb{R}^{d+d'}$. $\qquad\square$

Now we can finish the proof of our main result.

Proof of Theorem 1.2. Let $\mathbf{T}$ be the simplicial complex consisting of two isolated points. The complex $\mathbf{T}$ is topologically 1-representable and $\gamma_0(\mathbf{T}) = 1$. Let us set

$$\mathbf{J} = \mathbf{L} \star \underbrace{\mathbf{T} \star \cdots \star \mathbf{T}}_{d-2}.$$

---

[3]It would be perhaps possible to show that the complex is not 2-collapsible even if the newly introduced faces were 2-collapsible. Listing all 1-faces in the previous subsection seems, however, more convenient for the current purpose.

[4]If A and B are sets with $A \cap B \neq \emptyset$ then their disjoint union can be defined as $A \sqcup B := A \times \{1\} \cup B \times \{2\}$.

In topology, the complex $\mathbf{J}$ would be called $(d-2)$-tuple *suspension* of Ł. Then $\gamma_0(\mathbf{J}) = d+1$ due to Proposition 2.1 and Lemma 3.1. On the other hand, $\mathbf{J}$ is topologically d-representable due to Lemma 3.2. $\qquad\square$

4 Conclusion

In the spirit of Helly-type theorems we could ask whether there is at least some weaker bound for collapsibility of topologically d-representable complexes.

Question 4.1. *For which $d \geq 2$ there is a $d' \in \mathbb{N}$ (as least as possible) such that every topologically d-representable complex is d-collapsible?*

Using joins of multiple copies of Ł (instead of suspensions of Ł) we obtain the following bound.

Proposition 4.2. *For every $d \geq 2$ there is a simplicial complex which is topologically $2d$-representable but not $(3d-1)$-collapsible.*

Proof. Consider the complex $\underbrace{Ł \star \cdots \star Ł}_{d}$. $\qquad\square$

If there is a wider gap among these notions it will also reflect at the gap between d-representable and d-*Leray* complexes obtained (with a similar method) by Matoušek and the author [MT09].

Acknowledgement

References

[Hat01]   A. Hatcher. *Algebraic Topology.* Cambridge University Press, Cambridge, 2001.

[Hel23]   E. Helly. Über mengen konvexer Körper mit gemeinschaftlichen Punkten. *Jahresber. Deustch. Math.-Verein.*, 32:175–176, 1923.

[Hel30]   E. Helly. Über Systeme von abgeschlossenen Mengen mit gemeinschaftlichen Punkten. *Monaths. Math. und Physik*, 37:281–302, 1930.

[KM05]   G. Kalai and R. Meshulam. A topological colorful Helly theorem. *Adv. Math.*, 191(2):305–311, 2005.

[Mat03] J. Matoušek. *Using the Borsuk-Ulam Theorem.* Springer, Berlin etc., 2003.

[MT09]   J. Matoušek and M. Tancer. Dimension gaps between representability and collapsibility. *Discrete Comput. Geom.*, 42(4):631–639, 2009.

[Mun84] J. R. Munkres. *Elements of Algebraic Topology.* Addison - Wesley, 1984.

[Weg75] G. Wegner. d-collapsing and nerves of families of convex sets. *Arch. Math.*, 26:317–321, 1975.