NOTES ON ENTOURAGES AND LOCALIC GROUPS

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Abstract. The relation between the cover (Tukey type) uniformities and the entourage (Weil type) ones, in the point-free context, is studied and a transparent translation is presented. In particular the natural uniformities on localic groups are discussed, and the uniformity of localic group homomorphisms is proved.

Introduction

In the classical spaces, a *uniformity* on X is approached, basically, by two different (but equivalent) ways:

– one can take systems of special covers of X (reminiscent of the system of covers of a metric space by the ε-balls) — the Tukey mode (see e.g. [24, 11]),

– or one can consider systems of special "neighborhoods of the diagonal" (entourages) in the product $X \times X$ (reminiscent of the ε-entourages $\{(x, y) \mid \rho(x, y) < \varepsilon\}$ in a metric space) — the Weil mode ([25]).

Both can be extended to the point-free spaces (locales). Thus we have a quite extensive literature about the cover uniformities, starting with the pioneering Isbell's [12] (further see e.g. [1, 2, 3, 10, 22, 23], etc.). On the other hand the entourage and kindred types of uniformities were thoroughly studied e.g. in [19, 20, 21] and from another technical standpoint (a functional one in nature, based on a certain kind of Galois connections – axialities – rather than on entourages) e.g. in [8, 9] (for a discussion of the relation with the entourage technique see [7]).

There is a fundamental difference in these two extensions of the classical structures to the more general point-free context. While the cover description (see 1.2 below) is a straightforward generalization of the

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classical one (covers are covers, star-refinement corresponds to the classical one), the entourage definition is, rather, just an extension by analogy, or mimicking the classical definition in the category of locales: the (binary) product of locales (coproduct of frames $L \oplus L$) does not quite correspond to the product of spaces, and hence we can think of the entourages $E \in L \oplus L$ (see 1.4) only guardedly as of models of open sets containing the diagonal.

Nevertheless, the two approaches are, again, equivalent (which should come, in a way, as a bit of surprise).

Now while, as we have already said, there is abundant literature on both the cover and the entourage uniformities, the relation between the two has been somewhat neglected. The equivalence was proved by the first author in his Thesis ([19]; cf. [20]) using a certain technical detour (specifically, Lemma 3.1 of [20] about the behaviour of the composition operator on down-sets of $L \times L$); but this seems to be about all. Thus, one does not have, to our knowledge, a direct proof of the equivalence in the standard journal literature; one of the aims of this paper is to fill in this gap.

Further, we concentrate on the uniform structure of localic groups (analogues of topological groups in the point-free context). In the original article about this subject ([13]) it was shown that similarly like in the classical case one has natural cover uniformities induced by the group structure. In fact one has equally (if not even more) natural entourage ones ([21]). We describe them and show their relation to (of course, an equivalence with) the cover ones; while doing this we also discuss the semigroup of open parts of a localic group (which has not yet been presented in this detail). As an application we present an extremely simple proof of the fact that the localic group homomorphisms are uniform; it should be noted that this fact has so far, to our knowledge, not been proved in the literature by the cover methods, and even remaking our simple entourage proof to a cover one by translation seems to be rather complex. We see it as another corroboration of the usefulness of the entourage approach.

Only basic knowledge of classical topology and of category theory (as in the less involved parts of [18]) is assumed. The necessary definitions and facts concerning frames (locales) are presented in Preliminaries below.

1. Preliminaries

1.1. Recall that a *frame* is a complete lattice satisfying the distributive law

$$(\bigvee A) \wedge b = \bigvee\{a \wedge b \mid a \in A\}$$

for all subsets $A \subseteq L$ and all $b \in L$. A frame homomorphism $h : L \to M$ preserves all joins (including the void one, the bottom 0) and all *finite* meets (including the top 1). The resulting category will be denoted by

$$\mathbf{Frm}.$$

A typical frame is the lattice $\Omega(X)$ of all open sets of a topological space X; if $f : X \to Y$ is a continuous map then $\Omega(f) = (U \mapsto f^{-1}[U]) : \Omega(Y) \to \Omega(X)$ is a frame homomorphism. Thus one has a contravariant functor $\Omega : \mathbf{Top} \to \mathbf{Frm}$ (where **Top** is the category of topological spaces). Setting

$$\mathbf{Loc} = \mathbf{Frm}^{\mathrm{op}}$$

one obtains the *category of locales*. Then Ω becomes a contravariant functor $\mathbf{Top} \to \mathbf{Loc}$; furthermore, restricted to the subcategory **Sob** of sober spaces it is a full embedding. Thus one can think of locales as a generalization of (sober) topological spaces. For more about frames see e.g. [14, 23].

1.2. Cover (Tukey) uniformities. A *cover* of a frame L is a subset $U \subseteq L$ such that $\bigvee U = 1$. A cover U *refines* (or is a *refinement* of) a cover V if

$$\forall u \in U\ \exists v \in V \quad \text{such that} \quad u \leq v.$$

This is indicated by writing $U \leq V$.

For covers U, V we have the largest common refinement in the preorder $\leq$,

$$U \wedge V = \{u \wedge v \mid u \in U, v \in V\}.$$

If $U \subseteq L$ is a cover and $a \in L$ we set

$$Ua = \bigvee\{u \in U \mid u \wedge a \neq 0\}$$

and for covers U, V define

$$UV = \{Uv \mid v \in V\}.$$

Note that if U is a cover of L and if $h : L \to M$ is a frame homomorphism then

$$h[U]h(a) \leq h(Ua) \tag{1.2.1}$$

(if $h(u) \wedge h(a) \neq 0$ then $u \wedge a \neq 0$ and hence $h(u) \leq h(Ua)$ for $u \in U$).

Finally, for a set of covers $\mathcal{U}$ define a relation

$$b \lhd_{\mathcal{U}} a \quad \equiv_{\mathrm{df}} \quad \text{there is a } U \in \mathcal{U} \text{ such that} \quad Ub \leq a.$$

$\mathcal{U}$ is said to be *admissible* if

$$\forall a \in L, \quad a = \bigvee \{b \mid b \lhd_{\mathcal{U}} a\}.$$

A *(cover-) uniformity* on a frame L is an admissible non-empty system of covers $\mathcal{U}$ such that

(U1) if $U \in \mathcal{U}$ and $U \leq V$ then $V \in \mathcal{U}$,
(U2) if $U, V \in \mathcal{U}$ then $U \wedge V \in \mathcal{U}$,
(U3) for every $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that $VV \leq U$.

Note that if $\mathcal{U}$ is a uniformity then the relation $\lhd_{\mathcal{U}}$ interpolates, that is, if $a \lhd_{\mathcal{U}} b$ then there is a c such that $a \lhd_{\mathcal{U}} c \lhd_{\mathcal{U}} b$ (if $Ua \leq b$ take a V from (U3) and set $c = Va$; use the easy fact that $U(Vx) \leq (UV)x$ to obtain $Vc \leq b$).

1.3. Recall (see e.g. [14, 23]; it should be noted that the first construction of frame coproducts appeared in [6]) that the coproduct $L \oplus M$ in **Frm** (product in **Loc**) can be constructed as follows.

First take the Cartesian product $L \times M$ as a poset and $\mathfrak{D}(L \times M) = \{U \subseteq L \times M \mid \downarrow U = U \neq \emptyset\}$ (where $\downarrow U = \{(x, y) \mid (x, y) \leq (a, b) \in U\}$, as usual), and call a $U \in \mathfrak{D}(L \times M)$ *saturated* if

(1) for any subset $A \subseteq L$ and any $b \in M$, if $A \times \{b\} \subseteq U$ then $(\bigvee A, b) \in U$, and
(2) for any $a \in L$ and any subset $B \subseteq M$, if $\{a\} \times B \subseteq U$ then $(a, \bigvee B) \in U$.

The set A resp. B can be void; hence, in particular, each saturated set contains as a subset

$$\mathsf{n} = \{(0, b), (a, 0) \mid a \in L, b \in M\}.$$

It is easy to check that for each $(a, b) \in L \times M$,

$$a \oplus b = \downarrow(a, b) \cup \mathsf{n} \textit{ is saturated.}$$

To finish the construction of a coproduct one takes

$$L \oplus M = \{U \in \mathfrak{D}(L \times M) \mid U \text{ saturated}\}$$

with the coproduct injections

$$\iota_L = (a \mapsto a \oplus 1) : L \to L \oplus M, \quad \iota_M = (b \mapsto 1 \oplus b) : M \to L \oplus M.$$

Note that we have

(1.3.1) for each saturated U,

$$U = \bigvee \{a \oplus b \mid (a, b) \in U\} = \bigcup \{a \oplus b \mid (a, b) \in U\}, \quad \text{and}$$

(1.3.2) if $a \oplus b \leq c \oplus d$ and $b \neq 0$ then $a \leq c$.

1.4. Entourage (Weil) uniformities. An *entourage* in L is an element $E \in L \oplus L$ such that

$$\{u \mid u \oplus u \leq E\}$$

is a cover of L.

For entourages E, F of L set

$$E \circ F = \bigvee \{a \oplus c \mid \exists b \neq 0, \quad a \oplus b \leq E \text{ and } b \oplus c \leq F\} =$$
$$= \bigvee \{a \oplus c \mid \exists b \neq 0, \quad (a, b) \in E \text{ and } (b, c) \in F\}.$$

(Caution: unions of saturated sets are not necessarily saturated and the join above is typically bigger than the corresponding union.)

Further, for an entourage E set

$$E^{-1} = \{(a, b) \mid (b, a) \in E\}$$

(which is obviously an entourage again).

If E is an entourage (resp. $\mathcal{E}$ a set of entourages) write

$$b \lhd_E a \ \text{ if } \ E \circ (b \oplus b) \leq a \oplus a, \quad \text{and} \quad b \lhd_{\mathcal{E}} a \ \text{ if } \ \exists E \in \mathcal{E}, \ b \lhd_E a.$$

A set of entourages $\mathcal{E}$ is said to be *admissible* if

$$\forall a \in L, \quad a = \bigvee \{b \mid b \lhd_{\mathcal{E}} a\}.$$

An *entourage uniformity* on a frame L is an admissible set of entourages $\mathcal{E}$ such that

(E1) if $E \in \mathcal{E}$ and $E \leq F$ then $F \in \mathcal{E}$,
(E2) if $E, F \in \mathcal{E}$ then $E \cap F \in \mathcal{E}$,
(E3) if $E \in \mathcal{E}$ then E^{-1} is in $\mathcal{E}$, and
(E4) for every $E \in \mathcal{E}$ there is an $F \in \mathcal{E}$ such that $F \circ F \leq E$.

Note that obviously intersections of saturated elements are saturated; hence the $E \cap F$ in (E2) makes sense.

1.5. If $\mathcal{U}$ resp. $\mathcal{E}$ is a cover- resp. entourage-uniformity on L one speaks of $(L, \mathcal{U})$ resp. $(L, \mathcal{E})$ as a *cover-* resp. *entourage-uniform frame.* As a rule it is obvious which of the two it is, and one speaks simply of a *uniform frame.*

1.5.1. Let $(L, \mathcal{U})$, $(M, \mathcal{V})$ be cover-uniform frames. A frame homomorphism $h : L \to M$ is said to be *uniform* if

$$\forall U \in \mathcal{U}, \quad h[U] \in \mathcal{V}.$$

1.5.2. Let $(L, \mathcal{E})$, $(M, \mathcal{F})$ be entourage-uniform frames. A frame homomorphism $h : L \to M$ is said to be *uniform* if

$$\forall E \in \mathcal{E}, \quad (h \oplus h)(E) \in \mathcal{F}$$

(where $h \oplus h$ is the frame homomorphism $L \oplus L \to M \oplus M$ defined by $(h \oplus h)\iota_i = \iota_i h$ for $i = 1, 2$).

1.6. Bases of uniformities. A uniformity is often described by a *basis* $\mathcal{U}' \subseteq \mathcal{U}$ ($\mathcal{E}' \subseteq \mathcal{E}$), that is, a subset such that

$$\mathcal{U} = \{V \mid U \leq V \text{ for a } U \in \mathcal{U}'\} \quad \text{resp.} \quad \mathcal{E} = \{F \mid E \leq F \text{ for an } E \in \mathcal{E}'\}$$

(the $\leq$ in the former is the refinement preorder while in the latter it is the order in the frame $L \oplus L$).

Note that an entourage uniformity has for instance the basis constituted by the *symmetric* entourages, that is, entourages E such that $E^{-1} = E$: indeed, for a general E consider the $E^{-1} \cap E \leq E$.

1.6.1. One very often works, instead of with the whole uniformities with their bases, usually very naturally described. Then the formulas for homomorphisms are modified to

$$\begin{aligned} &\forall U \in \mathcal{U}\ \exists V \in \mathcal{V} \quad \text{such that} \quad h[U] \geq V, \\ \text{resp.}\quad &\forall E \in \mathcal{E}\ \exists F \in \mathcal{F} \quad \text{such that} \quad (h \oplus h)(E) \geq F. \end{aligned}$$

2. Translations

2.1. In this section we will prove that the two concepts of uniformity from 1.2 and 1.4 are equivalent in the sense that there are natural translations of one into the other, with satisfactory properties (for instance frame homomorphisms are uniform iff they are uniform when the uniformities are replaced by the associated ones).

Recall from the Introduction that while the cover uniformities are genuine generalization of the classical ones, the entourage ones are, rather, just an analogy of the classical definition (since products of locales – coproducts of frames – do not quite correspond to products of classical spaces). Hence the fact of equivalence is somewhat deeper than it sounds.

2.2. The following will play a crucial role.

Lemma. *Let U be a cover of L and let $x \oplus y \leq \bigvee\{u \oplus u \mid u \in U\}$. Let $y \neq 0$. Then $x \leq Uy$.*

Proof. We have

$$\begin{aligned} x \oplus y &= \bigvee\{(u \wedge x) \oplus (u \wedge y) \mid u \in U\} = \\ &= \bigvee\{(u \wedge x) \oplus (u \wedge y) \mid u \in U,\ u \wedge y \neq 0\} \leq (Uy \wedge x) \oplus y. \end{aligned}$$

Thus, if $y \neq 0$, $x \leq x \wedge Uy$, and finally $x \leq Uy$. ☐

2.3. Lemma. *If $U_E = \{x \mid x \oplus x \leq E\}$ is a cover then $E \leq E \circ E$.*

Proof. Let $a \oplus b \leq E$ and $b \neq 0$. We have

$$b = \bigvee\{b \wedge u \mid u \in U_E\} = \bigvee\{b \wedge u \mid u \in U_E,\ u \wedge b \neq 0\}.$$

Now we have, for $b \wedge u \neq 0$, $a \oplus (b \wedge u) \leq E$ and $(b \wedge u) \oplus (b \wedge u) \leq E$, hence $a \oplus (b \wedge u) \leq E \circ E$ and hence $a \oplus b = a \oplus \bigvee\{b \wedge u \mid u \in U_E\} = \bigvee\{a \oplus (b \wedge u) \mid u \in U_E\} \leq E \circ E$. ☐

2.4. For an entourage E define

$$\widetilde{E} = \bigvee\{u \oplus u \mid u \oplus u \le E\}.$$

2.4.1. Lemma. *Let F be a symmetric entourage and let $F \circ F \le E$. Then for each $a \oplus b \le F$ we have $(a \vee b) \oplus (a \vee b) \le E$*

Consequently, if $\mathcal{E}$ is a uniformity then for each $E \in \mathcal{E}$, $\widetilde{E} \in \mathcal{E}$.

Proof. Let $a \oplus b \le F$, so that also $b \oplus a \le F$ and hence

$$a \oplus a, b \oplus b \le F \circ F \le E;$$

by 2.3 also

$$a \oplus b, b \oplus a \le F \circ F \le E.$$

Thus $a \oplus (a \vee b) \le E$, $b \oplus (a \vee b) \le E$ and finally $(a \vee b) \oplus (a \vee b) \le E$ and we conclude that $a \oplus b \le (a \vee b) \oplus (a \vee b) \le \widetilde{E}$. □

2.5. Translations. For a cover U define an entourage E_U, and for an entourage E define a cover U_E as follows.

$$E_U = \bigvee\{x \oplus x \mid x \in U\},$$
$$U_E = \{x \mid x \oplus x \le E\}.$$

2.5.1. Lemma. (a) $U \le U_{E_U} \le UU$.

(b) $\widetilde{E} = E_{U_E} \le E$.

Proof. (a) If $x \oplus x \le E_U$ then for any $u_0 \in U$ such that $u_0 \wedge x \ne 0$ we have by 2.2

$$x \le U(u_0 \wedge x) \le Uu_0.$$

Thus,

$$U_{E_U} = \{x \mid x \oplus x \le E_U\} \le UU.$$

On the other hand, trivially, $U \le U_{E_U}$.

(b) We have, by the definitions, $E_{U_E} = \bigvee\{x \oplus x \mid x \oplus x \le E\} = \widetilde{E}$. □

2.5.2. Lemma. (a) $b \lhd_E a \quad \Rightarrow \quad U_E b \le a$.

(b) $Ub \le a \quad \Rightarrow \quad b \lhd_{E_U} a$.

Proof. (a): Let $u \in U_E$ and $u \wedge b \ne 0$. Then $u \oplus (u \wedge b) \le E_U$ and $(u \wedge b) \oplus b \le b \oplus b$, and hence $u \oplus b \le E \circ (b \oplus b) \le a \oplus a$; thus, as $b \ne 0$, $u \le a$ and we conclude that $U_E b \le a$.

(b): Let $Ub \le a$ and let $x \oplus y \le E_U$ and $y \oplus z \le b \oplus b$ for some $y \ne 0$. Then $x \oplus y \le \bigvee\{u \oplus u \mid u \in U\}$ and by 2.2, $x \le Uy$. Thus, $x \oplus z \le Uy \oplus b \le Ub \oplus b \le a \oplus a$. □

For an entourage uniformity $\mathcal{E}$ set $\mathcal{U}_\mathcal{E} = \{V \mid V \ge U_E,\ E \in \mathcal{E}\}$ and for a cover uniformity $\mathcal{U}$ define $\mathcal{E}_\mathcal{U} = \{F \mid F \ge E_U,\ U \in \mathcal{U}\}$.

2.5.3. Theorem. *The correspondences $\mathcal{E} \mapsto \mathcal{U}_{\mathcal{E}}$ and $\mathcal{U} \mapsto \mathcal{E}_{\mathcal{U}}$ constitute a one-one correspondence between the entourage uniformities and the cover ones. More explicitly, $\mathcal{E}_{\mathcal{U}_{\mathcal{E}}} = \mathcal{E}$ and $\mathcal{U}_{\mathcal{E}_{\mathcal{U}}} = \mathcal{U}$.*

Proof. By 2.5.2, $\lhd_{\mathcal{U}} = \lhd_{\mathcal{E}_{\mathcal{U}}}$ (and $\lhd_{\mathcal{E}} = \lhd_{\mathcal{U}_{\mathcal{E}}}$). Thus, if any of the associated uniformities is admissible then the other is as well.

By 2.5.1 we have the formulas $\mathcal{E}_{\mathcal{U}_{\mathcal{E}}} = \mathcal{E}$ and $\mathcal{U}_{\mathcal{E}_{\mathcal{U}}} = \mathcal{U}$.

Thus, it remains to be proved that the systems $\mathcal{U}_{\mathcal{E}}$ and $\mathcal{E}_{\mathcal{U}}$ are uniformities (of their types). We will prove (U3) and (E4), the other facts are straightforward.

To prove (U3) we will show that if F is symmetric and $(F \circ F) \circ (F \circ F) \leq E$ then $U_F U_F \leq U_E$. Set $F_1 = F \circ F$. Fix a $u \in U_F$ and take an arbitrary $v \in U_F$ such that $u \wedge v \neq 0$. Then because of $u \wedge v \neq 0$, $v \oplus u \in F_1 = F \circ F$ and since F_1 is saturated,

$$U_F u \oplus u = (\bigvee \{v \mid v \wedge u \neq 0\}) \oplus u \leq F_1.$$

Since F is symmetric, F_1 is symmetric as well and hence by 2.4.1 $U_F u \oplus U_F u \leq F_1 \circ F_1 \leq E$, and $U_F u \in U_E$.

To prove (E4) we will show that $E_U \circ E_U \leq E_{UU}$. Take an $x \oplus y \leq E_U \circ E_U$; hence there is a $y \neq 0$ such that $x \oplus y \leq E_U$ and $y \oplus z \leq E_U$. Choose a $u \in U$ such that $u \wedge y \neq 0$. Since $x \oplus (u \wedge y) \leq E_U$ and $(u \wedge y) \oplus z \leq E_U$ we obtain from 2.2 that $x \oplus z \leq Uu \oplus Uu \leq E_{UU}$. □

2.6. Proposition. *For the associated entourage and cover uniformities, the concepts of a uniform homomorphism coincide.*

Proof. I. Suppose that for each $E \in \mathcal{E}$ there exists an $F \in \mathcal{F}$ such that

$$(*) \qquad (h \oplus h)(E) \geq F.$$

Take $U \in \mathcal{U}$ and a $V \in \mathcal{U}$ such that $VV \leq U$. By $(*)$ (and 1.4) there exists in particular a $W \in \mathcal{V}$ such that

$$(h \oplus h)(E_V) \geq E_W.$$

Take a $w \in W$. Then $w \oplus w \leq \bigvee \{h(v) \oplus h(v) \mid v \in V\}$ and hence, by Lemma 2.2, if we take a $v_0 \in V$ such that $y = w \wedge h(v_0) \neq 0$ we obtain that $w \leq h[V]y \leq h[V]h(v_0) \leq h(Vv_0) \leq h(u)$ for some $u \in U$. Thus, $W \leq h[U]$.

II. Let for each $U \in \mathcal{U}$ there be a $V \in \mathcal{V}$ such that

$$h[U] \geq V.$$

Consider an $E \in \mathcal{E}$. There is an $F \in \mathcal{F}$ such that $h[U_E] \geq U_F$. Let $v \oplus v \leq F$. Then $v \in U_F$ and hence there is a $u \in U_E$ such that $v \leq h(u)$. Then

$$v \oplus v \leq h(u) \oplus h(u) \leq (h \oplus h)(u \oplus u) \leq (h \oplus h)(E)$$

and hence $(h \oplus h)(E) \geq \widetilde{F}$. Recall 2.4. □

3. Localic groups and the associated semigroups of open parts

3.1. Recall that a group in a category $\mathcal{C}$ with products is a collection of data (A, m, i, e) with

$$m : A \times A \to A, \quad i : A \to A, \quad e : T \to A \quad \text{morphisms in } \mathcal{C}$$

($T = A^0$ is the empty product, that is, the terminal object of $\mathcal{C}$, the object such that from every $X \in \mathcal{C}$ there is precisely one morphism $t_X : X \to T$) such that

$$\begin{aligned} &m(m \times \mathrm{id}) = m(\mathrm{id} \times m), \\ &m(e \times \mathrm{id}) = m(\mathrm{id} \times e) = \mathrm{id}, \quad \text{and} \\ &m(i \times \mathrm{id})\Delta = m(\mathrm{id} \times i)\Delta = et_A. \end{aligned}$$

Note that if $\mathcal{C}$ is the category of sets (where products are the Cartesian ones and $T = \{\emptyset\}$ is the one-point set this is the classical group $(A, \cdot, (-)^{-1}, e)$ and the identities above are the standard $x(yz) = (xy)z$, $xe = ex = x$ and $xx^{-1} = x^{-1}x = e$; or e.g. in **Top** we have operations such that the maps $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ are continuous (the classical topological group).

Since we will work with frames and frame homomorphisms rather than with morphisms in **Loc** we will think of a localic group as a system $(L, \mu, \iota, \varepsilon)$ where

$$\mu : L \to L \oplus L, \quad \gamma : L \to L, \quad \varepsilon : L \to \mathbf{2} = \{0, 1\}$$

are frame homomorphisms such that

$$\begin{aligned} &(\mu \oplus \mathrm{id})\mu = (\mathrm{id} \oplus \mu)\mu, \\ &(\varepsilon \oplus \mathrm{id})\mu = (\mathrm{id} \oplus \varepsilon)\mu = \mathrm{id}, \quad \text{and} \\ &\nabla(\gamma \oplus \mathrm{id})\mu = \nabla(\mathrm{id} \oplus \gamma)\mu = \sigma_L\varepsilon \end{aligned}$$

where $\sigma_L : \mathbf{2} \to L$ sends 0 to 0 and 1 to 1, and ∇ is the codiagonal $L \oplus L \to L$ defined by $\nabla\iota_i = \mathrm{id}$, $i = 1, 2$.

3.1.2. Note. A topological group does not always transform by the functor Ω to a localic group, as one might expect. The map $\Omega(m) : \Omega(X) \to \Omega(X \times X)$ is not a homomorphism with the target $\Omega(X) \oplus \Omega(X)$ and in general cannot be lifted to one. In special cases where Ω respects products (locally compact groups, complete metric ones, etc.) it does. See [13].

3.2. By Kock's theorem ([17]) the counterpart of each identity that can be deduced in a classical equational class (variety) of algebras holds in the associated category of algebras in $\mathcal{C}$.

We will need the following identities ($\tau : L \oplus L \to L \oplus L$ is the homomorphism defined by $\tau\iota_i = \iota_{3-1}$, $i = 1, 2$):

(3.2.1) $\gamma\gamma = \mathrm{id}$ (corresponding to $(x^{-1})^{-1} = x$),

(3.2.2) $\varepsilon\gamma = \varepsilon$ (corresponding to $e^{-1} = e$),
(3.2.3) $(\gamma \oplus \gamma)\mu = \tau\mu\gamma$ (corresponding to $(xy)^{-1} = y^{-1}x^{-1}$),
(3.2.4) and the fact that $\alpha = (\mathrm{id} \oplus \nabla)(\mu \oplus \gamma)$ satisfies $\alpha\alpha = \mathrm{id}$ and $\alpha\iota_1 = \mu$ (α corresponds to the mapping $(x, y) \mapsto (xy, y^{-1})$).

Remark. Needless to say these identities can be deduced directly (by a somewhat lengthy and tedious computation).

3.3. A frame L is not necessarily *spatial*, that is, isomorphic to an $\Omega(X)$. The possible points are modeled by homomorphisms $h : L \to \mathbf{2}$ (mimicking the maps $f : P \to X$ where P is the one-point space). Thus one obtains the *spectrum* ΣL of L, with open sets $\Sigma_a = \{h : L \to \mathbf{2} \mid h(a) = 1\}$, $a \in L$. Each localic group has at least one point, namely $\varepsilon : L \to \mathbf{2}$ (this may be the only one, even if L is large, see [13]); the set

$$N_L = \{a \mid \varepsilon(a) = 1\}$$

can be viewed as the set of (representatives of) the neighbourhoods of the unit.

3.4. Recall that monotone maps $f : (X, \leq) \to (Y, \leq)$ are (Galois) adjoint (f is the left one and g the right one) if

$$f(x) \leq y \quad \text{iff} \quad x \leq g(y).$$

We will denote the left adjoint f of g by

$$g_{\#}.$$

If X, Y are complete lattices f is a left adjoint (resp. g is a right adjoint) iff it preserves all suprema (resp. infima) (see any text on partially ordered sets, e.g. [5]). Thus in particular each frame homomorphism has a right adjoint. We have, though,

3.4.1. Proposition. (a) *The multiplication μ in a localic group has a left adjoint.*

(b) *Also $\mu \oplus \mathrm{id}$ and $\mathrm{id} \oplus \mu$ have left adjoints and there holds*

$$(\mu \oplus \mathrm{id})_{\#}(a \oplus b) = \mu_{\#}(a) \oplus b \quad \textit{and} \quad (\mathrm{id} \oplus \mu)_{\#}(a \oplus b) = a \oplus \mu_{\#}(b).$$

Proof. (a) Recall (3.2.4). We have $\mu = \alpha\iota_1$; α, as an isomorphism, is its own adjoint, and ι_1 has, as it is easy to check, the left adjoint

$$(\iota_1)_{\#}(u) = \bigvee\{x \mid \exists y \neq 0,\ x \oplus y \leq u\}.$$

Thus, we have $\mu_{\#} = (\iota_1)_{\#}\alpha$.

(b) It is easy to check that if $E \in L \oplus L$ is saturated then the *union* $\bigcup\{\downarrow(\mu(x), y) \mid x \oplus y \leq E\}$ is saturated (recall 1.3) so that

$$(\mu \oplus \mathrm{id})(E) = \bigcup\{\downarrow(\mu(x), y \mid x \oplus y \leq E\}.$$

Hence

$$\begin{aligned}
&F \le (\mu \oplus \mathrm{id})(E)\\
\text{iff}\quad &\forall a \oplus b \le F,\ \ (a,b) \in \bigcup\{{\downarrow}(\mu(x), y) \mid x \oplus y \le E\}\\
\text{iff}\quad &\forall a \oplus b \le F\ \ \exists x \oplus y \le E,\ a \le \mu(x) \text{ and } b \le y\\
\text{iff}\quad &\forall a \oplus b \le F\ \ \exists x \oplus y \le E,\ \mu_\#(a) \le x \text{ and } b \le y\\
\text{iff}\quad &\forall a \oplus b \le F\ \ \mu_\#(a) \oplus b \le E\\
\text{iff}\quad &\varphi(F) = \bigvee\{a \oplus b \mid \mu_\#(a) \oplus b \le E\} \le E.
\end{aligned}$$

In particular $(\mu \oplus \mathrm{id})_\#(a \oplus b) = \varphi(a \oplus b) = \mu_\#(a) \oplus b$. □

Note. The first statement is a part of Johnstone's stronger observation ([15]) that μ is an open homomorphism (open homomorphisms are counterparts of open continuous maps, and are characterized as Heyting homomorphisms that have left adjoint ([16]). The L in a localic group is always regular, and hence the Heyting part follows from the adjointness ([14, 23]).

3.5. The "semigroup of open parts". The algebra $(L, *, (-)^{-1})$ to be introduced is a counterpart of the semigroup (with involution) of open subsets of a topological group, with the operations

$$UV = \{uv \mid u \in U, v \in V\}, \quad U^{-1} = \{u^{-1} \mid u \in U\}.$$

It appeared in passing in [4] with the proofs of the properties just hinted. Here we will be more explicit.

On L define a (classical) binary operation $*$ and a unary operation $(-)^{-1}$ on L by setting

$$x * y = \mu_\#(x \oplus y), \qquad x^{-1} = \gamma(x).$$

3.5.1. Observation. Since ι_1 is one-one, $\mu = \alpha\iota_1$ is one-one and we easily infer that

$$\mu\mu_\# \ge \mathrm{id} \quad \text{and} \quad \mu_\#\mu = \mathrm{id}.$$

In particular

$$\mu_\#(0) = 0.$$

3.5.2. Proposition. (1) *If $x' \le x$ and $y' \le y$ then $x' * y' \le x * y$.*
(2) *If $x = 0$ or $y = 0$ then $x * y = 0$.*
(3) *The operation $*$ is associative.*
(4) *If $y \in N$ then $x * y \ge x$ and $y * x \ge x$.*
(5) *If $x \wedge y \ne 0$ then $x * y^{-1} \in N$.*
(6) *$(x * y)^{-1} = y^{-1} * x^{-1}$.*
(7) *If $x \in N$ then $x^{-1} \in N$.*

Proof. (1) and (2) are trivial.

(3) Using 3.4.1 we obtain

$$\begin{aligned} a*(b*c) &= \mu_\#(a\oplus\mu_\#(b\oplus c)) = \mu_\#(\mathrm{id}\oplus\mu)_\#(a\oplus b\oplus c) = \\ &= ((\mathrm{id}\oplus\mu)\mu)_\#(a\oplus b\oplus c) = ((\mu\oplus\mathrm{id})\mu)_\#(a\oplus b\oplus c) = \\ &= \mu_\#(\mu\oplus\mathrm{id})_\#(a\oplus b\oplus c) = \mu_\#(\mu_\#(a\oplus b)\oplus c) = (a*b)*c. \end{aligned}$$

(4) Applying $\mathrm{id}\oplus\varepsilon$ on both sides of $\mu\mu_\#(x\oplus y)\geq x\oplus y$ we obtain

$$x*y\geq(\mathrm{id}\oplus\varepsilon)(x\oplus y) = x\oplus\varepsilon(y) = x\wedge\varepsilon(y).$$

(5) Since we have

$$\begin{aligned} \sigma\varepsilon(x*y^{-1}) &= \nabla(\mathrm{id}\oplus\gamma)\mu\mu_\#(\mathrm{id}\oplus\gamma)(x\oplus y)\geq \\ &\geq \nabla(\mathrm{id}\oplus\gamma)(\mathrm{id}\oplus\gamma)(x\oplus y) = x\wedge y\neq 0, \end{aligned}$$

$\varepsilon(x*y^{-1})$ cannot be 0.

(6) Since $\tau_\# = \tau$ and $\gamma_\# = \gamma$, we obtain from (3.2.3) that

$$\mu_\#(\gamma\oplus\gamma) = \gamma\mu_\#\tau,$$

and as $\tau(x\oplus y) = y\oplus x$ we conclude

$$x^{-1}*y^{-1} = \mu_\#(\gamma(x)\oplus\gamma(y)) = \gamma\mu_\#(y\oplus x) = (y*x)^{-1}.$$

(7) follows from (3.2.2). □

4. Uniformities on localic groups

4.1. More on the semigroup $(L,*,(-)^{-1})$. Recall the system $N = N_L = \{a \mid \varepsilon(a) = 1\}$ of "neighbourhoods of unit" from 3.3. Obviously

$$a,b\in N \;\;\Rightarrow\;\; a\wedge b\in N \quad\text{and}\quad a\in N \;\;\Rightarrow\;\; a^{-1} = \gamma(a)\in N.$$

Here are some more facts about the multiplication.

4.1.1. Lemma. *If $c*b\leq a$, $u*u^{-1}\leq c$ and $v\wedge b\neq 0$ then $u\leq a$.*

Proof. We have $c\oplus b\leq\mu(a)$ and $u\oplus\gamma(u)\leq\mu(c)$ so that

$$u\oplus\gamma(u)\oplus b\leq\mu(c)\oplus b = (\mu\oplus\mathrm{id})(c\oplus b)\leq(\mu\oplus\mathrm{id})\mu(a) = (\mathrm{id}\oplus\mu)\mu(a).$$

Applying $\mathrm{id}\oplus\nabla(\gamma\oplus\mathrm{id})$ we obtain on the leftmost side $u\oplus(u\wedge b)$ (as $\nabla(x\oplus y) = x\wedge y$) and on the rightmost one

$$\begin{aligned} (\mathrm{id}\oplus(\nabla(\gamma\oplus\mathrm{id})\mu)\mu(a) &= (\mathrm{id}\oplus\sigma\varepsilon)\mu(a) = \\ &= (\mathrm{id}\oplus\sigma)(\mathrm{id}\oplus\varepsilon)\mu(a) = (\mathrm{id}\oplus\sigma)(a) = a\oplus 1. \end{aligned}$$

Thus $u\oplus(u\wedge b)\leq a\oplus 1$ and since $u\wedge b\neq 0$, $u\leq a$. □

4.1.2. Lemma. *For each $a\in N$ there are $b,c\in N$ such that $b*b\leq a$ and $c*c^{-1}\leq a$*

Proof. Any L can be viewed as a coproduct

$$\mathbf{2}\xrightarrow{\;\sigma_L\;} L\xleftarrow{\;\mathrm{id}_L\;} L$$

and since $\sigma_{\mathbf{2}} = \mathrm{id}_{\mathbf{2}}$ we have

$$\varepsilon = \mathrm{id}_{\mathbf{2}} \oplus \varepsilon : L = \mathbf{2} \oplus L \to \mathbf{2} \oplus \mathbf{2} = \mathbf{2}.$$

Hence, $\varepsilon = (\mathrm{id}_{\mathbf{2}} \oplus \varepsilon)(\varepsilon \oplus \mathrm{id}_L)\mu = (\varepsilon \oplus \varepsilon)\mu$ and we obtain, for $a \in N$,

$$1 = \varepsilon(a) = \bigvee\{\varepsilon(x) \oplus \varepsilon(y) \mid x \oplus y \leq \mu(a)\} = \bigvee\{\varepsilon(x) \oplus \varepsilon(y) \mid x * y \leq a\}$$

so that there are x, y such that $x * y \leq a$ and $\varepsilon(x) = \varepsilon(y) = 1$. Set $b = x \wedge y$ and $c = x \wedge \gamma(y)$. □

4.2. For an $a \in N$ set

$$U(a) = \{x \in L \mid x \oplus \gamma(x) \leq \mu(a)\} = \{x \in L \mid x * x^{-1} \leq a\},$$
$$V(a) = \{x \in L \mid \gamma(x) \oplus x \leq \mu(a)\} = \{x \in L \mid x^{-1} * x \leq a\}$$

and consider the systems

$$\mathcal{U} = \{U \mid U \geq U(a),\ \varepsilon(a) = 1\} \text{ and } \mathcal{V} = \{V \mid V \geq V(a)\ \varepsilon(a) = 1\}.$$

4.2.1. Proposition. *$\mathcal{U}$ and $\mathcal{V}$ are uniformities on L.*

Proof. It will be done for $\mathcal{U}$.

I. *Each $U(a)$ is a cover.* We have

$$U(a) = \{x \wedge y \mid x \oplus y \leq (\mathrm{id} \oplus \gamma)\mu(a)\}.$$

(Indeed, if $x \oplus \gamma(x) \leq \mu(a)$ then $x \oplus x \leq (\mathrm{id} \oplus \gamma)(x \oplus \gamma(x)) \leq (\mathrm{id} \oplus \gamma)\mu(a)$. On the other hand, if $x \oplus y \leq (\mathrm{id} \oplus \gamma)\mu(a)$ then $(x \wedge y) \oplus \gamma(x \wedge y) \leq (\mathrm{id} \oplus \gamma)(x \oplus y) \leq (\mathrm{id} \oplus \gamma)(\mathrm{id} \oplus \gamma)\mu(a) = \mu(a)$.)

Thus,

$$\bigvee U(a) = \bigvee\{x \wedge y \mid x \oplus y \leq (\mathrm{id} \oplus \gamma)\mu(a)\} =$$
$$= \bigvee\{\nabla(x \oplus y) \mid x \oplus y \leq (\mathrm{id} \oplus \gamma)\mu(a)\} =$$
$$= \nabla \bigvee\{x \oplus y \mid x \oplus y \leq (\mathrm{id} \oplus \gamma)\mu(a)\} = \nabla(\mathrm{id} \oplus \gamma)\mu(a) = \sigma\varepsilon(a) = 1.$$

II. *The system $\mathcal{U}$ is admissible.* By 4.1.1, if $c * b \leq a$ then $U(c)b \leq a$. We have $a = (\varepsilon \oplus \mathrm{id})\mu(a) = \bigvee\{\varepsilon(c) \oplus b \mid c \oplus b \leq \mu(a)\} = \bigvee\{b \mid u * b \leq a,\ c \in N\} \leq \bigvee\{b \mid U(c)b \leq a,\ c \in N\}$.

III. Trivially $U(a \wedge b) \leq U(a) \wedge U(b)$.

IV. For $a \in N$ choose, by 4.1.2, a $b \in N$ such that $b * b * b^{-1} * b^{-1} \leq a$. We will show that $U(b)U(b) \leq U(a)$.

Fix an $x \in U(b)$ and consider any $u \in U(b)$ such that $u \wedge x \neq 0$. Thus, $x * x^{-1} \leq b$ and $u * u^{-1} \leq b$ and, by 3.5.2, $(u \wedge x)^{-1} * (u \wedge x) \in N$. Thus,

$$u \leq u * (u \wedge x)^{-1} * (u \wedge x) \leq u * u^{-1} * x \leq b * x$$

and hence $U(b)x \leq b * x$ and finally, since also $b^{-1} \in N$, again by 3.5.2,

$$U(b)x * (U(b)x)^{-1} \leq b * x * x^{-1} * b^{-1} \leq b * b * b^{-1} \leq b * b * b^{-1} * b^{-1} \leq a$$

and $U(b)x \in U(a)$. □

4.2.2. The uniformity $\mathcal{U}$ (resp. $\mathcal{V}$) is called the *left uniformity* (resp. *right uniformity*) on the localic group.

4.3. An alternative description by entourages. For an $a \in N_L$ set

$$E(a) = (\mathrm{id} \oplus \gamma)\mu(a) \quad \text{and} \quad F(a) = (\gamma \oplus \mathrm{id})\mu(a);$$

hence

$$E(a) = \bigvee\{x \oplus y \mid x \oplus y \leq (\mathrm{id} \oplus \gamma)\mu(a)\} = \bigvee\{x \oplus y \mid x * y \leq a\},$$

and similarly for $F(a)$.

Observation. *$E(a)$ and $F(a)$ are entourages.*

Proof. For $x \in U(a)$ we have $x \oplus \gamma(x) \leq \mu(a)$. Hence

$$\bigvee\{x \mid x \oplus x \leq E(a)\} \geq \bigvee U(a) = 1$$

since $U(a)$ is a cover, as we already know. □

4.4. Denote by $\mathcal{E}'$ resp. $\mathcal{F}'$ the system of entourages $\{E(a) \mid a \in N\}$ resp. $\{F(a) \mid a \in N\}$, and set

$$\mathcal{E} = \{E \mid E \text{ entourage}, \ E \geq E(a) \in \mathcal{E}'\},$$
$$\mathcal{F} = \{E \mid E \text{ entourage}, \ E \geq F(a) \in \mathcal{F}'\}.$$

Proposition. *The systems $\mathcal{E}$ and $\mathcal{F}$ are entourage-uniformities and we have, in the notation of 4.2 and 2.5, $\mathcal{E} = \mathcal{E}_{\mathcal{U}}$ and $\mathcal{F} = \mathcal{E}_{\mathcal{V}}$.*

Proof. We will show that $\mathcal{E} = \{E \mid E \text{ entourage}, \ E \geq E(a) \in \mathcal{E}'\} = \mathcal{E}_{\mathcal{U}} = \{E \mid E \text{ entourage}, \ E \geq E_{U(a)}, a \in N\}$.

We have $E_{U(a)}(= \bigvee\{x \oplus x \mid x \oplus \gamma(x) \leq \mu(a)\}) \leq E(a)$.

To obtain an estimate from the other side, choose by 4.1.2 $b, c \in N$ such that $b * b^{-1} \leq c$ and $c * c^{-1} \leq a$. Let $x \oplus y \leq E(b)$. We can assume $x \oplus y \neq 0$, hence $x \neq 0 \neq y$. First, as $y \neq 0$, we have by 3.5.2 ((4), (5) and (7)),

$$x * x^{-1} \leq x * y^{-1} * y * x^{-1} \leq b * b^{-1} \leq c \quad \text{and} \quad x * y^{-1} \leq b * b^{-1} \leq c$$

and hence $(x, x), (x, y) \in E(c)$ and since $E(c)$ is saturated (recall 1.3) we have, for $z = x \vee y$,

$$(x, z) \leq E(c), \quad \text{that is,} \quad x * z^{-1} \leq c.$$

Now $(x*z^{-1})*(x*z^{-1})^{-1} \leq c*c^{-1} \leq a$, hence $(x*z^{-1})\oplus(x*z^{-1}) \leq E_{U(a)}$ and $(x * z^{-1}) * (x * z^{-1})^{-1} \leq \mu(E_{U(a)})$. Since $x \wedge z \neq 0$ we have $(x * z^{-1})^{-1} \in N$ by 3.5.2(5), and by 3.5.2(4) we obtain

$$x * z^{-1} \leq x * z^{-1} * (x * z^{-1})^{-1} \leq \mu(E_{U(a)})$$

so that $x \oplus y \leq x \oplus z \leq E_{U(a)}$. Thus, $E(b) \leq E_{U(a)}$. □

4.5. Localic group homomorphisms. A *localic group homomorphism* (briefly, *LG-homomorphism*)

$$h : (L, \mu_L, \gamma_L, \varepsilon_L) \to (M, \mu_M, \gamma_M, \varepsilon_M)$$

is a frame homomorphism $h : L \to M$ such that

(hom) $\quad \mu_M h = (h \oplus h)\mu_L, \quad \gamma_M h = h\gamma_L \quad \text{and} \quad \varepsilon_M h = \varepsilon_L$

(that is, it is the standard homomorphism between algebras defined in the category **Loc** represented in the category of frames).

4.5.1. Proposition. *Each LG-homomorphism* $h : (L, \mu_L, \gamma_L, \varepsilon_L) \to (M, \mu_M, \gamma_M, \varepsilon_M)$ *is uniform with respect to both the left and the right uniformities.*

Proof. We will prove it for the left uniformity. By 4.4 and 2.6 we can choose whether we will prove it for the $\mathcal{U}$ or for the entourage uniformity $\mathcal{E}$. We will do it for the latter. By (hom) we have

$$\begin{aligned}(h \oplus h)(E(a)) &= (h \oplus h)(\mathrm{id}_L \oplus \gamma)\mu_L(a) = \\ &= (\mathrm{id}_M \oplus \gamma_M)(h \oplus h)\mu_L(a) = (\mathrm{id}_M \oplus \gamma_M)\mu_M(h(a)) = E(h(a))\end{aligned}$$

(since $\varepsilon_M h = \varepsilon_L$, $h(a) \in N_M$, and $E(h(a))$ makes sense). □

4.6. Remarks. (1) Note that we did not have to prove that $\mathcal{E}$ is a uniformity. It followed from the fact that $\mathcal{E}_{\mathcal{U}}$ is one.

(2) The proof of 4.5.1 shows an advantage of the entourage approach. A proof of the fact based on the cover description seems to be rather difficult.

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